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ON S-WEAKLY 1-ABSORBING PRIMARY IDEALS AND SUBMODULES

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ABSTRACT. In this paper, we investigate the concept of S-weakly 1-absorbing primary ideals and submodules in commutative rings and modules. We introduce and analyze the properties of S-weakly 1-absorbing primary ideals, which generalize the weakly 1-absorbing primary ideals, by focusing on ideals that satisfy specific multiplication conditions relative to a multiplicative subset S of the ring. Additionally, we extend the study to S-weakly 1-absorbing primary submodules in R-modules and explore their behavior under various operations, including trivial ring extensions and amalgamations of rings. Key results include characterizations of S-weakly 1-absorbing primary submodules, as well as the transfer of these properties through homomorphisms, factor modules, and localizations. These findings provide further insight into the structure of absorbing-type ideals in algebraic settings.

1. INTRODUCTION

Throughout this study, we focus solely on commutative rings with unity and nonzero unital modules. Let R always denote such a ring, and let M denote such an R-module. We begin by recalling some background material. A proper ideal I of R is an ideal $I \neq R$.

Before presenting results, we introduce some notation and terminology. We denote the set of units of R by $U(R)$, and the set of zero-divisors by $Z(R) = R\setminus \operatorname{Reg}(R)$. Define $Z_I(R) = \{a \in R \mid as \in I \text{ for some } s \in R\setminus I\}$. For an R-module M, the annihilator of $x \in M$ in R is $\operatorname{Ann}_R(x) = \{a \in R \mid ax = 0\}$.

Recall that a nonempty subset S of R is said to be a multiplicative set if $0 \notin S$, $1 \in S$, and S is closed under multiplication. For a multiplicative subset S of R, the quotient ring of R at S is denoted by $S^{-1}R = \left\{\frac{a}{s} \mid a \in R, s \in S\right\}$.

By $\sqrt{I}$, we mean the radical of I, that is, $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$. For two ideals I and J of R, the residual division of I and J is defined as $(I : J) = \{a \in R \mid aJ \subseteq I\}$.

Let M be an R-module and N a submodule of M. We define $(N :_R M) = \{a \in R \mid am \in N \text{ for all } m \in M\}$, which is an ideal of R, called the residual ideal of N in M.

Finally, by $\operatorname{rad}_M(N)$, we denote the radical of N in M, which is the intersection of all prime submodules of M containing N.

Consider a multiplicative subset S of a ring R, and let I be a proper ideal of R that is disjoint from S. Recently, S-versions of prime and primary ideals, along

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with their properties, have been thoroughly researched in various studies. Moreover, absorbing-type ideals have been proposed and investigated. From the literature, we have gathered the following definitions (see [6, 16, 19]) that we are interested in:

(1) We say that I is S-weakly prime if there exists $s \in S$ such that for all $a, b \in R$, if $0 \neq ab \in I$, then either $sa \in I$ or $sb \in I$.
(2) We call I S-1-absorbing primary if there exists $s \in S$ such that whenever $abc \in I$ for some nonunit elements $a, b, c \in R$, then either $sab \in I$ or $sc \in \sqrt{I}$.
(3) I is called S-weakly primary if there exists $s \in S$ such that whenever $0 \neq ab \in I$ for $a, b \in R$, then either $sa \in I$ or $sb \in \sqrt{I}$.
(4) I is called an S-1-absorbing prime ideal if there exists $s \in S$ such that whenever $abc \in I$ for all nonunit elements $a, b, c \in R$, then either $sab \in I$ or $sc \in I$.

Some of our results use the $R \ltimes M$ construction. Let R be a ring and M an R-module. Then $R \ltimes M$, the trivial ring extension of R by M, is the ring whose additive structure is that of the external direct sum $R \oplus M$, and whose multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2+r_2m_1)$ for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in the books [12, 14]. Trivial ring extensions have been particularly useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See, for instance, [2, 10, 17, 18].

Let (R_1, R_2) be a pair of commutative rings with unity, $f : R_1 \longrightarrow R_2$ a ring homomorphism, and J an ideal of R_2. In this setting, we consider the following subring of $R_1 \times R_2$:

$$R_1 \bowtie^f R_2 = \{(a, f(a) + j) \mid a \in R_1 \text{ and } j \in J\},$$

called the amalgamation of R_1 and R_2 along J with respect to f. This construction was defined and examined by D'Anna et al., who studied these constructions in the framework of pullbacks. This allowed them to establish numerous results on the transfer of various ideal and ring-theoretic properties from A and $f(R_1) + J$ to $R_1 \bowtie^f J$.

The concept of amalgamation is an important and interesting topic that has received considerable attention over the last few decades. Motivations and additional applications of amalgamations are discussed in detail in [9], and have since been extensively investigated in various papers [7, 8, 11, 22].

In [21], the concept of a weakly 1-absorbing primary ideal, which generalizes the notion of a 1-absorbing ideal, was introduced. A proper ideal I of a ring R is called a weakly 1-absorbing primary ideal if, whenever nonunit elements $a, b, c \in R$ satisfy $0 \neq abc \in I$, then either $ab \in I$ or $c \in \sqrt{I}$. The authors studied various results concerning weakly 1-absorbing primary ideals and provided examples. In [5], a proper submodule N of an R-module M is defined as a weakly 1-absorbing primary submodule if, for nonunit elements $a, b \in R$, $m \in M$, and $0 \neq abm \in N$, we have either $ab \in (N :_R M)$ or $m \in M\text{-rad}(N)$.

The main goal of this paper is to extend the study of ideals by introducing and analyzing the concept of S-weakly 1-absorbing primary ideals (and submodules) of a commutative ring, which generalizes all previous results concerning weakly 1-absorbing primary ideals (and submodules). Let R be a commutative ring, S a multiplicative subset, and I a proper ideal of R disjoint from S. We say that I is an S-weakly 1-absorbing primary ideal if there exists a fixed $s \in S$ such that, for all nonunit elements $a, b, c \in R$ and $0 \neq abc \in I$, either $sab \in I$ or $sc \in \sqrt{I}$ holds.

In Section 2, we explore the basic properties of S-weakly 1-absorbing primary ideals. In Proposition 2.2, we provide several characterizations of S-weakly 1-absorbing primary ideals. Example 2.3 presents an S-weakly 1-absorbing primary ideal that is not a weakly 1-absorbing primary ideal. Proposition 2.6 states that I is an S-weakly 1-absorbing primary ideal of R if and only if $(I : s)$ is a weakly 1-absorbing primary ideal of R for some $s \in S$.

For an S-weakly 1-absorbing primary ideal I of a ring R, we show (Proposition 2.7) that $S^{-1}I$ is a weakly 1-absorbing primary ideal of $S^{-1}R$ for every multiplicative subset S of R that is disjoint from I, and we prove that the converse holds if $S \cap Z(R) = \emptyset$ and $S \cap Z_I(R) = \emptyset$. In Theorem 2.9, we present several characterizations for S-weakly 1-absorbing primary ideals of a commutative ring.

The third section is dedicated to S-weakly 1-absorbing primary submodules. We call a submodule N of M with $(N :_R M) \cap S = \emptyset$ an S-weakly 1-absorbing primary submodule if there exists $s \in S$ such that for all nonunit elements $a, b \in R$ and $m \in M$, if $0 \neq abm \in N$, then either $sab \in (N :_R M)$ or $sm \in M$-rad(N). We establish the relationships between this new concept and other classical types of submodules (see Example 3.3), and we provide some characterizations of this concept (Theorem 3.7). We also investigate the behavior of S-weakly 1-absorbing primary submodules under homomorphisms, in factor modules, and in modules of fractions (see Proposition 3.12).

Finally, Section 4 is devoted to the study of S-weakly 1-absorbing primary ideals of the form $I \ltimes M$ in trivial ring extensions (see Theorem 4.2) and the form of S-weakly 1-absorbing primary ideals in the amalgamation of R_1 with R_2 along an ideal J with respect to f (Theorems 4.4, 4.6). Such amalgamations are denoted by $R_1 \bowtie^f J$, a concept introduced and studied by [7].

Any undefined notation or terminology can be found in [23].

2. On S-weakly 1-absorbing primary ideals

The purpose of this section is to introduce the notion of S-weakly 1-absorbing primary ideals, and we provide some basic properties.

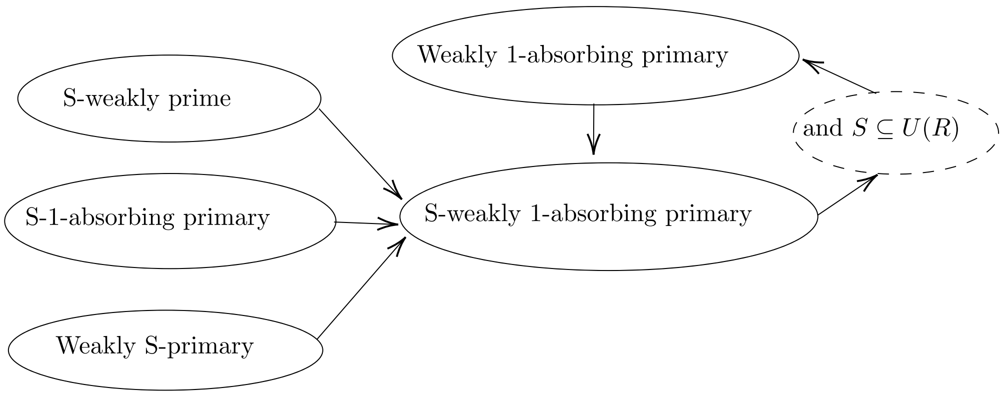
Definition 2.1. Let S be a multiplicative set of a ring R. We call a proper ideal I of R that is disjoint from S an S-weakly 1-absorbing primary ideal if there exists a fixed element $s \in S$ such that whenever $0 \neq abc \in I$ for all nonunit elements $a, b, c \in R$, either $sab \in I$ or $sc \in \sqrt{I}$. In this case, the fixed element $s \in S$ is called an S-element of I.

We begin with the following basic result:

Proposition 2.2. *Let I be a proper ideal of a ring R, and let S be a multiplicative set of R. Then the following statements hold:*

(1) *If I is an S-weakly prime ideal, then I is an S-weakly 1-absorbing primary ideal.*
(2) *If I is a weakly S-primary ideal, then I is an S-weakly 1-absorbing primary ideal.*
(3) *If I is an S-1-absorbing primary ideal, then I is an S-weakly 1-absorbing primary ideal.*

Next, from Proposition 2.2, we present the following diagram, which clarifies the position of S-weakly 1-absorbing primary ideals within the hierarchy of ideal types.



Notice that, from the previous definition, every weakly 1-absorbing primary ideal is clearly an S-weakly 1-absorbing primary ideal. In particular, when S consists entirely of units, the converse is true. However, this implication is not reversible in general, as demonstrated by the following example.

Example 2.3. Consider the polynomial ring $R = \mathbb{Z}[X]$ and the multiplicative subset $S = \{2^n \mid n \in \mathbb{N}\}$ of R. The ideal $I = 2^2X\mathbb{Z}[X]$ is an S-weakly 1-absorbing primary ideal of R. Indeed, take $s = 2^2 \in S$. Then $(I : s) = X\mathbb{Z}[X]$ is a prime ideal of R, and by [13, Proposition 1], I is an S-prime ideal of R and thus S-weakly 1-absorbing primary. However, it is not a weakly 1-absorbing primary ideal of R since $0 \neq 2 \cdot 2 \cdot X \in I$, but neither $4 \in I$ nor $X \in 2X\mathbb{Z}[X] = \sqrt{I}$.

Next, we present some results concerning this concept.

Proposition 2.4. *Let R be a ring and S a multiplicative set of R. Then the following statements hold:*

(1) Let J be an ideal of R such that $J \cap S \neq \emptyset$. If I is an S-weakly 1-absorbing primary ideal of R, then so is JI.

(2) Let $R \subseteq R'$ be an extension of rings. If Q is an S-weakly 1-absorbing primary ideal of R', then $Q \cap R$ is an S-weakly 1-absorbing primary ideal of R.

Proof. (1) It is obvious that $JI \cap S = \emptyset$ since $JI \subseteq I$ and $I \cap S = \emptyset$. As I is an S-weakly 1-absorbing primary ideal of R, there exists $s \in S$ such that for all nonunit $a, b, c \in R$, if $0 \neq abc \in I$, then $sab \in I$ or $sc \in \sqrt{I}$. Pick $t \in J \cap S$. Let $s' = ts \in S$. Now, let a', b', c' be nonunit elements of R such that $0 \neq a'b'c' \in JI$. Since $JI \subseteq I$, we have $a'b'c' \in I$. If $sa'b' \in I$, then $s'a'b' = t(sa'b') \in JI$. Now, assume that $sc' \in \sqrt{I}$. Then $s^n c'^n = (sc')^n \in I$ for some integer $n \geq 1$. Thus $(s'c')^n = t^n s^n c'^n \in JI$. It follows that $s'c' \in \sqrt{JI}$. Therefore, JI is an S-weakly 1-absorbing primary ideal of R.

(2) As Q is an S-weakly 1-absorbing primary ideal of R', there exists $s \in S$ such that for all nonunit $a, b, c \in R'$, if $0 \neq abc \in Q$, then $sab \in Q$ or $sc \in \sqrt{Q}$. Let a', b', c' be nonunit elements of R such that $0 \neq a'b'c' \in Q \cap R$. Since $Q \cap R \subseteq Q$, we have $a'b'c' \in Q$. If $sa'b' \in Q$, then clearly $sa'b' \in Q \cap R$. Now suppose that

$sc' \in \sqrt{Q}$. Then $(sc')^n \in Q$ for some integer $n \geq 1$. Thus $(sc')^n \in Q \cap R$, and so $sc' \in \sqrt{Q \cap R}$. It follows that $Q \cap R$ is an S-weakly 1-absorbing primary ideal of R. □

Theorem 2.5. *Let R be a reduced ring and S a multiplicative set of R. If I is an S-weakly 1-absorbing primary ideal of R, then $\sqrt{I}$ is an S-prime ideal of R.*

Proof. Assume that $0 \neq ab \in \sqrt{I}$ for some nonunit elements $a, b \in R$. Then, there exists an even positive integer $n = 2m$ $(m \geq 1)$ such that $(ab)^n \in I$. Since $Nil(R) = \{0\}$, we have $(ab)^n \neq 0$. Hence, $0 \neq a^{2m}b^n \in I$. As I is an S-weakly 1-absorbing primary ideal, there exists $s \in S$ such that either $sa^n \in I$ or $sb^n \in \sqrt{I}$. Thus, either $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. Therefore, $\sqrt{I}$ is an S-weakly prime ideal of R. Since $\sqrt{I}^2 \neq 0$, we conclude that $\sqrt{I}$ is an S-prime ideal of R (see [1, Proposition 4]). □

Our next proposition characterizes S-weakly 1-absorbing primary ideals of a ring R. But first, recall that if I is an ideal of R and $s \in R$, then $(I : s) = \{x \in R \mid sx \in I\}$ is an ideal of R containing I.

Proposition 2.6. *Let S be a multiplicative subset of a ring R, and let I be a proper ideal of R disjoint from S. Then the following assertions are equivalent:*

(1) I is an S-weakly 1-absorbing primary ideal of R.
(2) $(I : s)$ is a weakly 1-absorbing primary ideal of R for some $s \in S$.

Proof. (1) ⇒ (2) Since I is an S-weakly 1-absorbing primary ideal of R, there exists $s \in S$ such that for all nonunit elements $a, b, c \in R$, if $0 \neq abc \in I$, then either $sab \in I$ or $sc \in \sqrt{I}$. We claim that $(I : s)$ is a weakly 1-absorbing primary ideal of R. Let $x, y, z \in R$ such that $0 \neq xyz \in (I : s)$. Then $sxyz \in I$. It follows that either $s^2xy \in I$ or $sz \in \sqrt{I}$. If $s^2xy \in I$, then $sxy \in I$ or $s^3 \in \sqrt{I}$. Note that if $s^3 \in \sqrt{I}$, then $s^{3n} \in I$ for some integer $n \geq 1$, and so $s^{3n} \in I \cap S$, which is a contradiction since $I \cap S = \emptyset$. Therefore, $sxy \in I$, and consequently $xy \in (I : s)$. Now, if $sz \in \sqrt{I}$, then $s^m z^m = (sz)^m \in I$ for some integer $m \geq 1$. Thus, $sz^m \in I$ or $s^{m+1} \in \sqrt{I}$. The fact that $s^{m+1} \in \sqrt{I}$ is impossible, as otherwise I would meet S, which contradicts the assumption made on I. Therefore, $sz^m \in I$, and hence $z^m \in (I : s)$, or equivalently, $z \in \sqrt{(I : s)}$. Thus, we have demonstrated that $(I : s)$ is a weakly 1-absorbing primary ideal of R, as claimed.

(2) ⇒ (1) Assume (2) holds. Let $a, b, c \in R$ such that $0 \neq abc \in I$. Since $(I : s)$ is a weakly 1-absorbing primary ideal of R and $abc \in (I : s)$, either $ab \in (I : s)$, in which case $sab \in I$, or $c \in \sqrt{(I : s)}$, in which case $sc^n \in I$ for some integer $n \geq 1$. Therefore, $(sc)^n = s^n c^n \in I$, which implies that $sc \in \sqrt{I}$. Thus, I is an S-weakly 1-absorbing primary ideal of R, as desired. □

Now, we study the localization of the S-weakly 1-absorbing primary ideal property.

Proposition 2.7. *Let S be a multiplicative set of a ring R and I a proper ideal of R. Then, the following statements hold:*

(1) If I is an S-weakly 1-absorbing primary ideal of R, then $S^{-1}I$ is a weakly 1-absorbing primary ideal of $S^{-1}R$.
(2) If $S^{-1}I$ is an S-weakly 1-absorbing primary ideal of $S^{-1}R$, and $S \cap Z(R) = \emptyset$ and $S \cap Z_I(R) = \emptyset$, then I is a weakly 1-absorbing primary ideal of R.

Proof. (1) Since $S \cap I = \emptyset$, we have $S^{-1}I \neq S^{-1}R$. Suppose that $\frac{0}{1} \neq \frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} \in S^{-1}I$ for some nonunit elements $a, b, c \in R$ and $s_1, s_2, s_3 \in S$. Then $\frac{abc}{s_1 s_2 s_3} = \frac{p}{t_1}$ for some $p \in I$ and $t \in S$. So, we get $0 \neq abct_1t_2 = ps_1s_2s_3t_2$ for some $t_2 \in S$. Set $t := t_1t_2$. Then $0 \neq abct \in I$. Since I is an S-weakly 1-absorbing primary ideal of R, there exists $s \in S$ such that either $sab \in I$ or $stc \in \sqrt{I}$. If $sab \in I$, then $\frac{ab}{s_1s_2} = \frac{sab}{ss_1s_2} \in S^{-1}I$. Now, if $stc \in \sqrt{I}$, then $(stc)^n \in I$ for some integer $n \geq 1$. This implies that $\left(\frac{c}{s_3}\right)^n = \left(\frac{stc}{sts_3}\right)^n \in S^{-1}I$, hence $\frac{c}{s_3} \in \sqrt{S^{-1}I}$, and so we have the desired result.

(2) Assume that $0 \neq abc \in I$ for some nonunit elements $a, b, c \in R$. Hence, $\frac{0}{1} \neq \frac{abc}{1} = \frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{-1}I$, as $S \cap Z_I(R) = \emptyset$. Since $S^{-1}I$ is an S-weakly 1-absorbing primary ideal, we have either $\frac{s}{1}\frac{a}{1}\frac{b}{1} \in S^{-1}I$ or $\frac{s}{1}\frac{c}{1} \in \sqrt{S^{-1}I} = S^{-1}\sqrt{I}$. If $\frac{s}{1}\frac{a}{1}\frac{b}{1} \in S^{-1}I$, then $usab \in I$ for some $u \in S$. Since $S \cap Z_I(R) = \emptyset$, we conclude that $ab \in I$. If $\frac{s}{1}\frac{c}{1} \in S^{-1}\sqrt{I}$, then $(tsc)^n \in I$ for some positive integer $n \geq 1$ and $t \in S$. Since t^n and s^n do not belong to $Z_I(R)$, we have $c^n \in I$, i.e., $c \in \sqrt{I}$. Thus, I is a weakly 1-absorbing primary ideal of R. □

Let R be a ring and S a multiplicative subset of R. Set

$$S^* = \left\{ r \in R \mid \frac{r}{1} \text{ is a unit of } S^{-1}R \right\}.$$

Note that S^* is a multiplicative subset containing S. Our next result examines the relationship between the notions of S^*-weakly 1-absorbing primary and S-weakly 1-absorbing primary ideals.

Proposition 2.8. *Let R be a ring, S a multiplicative set of R, and I an ideal of R disjoint from S. Then the following assertions hold:*

(1) *Assume that $S_1 \subseteq S_2$ are multiplicative subsets. If I is S_1-weakly 1-absorbing primary and $I \cap S_2 = \emptyset$, then I is an S_2-weakly 1-absorbing primary ideal.*
(2) *I is S-weakly 1-absorbing primary if and only if I is S^*-weakly 1-absorbing primary.*

Proof. (1) This is straightforward.

(2) ($\Rightarrow$) First, we show that $I \cap S^* = \emptyset$. By contradiction, assume $u \in I \cap S^*$. Then $\frac{u}{1}$ is a unit of $S^{-1}R$, so $\frac{u}{1} \cdot \frac{v}{t} = 1$ for some $v \in R$ and $t \in S$. This implies $uvs \in I \cap S$ for some $s \in S$, a contradiction. Since S^* contains S, the conclusion follows.

($\Leftarrow$) Let a, b, c be nonunit elements of R with $0 \neq abc \in I$. Since I is S^*-weakly 1-absorbing primary, there exists $s_1 \in S^*$ such that $s_1ab \in I$ or $s_1c \in \sqrt{I}$. On the other hand, there exist $r \in R$ and $s \in S$ such that $\frac{s_1}{1} \cdot \frac{r}{s} = \frac{1}{1}$. So $us_1r = us$ for some $u \in S$. Set $t = us$. Then $tab \in I$ or $tc \in \sqrt{I}$, as desired. □

In the following, we study some characterizations of S-weakly 1-absorbing primary ideals.

Theorem 2.9. *Let R be a commutative ring, S a multiplicative set of R, and I a proper ideal of R. Then the following assertions are equivalent:*

(1) *I is an S-weakly 1-absorbing primary ideal of R.*
(2) *There exists $s \in S$ such that for every pair of nonunit elements $a, b \in R$ with $ab \notin (I : s)$, we have either $(I : ab) = (0 : ab)$ or $s(I : ab) \subseteq \sqrt{I}$.*

(3) There exists $s \in S$ such that for every nonunit element $a \in R$, and every ideal I_1 of R with $aI_1 \nsubseteq (I : s)$, we have $(I : aI_1) = (0 : aI_1)$ or $s\,(I : aI_1) \subseteq \sqrt{I}$.

(4) There exists $s \in S$ such that for every pair of proper ideals I_1 and I_2 of R with $I_1I_2 \nsubseteq (I : s)$, we have $(I : I_1I_2) = (0 : I_1I_2)$ or $s\,(I : I_1I_2) \subseteq \sqrt{I}$.

(5) There exists $s \in S$ such that for every set of ideals I_1, I_2, I_3 of R with $0 \neq I_1I_2I_3 \subseteq I$, then either $sI_1I_2 \subseteq I$ or $sI_3 \subseteq \sqrt{I}$.

Proof. (1) $\Rightarrow$ (2) Suppose that I is an S-weakly 1-absorbing primary ideal of R, and let a and b be two nonunit elements of R such that $sab \notin I$. Suppose that $(I : ab) \neq (0 : ab)$. Since $(0 : ab) \subseteq (I : ab)$, there exists $d \in (I : ab)$ with $abd \neq 0$. Thus $0 \neq abd \in I$, and so $sd \in \sqrt{I}$ because I is S-weakly 1-absorbing primary and $sab \notin I$. Now, let $c \in (I : ab)$. Then $abc \in I$. Since $sab \notin I$, c is a nonunit. If $0 \neq abc \in I$, then, because I is S-weakly 1-absorbing primary and $sab \notin I$, we have $sc \in \sqrt{I}$. Now, suppose that $abc = 0$, then $0 \neq abd = ab(c + d) \in I$. Hence, $s(c+d) \in \sqrt{I}$, and therefore $sc = s(c+d) - sd \in \sqrt{I}$. Consequently, $s(I : ab) \subseteq \sqrt{I}$.

(2) $\Rightarrow$ (3) Assume that $(I : aI_1) \nsubseteq (0 : aI_1)$ and $b \in I_1$ such that $ab \in (I : s)$. Let $x \in (I : aI_1) \setminus (0 : aI_1)$, so $axI_1 \subseteq I$ and $axI_1 \neq 0$.

Case 1. If $axb \neq 0$, then $x \in (I : ab) \setminus (0 : ab)$. Hence $(I : ab) \neq (0 : ab)$, and by the hypothesis, $s\,(I : ab) \subseteq \sqrt{I}$. This implies $s\,(I : aI_1) \subseteq s\,(I : ab) \subseteq \sqrt{I}$.

Case 2. If $axb = 0$ and $axI_1 \neq 0$, let $b' \in I_1$ such that $axb' \neq 0$, then $ax(b + b') = axb' \in I\backslash\{0\}$. If $sab' \notin I$, then $s\,(I : ab') \subseteq \sqrt{I}$, so $s\,(I : aI_1) \subseteq s\,(I : ab') \subseteq \sqrt{I}$. If $sab' \in I$, then $sa(b+b') \in I$ (as $sab \notin I$), and $xa(b+b') \in I\backslash\{0\}$. So, we get $a(b + b') \notin (I : s)$ and $(0 : a(b + b')) \neq (I : a(b + b'))$. By (2), we have $s\,(I : a(b + b')) \subseteq \sqrt{I}$. Thus, $(I : aI_1) \subseteq s\,(I : a(b + b')) \subseteq \sqrt{I}$.

(3) $\Rightarrow$ (4) Suppose that $(I : I_1I_2) \neq (0 : I_1I_2)$. Let $a \in I_2$ such that $aI_1 \nsubseteq (I : s)$ and take $x \in (I : I_1I_2) \setminus (0 : I_1I_2)$. So, we get $xI_1I_2 \subseteq I\backslash\{0\}$.

Case 1. If $axI_1 \neq 0$, then $x \in (I : aI_1) \setminus (0 : aI_1)$. Hence, $(I : aI_1) \neq (0 : aI_1)$, and so $s\,(I : aI_1) \subseteq \sqrt{I}$. Thus, $s\,(I : I_1I_2) \subseteq s\,(I : aI_1) \subseteq \sqrt{I}$.

Case 2. If $axI_1 = 0$ and $xI_1I_2 \neq 0$, there exists $a' \in I_2$ such that $xa'I_1 \neq 0$. Then $x(a+a')I_1 = xa'I_1 \subseteq I\backslash\{0\}$. First, if $a'I_1 \nsubseteq (I : s)$ and $x \in (I : a'I_1) \setminus (0 : a'I_1)$, we get that $s\,(I : a'I_1) \subseteq \sqrt{I}$ according to (3), and hence $s\,(I : I_1I_2) \subseteq s\,(I : a'I_1) \subseteq \sqrt{I}$. Secondly, if $a'I_1 \subseteq (I : s)$, then $(a + a')I_1 \nsubseteq (I : s)$. Since $a'I_1 \subseteq (I : s)$, this shows that $s(a + a')I_1 \subseteq I\backslash\{0\}$, and so $(I : (a + a')I_1) \neq (0 : (a + a')I_1)$. Hence, $s\,(I : (a + a')I_1) \subseteq \sqrt{I}$ according to (3). Therefore, $s\,(I : I_1I_2) \subseteq \sqrt{I}$.

(4) $\Rightarrow$ (5) Let I_1, I_2, and I_3 be proper ideals of R such that $0 \neq I_1I_2I_3 \subseteq I$. Assume that $sI_1I_2 \nsubseteq I$, so $I_1I_2 \nsubseteq (I : s)$, and by (4), $(I : I_1I_2) = (0 : I_1I_2)$ or $s\,(I : I_1I_2) \subseteq \sqrt{I}$. Since $0 \neq I_1I_2I_3 \subseteq I$, we have $I_3 \subseteq (I : I_1I_2) \setminus (0 : I_1I_2)$, and it follows that $(I : I_1I_2) \neq (0 : I_1I_2)$. Hence, $s\,(I : I_1I_2) \subseteq \sqrt{I}$. Therefore, $sI_3 \subseteq \sqrt{I}$ as $I_3 \subseteq (I : I_1I_2)$.

(5) $\Rightarrow$ (1) Choose $s \in S$ as in (5), and let $a, b, c \in R$ be nonunit elements with $0 \neq abc \in I$. Set $I_1 = \langle a\rangle$, $I_2 = \langle b\rangle$, and $I_3 = \langle c\rangle$ in (5). Thus, we obtain the desired result. $\square$

The next proposition examines the S-weakly 1-absorbing primary property under homomorphic images.

Proposition 2.10. *Let $f : R_1 \to R_2$ be a ring homomorphism such that $f(a)$ is nonunit in R_2 for every nonunit $a \in R_1$, and let S be a multiplicative set of R_1 such that $f(S)$ does not contain zero. Then the following statements hold:*

(1) If f is onto and I is an S-weakly 1-absorbing primary ideal of R_1 with $\mathrm{Ker}(f) \subseteq I$, then $f(I)$ is an $f(S)$-weakly 1-absorbing primary ideal of R_2.
(2) If J is an $f(S)$-weakly 1-absorbing primary ideal of R_2, then $f^{-1}(J)$ is an S-weakly 1-absorbing primary ideal of R_1.

Proof. (1) Let $r \in f(S) \cap f(I)$. Then, $r = f(s) = f(p)$ for some $s \in S$ and $p \in I$. So, $s - p \in \mathrm{Ker}(f) \subseteq I$, which implies that $s \in I$, a contradiction. Hence, $f(S) \cap f(I) = \emptyset$. Now, suppose that f is onto, I is an S-weakly 1-absorbing primary ideal of R_1, and let $0 \neq xyz \in f(I)$ for some nonunit elements $x, y, z \in R_2$. Since f is onto, there exist nonunit elements $a, b, c \in R_1$ such that $x = f(a)$, $y = f(b)$, and $z = f(c)$. Therefore, $0 \neq f(abc) = f(a)f(b)f(c) = xyz \in f(I)$. Since $\mathrm{Ker}(f) \subseteq I$, we conclude that $abc \in I$. Thus, there exists $s \in S$ such that $sab \in I$ or $sc \in \sqrt{I}$. This means that $f(s)xy \in f(I)$ or $f(s)z \in f(\sqrt{I})$. Since f is onto and $\mathrm{Ker}(f) \subseteq I$, we have $f(\sqrt{I}) = \sqrt{f(I)}$. Thus, $f(I)$ is an $f(S)$-weakly 1-absorbing primary ideal of R_2.

(2) Suppose that J is an $f(S)$-weakly 1-absorbing primary ideal of R_2. Now, we claim that $f^{-1}(J) \cap S = \emptyset$. Indeed, assume the contrary and let $s \in f^{-1}(J) \cap S$. Then $f(s) \in J \cap f(S)$, a contradiction, since J is an $f(S)$-weakly 1-absorbing primary ideal of R_2, and thus $J \cap f(S)$ should be empty. Now, let $0 \neq abc \in f^{-1}(J)$ for some nonunit elements $a, b, c \in R_1$. Then $0 \neq f(abc) = f(a)f(b)f(c) \in J$, which implies that there exists $s \in S$ such that $f(s)f(a)f(b) \in J$ or $f(s)f(c) \in \sqrt{J}$. If $f(s)f(a)f(b) \in J$, then $sab \in f^{-1}(J)$. If $f(s)f(c) \in \sqrt{J}$, then $f(sc) \in \sqrt{J}$. Therefore, $f((sc)^n) = (f(sc))^n \in J$ for some integer $n \geq 1$. It follows that $(sc)^n \in f^{-1}(J)$. Hence, $sc \in \sqrt{f^{-1}(J)}$. Thus, $f^{-1}(J)$ is an S-weakly 1-absorbing primary ideal of R_1. The proof is complete. □

Let R be a ring, S a multiplicative subset of R, and I an ideal of R disjoint from S. Let $s \in S$, and denote by $\bar{s}$ the equivalence class of s in R/J. Let $\bar{S} = \{\bar{s} = s + J \mid s \in S\}$, then $\bar{S}$ is a multiplicative subset of R/J. One can easily check that $J \cap S = \emptyset$ if and only if $(J/I) \cap \bar{S} = \emptyset$. We conclude the following result for $\bar{S}$-weakly 1-absorbing primary ideals of R/J.

Theorem 2.11. *Let S be a multiplicative subset of a ring R, and let I be a proper ideal of R. Then the following assertions hold:*

(1) If J is a proper ideal of R with $J \subseteq I$, and I is an S-weakly 1-absorbing primary ideal of R, then I/J is a $\bar{S}$-weakly 1-absorbing primary ideal of R/J.
(2) If J is a proper ideal of R with $J \subseteq I$ such that $U(R/J) = \{a + J \mid a \in U(R)\}$, and if J is an S-1-absorbing primary ideal of R and I/J is a $\bar{S}$-weakly 1-absorbing primary ideal of R/J, then I is an S-1-absorbing primary ideal of R.
(3) If $\{0\}$ is an S-1-absorbing primary ideal of R and I is an S-weakly 1-absorbing primary ideal of R, then I is an S-1-absorbing primary ideal of R.
(4) If J is a proper ideal of R with $J \subseteq I$ such that $U(R/J) = \{a + J \mid a \in U(R)\}$, and if J is an S-weakly 1-absorbing primary ideal of R and

I/J is a $\bar{S}$-weakly 1-absorbing primary ideal of R/J, then I is an S-weakly 1-absorbing primary ideal of R.

Proof. (1) Consider the natural epimorphism $\pi : R \longrightarrow R/J$. Then $\pi(I) = I/J$. So, we conclude by Proposition 2.10.

(2) Assume that $abc \in I$ for some nonunit elements $a, b, c \in R$. If $abc \in J$, then $sab \in J \subseteq I$ or $sc \in \sqrt{J} \subseteq \sqrt{I}$ as J is an S-1-absorbing primary ideal of R. Now, suppose that $abc \notin J$. Thus, $J \neq (a+J)(b+J)(c+J) \in I/J$, where $a+J, b+J, c+J$ are nonunit elements of R/J by hypothesis. Then $(s+J)(a+J)(b+J) \in I/J$ or $(s+J)(c+J) \in \sqrt{I/J}$. Hence, we obtain $sab \in I$ or $sc \in \sqrt{I}$.

(3) The proof follows from (2).

(4) Suppose that $0 \neq abc \in I$ for some nonunit elements $a, b, c \in R$. If $0 \neq abc \in J$, then $sab \in J \subseteq I$ or $sc \in \sqrt{J} \subseteq \sqrt{I}$ as J is an S-weakly 1-absorbing primary ideal of R. Now, assume that $abc \notin J$. Thus, $J \neq (a+J)(b+J)(c+J) \in I/J$, where $a+J, b+J, c+J$ are nonunit elements of R/J by hypothesis. Then $(s+J)(a+J)(b+J) \in I/J$ or $(s+J)(c+J) \in \sqrt{I/J}$. Therefore, $sab \in I$ or $sc \in \sqrt{I}$, completing the proof. □

Let R be a commutative ring. R is called decomposable if $R = R_1 \times R_2$ for some commutative rings R_1 and R_2. If I_1 is an ideal of R_1, then $I_1 \times R_2$ is an ideal of $R_1 \times R_2$, and $\sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2$. Similarly, if I_2 is an ideal of R_2, then $R_1 \times I_2$ is an ideal of $R_1 \times R_2$, and $\sqrt{(R_1 \times I_2)} = R_1 \times \sqrt{I_2}$. Now, we establish the following result.

Theorem 2.12. *Let R_1 and R_2 be commutative rings, and let S_1 and S_2 be multiplicative subsets of R_1 and R_2, respectively. Set $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Then the following hold:*

(1) I_1 is an S_1-weakly 1-absorbing primary ideal of R_1 if and only if $I_1 \times R_2$ is an S-weakly 1-absorbing primary ideal of R.

(2) I_2 is an S_2-weakly 1-absorbing primary ideal of R_2 if and only if $R_1 \times I_2$ is an S-weakly 1-absorbing primary ideal of R.

Proof. First, notice that $I_1 \cap S_1 = \emptyset$ if and only if $(I_1 \times R_2) \cap S = \emptyset$, and $I_2 \cap S_2 = \emptyset$ if and only if $(R_1 \times I_2) \cap S = \emptyset$.

(1) ($\Rightarrow$) Let $(a, b), (c, d), (e, f)$ be nonunit elements of $R_1 \times R_2$ such that $0 \neq (a, b)(c, d)(e, f) \in I_1 \times R_2$. Since I_1 is an S_1-weakly 1-absorbing primary ideal of R_1 and $0 \neq abc \in I_1$, there exists $s_1 \in S_1$ such that $s_1 ab \in I_1$ or $s_1 c \in \sqrt{I_1}$. Hence, $(s_1, 1)(a, b)(c, d) \in I_1 \times R_2$ or $(s_1, 1)(e, f) \in \sqrt{I_1} \times R_2 = \sqrt{(I_1 \times R_2)}$. Therefore, $I_1 \times R_2$ is an S-weakly 1-absorbing primary ideal of R.

($\Leftarrow$) Suppose that $I_1 \times R_2$ is an S-weakly 1-absorbing primary ideal of R. Let a, b, c be nonunit elements of R_1 with $0 \neq abc \in I_1$. By hypothesis, there exists $s = (s_1, s_2) \in S$ such that $s(a, 1)(b, 1) \in I_1 \times R_2$ or $s(c, 1) \in \sqrt{I_1} \times R_2 = \sqrt{(I_1 \times R_2)}$. Thus, $s_1 ab \in I_1$ or $s_1 c \in \sqrt{I_1}$.

(2) The proof follows similarly to (1). □

Next, we study the stability of the S-weakly 1-absorbing primary property under direct products.

Corollary 2.13. *Let $R = \prod_{i=1}^{n} R_i$, where each R_i is a commutative ring, and let $S = \prod_{i=1}^{n} S_i$, where S_i is a multiplicative subset of R_i for each i. If for some*

$1 \leq j \leq n$, I_j is an S_j-weakly 1-absorbing primary ideal of R_j, then $R_1 \times \cdots \times R_{j-1} \times I_j \times R_{j+1} \times \cdots \times R_n$ is an S-weakly 1-absorbing primary ideal of R.

We conclude this section by showing that the intersection of a family of S-weakly 1-absorbing primary ideals is an S-weakly 1-absorbing primary ideal.

Proposition 2.14. *Let S be a multiplicative set of R, and let I_i be an ideal of R with $I_i \cap S = \emptyset$ for every $i \in \{1, \dots, n\}$. If each I_i is an S-weakly 1-absorbing primary ideal of R, and $\sqrt{I_i} = \sqrt{I_j}$ for every distinct $i, j \in \{1, \dots, n\}$, then $I = \bigcap_{i=1}^{n} I_i$ is an S-weakly 1-absorbing primary ideal of R.*

Proof. Let $i \in \{1, \dots, n\}$. Since I_i is an S-weakly 1-absorbing primary ideal of R, there exists $s_i \in S$ such that for nonunit elements $a, b, c \in R$ with $0 \neq abc \in I_i$, we have either $s_i ab \in I_i$ or $s_i c \in \sqrt{I_i}$. Set $s = \prod_{i=1}^{n} s_i$, so $s \in S$. Let $a, b, c \in R$ such that $0 \neq abc \in I = \bigcap_{i=1}^{n} I_i$. Suppose that $sab \notin I$. Then $sab \notin I_k$ for some $k \in \{1, \dots, n\}$. Hence, $s_k ab \notin I_k$. Since $0 \neq abc \in I_k$, it follows that $s_k c \in \sqrt{I_k}$. Therefore, $sc \in \sqrt{I_k}$. By the hypothesis, $\sqrt{I_1} = \sqrt{I_i}$ for all $i \in \{1, \dots, n\}$. Hence, $sc \in \sqrt{I_1} = \bigcap_{i=1}^{n} \sqrt{I_i} = \sqrt{\bigcap_{i=1}^{n} I_i}$. This shows that there exists an element $s \in S$ such that for nonunit elements $a, b, c \in R$ with $0 \neq abc \in \bigcap_{i=1}^{n} I_i$, we have either $sab \in \bigcap_{i=1}^{n} I_i$ or $sc \in \sqrt{\bigcap_{i=1}^{n} I_i}$. This completes the proof. □

3. On S-weakly 1-absorbing primary submodules

We begin this section by introducing the concept of S-weakly 1-absorbing primary submodules.

Definition 3.1. Let S be a multiplicative set of a ring R, and let M be an R-module. We call a submodule N of M with $(N :_R M) \cap S = \emptyset$ an S-weakly 1-absorbing primary submodule if there exists $s \in S$ such that for all nonunit elements $a, b \in R$ and $m \in M$, if $0 \neq abm \in N$, then $sab \in (N :_R M)$ or $sm \in \mathrm{rad}_M(N)$. This fixed element $s \in S$ is called an S-element of N.

Recall from [24] that an R-module M is said to be uniformly S-torsion provided that there exists an element $s \in S$ such that $sM = 0$.

Remark 3.2. If M is a uniformly S-torsion module, then every submodule N of M is an S-weakly 1-absorbing primary submodule.

Clearly, every weakly 1-absorbing primary submodule is an S-weakly 1-absorbing primary submodule for any multiplicative set S of R disjoint from $(N :_R M)$. Moreover, it is evident that the classes of weakly 1-absorbing primary submodules and S-weakly 1-absorbing primary submodules coincide if $S \subseteq U(R)$. However, the converse is not true in general, as illustrated by the following example.

Example 3.3. Let $R = \mathbb{Z} \times \mathbb{Z}$, $M = \mathbb{Z}_6 \times \mathbb{Z}_{10}$, and $N = (\overline{3}) \times (\overline{5})$. First, note that $(N :_R M) = 3\mathbb{Z} \times 5\mathbb{Z}$ and $\mathrm{rad}_M(N) = (\overline{3}) \times (\overline{5}) = N$. For the multiplicative subset $S = \{(6^n, 10^n) \mid n \in \mathbb{N}\}$, N is an S-weakly 1-absorbing primary submodule of M. Indeed, take $s = (6, 10)$, and let $a, b \in R$ and $m \in M$ such that $0 \neq abm \in M$. Then $0 = sm \in \mathrm{rad}_M(N)$. However, N is not a weakly 1-absorbing primary submodule of M, since $(\overline{0}, \overline{0}) \neq (3, 1)(0, 1)(\overline{1}, \overline{5}) = (\overline{0}, \overline{5}) \in N$, but $(3, 1)(0, 1) \notin (N :_R M)$ and $(\overline{1}, \overline{5}) \notin \mathrm{rad}_M(N)$.

Remark 3.4. Let $T \subseteq S$ be two multiplicative subsets of a ring R, and let N be a submodule of an R-module M. If N is a T-weakly 1-absorbing primary submodule such that $(N :_R M) \cap S = \emptyset$, then N is an S-weakly 1-absorbing primary submodule of M.

Before proving Theorem 3.7, we need the following lemma.

Lemma 3.5. *Let S be a multiplicative subset of a ring R, and let N be a proper submodule of an R-module M satisfying $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:*

(1) N is an S-weakly 1-absorbing primary submodule of M.
(2) There exists $s \in S$ such that whenever $0 \neq Iam \subseteq N$ for some ideal I of R, $a \in R$, and $m \in M$, we have either $saI \subseteq (N :_R M)$ or $sm \in \mathrm{rad}_M(N)$.

Proof. (1) $\Longrightarrow$ (2) Suppose that N is an S-weakly 1-absorbing primary submodule of M. Then there exists $s \in S$ such that $0 \neq xyu \in N$ for some $x, y \in R$ and $u \in M$ implies $sxy \in (N :_R M)$ or $sm \in \mathrm{rad}_M(N)$. Let $0 \neq Iam \subseteq N$ for some ideal I of R, $a \in R$, and $m \in M$. Assume that $saI \nsubseteq (N :_R M)$. So, there exists $b \in I$ such that $sab \notin (N :_R M)$. If $0 \neq bam \in N$, since N is an S-weakly 1-absorbing primary submodule and $sab \notin (N :_R M)$, we get that $sm \in \mathrm{rad}_M(N)$. Now, if $bam = 0$ and $0 \neq Iam \subseteq N$, there exists $b' \in I$ such that $0 \neq b'am \in N$. If $sab' \notin (N :_R M)$, then since N is an S-weakly 1-absorbing primary submodule, we have $sm \in \mathrm{rad}_M(N)$. If $sab' \in (N :_R M)$, then $sa(b + b') \notin (N :_R M)$ and $0 \neq b'am = (b + b')am \in N$. Hence, $sm \in \mathrm{rad}_M(N)$ since N is an S-weakly 1-absorbing primary submodule. Thus, in all cases, we have $sm \in \mathrm{rad}_M(N)$.

(2) $\Longrightarrow$ (1) This implication is obvious. □

Lemma 3.6. *Let S be a multiplicative subset of a ring R, and let N be a proper submodule of an R-module M satisfying $(N :_R M) \cap S = \emptyset$. Then, the following statements are equivalent:*

(1) N is an S-weakly 1-absorbing primary submodule of M.
(2) There exists $s \in S$ such that if $0 \neq IJm \subseteq N$ for some ideals I, J of R and $m \in M$, then we have $sIJ \subseteq (N :_R M)$ or $sm \in \mathrm{rad}_M(N)$.

Proof. (1) $\Longrightarrow$ (2) Suppose that N is an S-weakly 1-absorbing primary submodule of M. Then, according to Lemma 3.5, there exists $s \in S$ such that whenever $0 \neq Iam \subseteq N$ for some ideal I of R, $a \in R$, and $m \in M$, we have either $saI \subseteq (N :_R M)$ or $sm \in \mathrm{rad}_M(N)$. Now, let $0 \neq IJm \subseteq N$ for some ideals I, J of R and $m \in M$. Assume that $sIJ \nsubseteq (N :_R M)$. So, there exists $a \in J$ such that $saI \notin (N :_R M)$. If $0 \neq aIm \in N$, since $saI \notin (N :_R M)$, we get that $sm \in \mathrm{rad}_M(N)$. Now, if $aIm = 0$ and $0 \neq IJm \subseteq N$, there exists $a' \in J$ such that $0 \neq a'Im \in N$. If $sa'I \notin (N :_R M)$, then, by Lemma 3.5 and the fact that N is an S-weakly 1-absorbing primary submodule, we have $sm \in \mathrm{rad}_M(N)$. If $sa'I \in (N :_R M)$, then $s(a + a')I \notin (N :_R M)$ and $0 \neq a'Im = (a + a')Im \in N$, which implies that $sm \in \mathrm{rad}_M(N)$. Thus, in all cases, we have $sm \in \mathrm{rad}_M(N)$.

(2) $\Longrightarrow$ (1) This implication is straightforward. □

Next, we provide a characterization for S-weakly 1-absorbing primary submodules of an R-module M.

Theorem 3.7. *Let S be a multiplicative subset of a ring R, and let N be a proper submodule of an R-module M satisfying $(N :_R M) \cap S = \emptyset$. Then N is an S-weakly 1-absorbing primary submodule of M if and only if there exists $s \in S$ such that if*

$0 \neq IJK \subseteq N$ for some proper ideals I, J of R and a submodule K of M, then either $sIJ \subseteq (N :_R M)$ or $sK \subseteq \text{rad}_M(N)$.

Proof. Suppose that N is an S-weakly 1-absorbing primary submodule of M. By Lemma 3.6, there exists $s \in S$ such that whenever $0 \neq IJm \subseteq N$ for some ideals I, J of R and $m \in M$, we have either $sIJ \subseteq (N :_R M)$ or $sm \in \text{rad}_M(N)$. Now, let $0 \neq IJK \subseteq N$ for some proper ideals I, J of R and a submodule K of M. Assume that $sIJ \nsubseteq (N :_R M)$. We aim to show that $sK \subseteq \text{rad}_M(N)$. Let $m \in K$. If $IJm \neq 0$, then since $0 \neq IJm \subseteq N$ and $sIJ \nsubseteq (N :_R M)$, we get that $sm \in \text{rad}_M(N)$. Now, if $IJm = 0$, since $IJK \neq 0$, there exists $m' \in K$ such that $IJm' \neq 0$. Then, since $0 \neq IJm' \subseteq N$ and $sIJ \nsubseteq (N :_R M)$, we get that $sm' \in \text{rad}_M(N)$. Furthermore, since $0 \neq IJm' = IJ(m + m') \subseteq (N :_R M)$, it follows that $s(m - m') \in \text{rad}_M(N)$. Hence, $sm = sm' + s(m - m') = s(m - m') \in \text{rad}_M(N)$. Thus, in all cases, we have $sm \in \text{rad}_M(N)$. Therefore, $sK \subseteq \text{rad}_M(N)$. The converse follows immediately. □

Recall from [20, Lemma 2.4] that if N is a proper submodule of a finitely generated multiplication R-module M, then $(\text{rad}_M(N) :_R M) = \sqrt{(N :_R M)}$.

Theorem 3.8. *Let S be a multiplicative subset of a ring R, and let M be a finitely generated faithful R-module. Suppose N is a proper submodule of M. If N is an S-weakly 1-absorbing primary submodule of M, then $(N :_R M)$ is an S-weakly 1-absorbing primary ideal of R.*

Proof. Assume that N is an S-weakly 1-absorbing primary submodule of M with respect to some $s \in S$. Let $0 \neq abc \in (N :_R M)$ for some nonunit elements $a, b, c \in R$. Since M is faithful, we conclude that $0 \neq ab(cM) \subseteq N$. If $sab \in (N :_R M)$, then we are done. So assume that $sab \notin (N :_R M)$. Choose an arbitrary element $m_1 \in M$.

First, if $abcm_1 \neq 0$, then since N is an S-weakly 1-absorbing primary submodule, we conclude that $scm_1 \in \text{rad}_M(N)$.

Second, if $abcm_1 = 0$, since $abcM \neq 0$, pick $m_2 \in M$ such that $abcm_2 \neq 0$. Since N is an S-weakly 1-absorbing primary submodule, we have $scm_2 \in \text{rad}_M(N)$. On the other hand, since $0 \neq abc(m_1 + m_2) \in N$, we get $sc(m_1 + m_2) \in \text{rad}_M(N)$, implying that $scm_1 \in \text{rad}_M(N)$. Therefore, we have $scM \subseteq \text{rad}_M(N)$, which implies that $sc \in (\text{rad}_M(N) :_R M) = \sqrt{(N :_R M)}$. Thus, $(N :_R M)$ is an S-weakly 1-absorbing primary ideal of R, as desired. □

Proposition 3.9. *Let S be a multiplicative subset of a ring R, and let M be a multiplication R-module. If $\{N_i\}_{i=1}^k$ is a family of S-weakly 1-absorbing primary submodules of M with the same M-radical, then $\bigcap_{i=1}^k N_i$ is an S-weakly 1-absorbing primary submodule of M.*

Proof. Let $s_i \in S$ be an S-element of N_i for each $i = 1, \ldots, k$. First, note that $\left(\bigcap_{i=1}^k N_i\right) \cap S = \emptyset$. Put $s = s_1 \cdots s_k$. Suppose that $0 \neq abm \in \bigcap_{i=1}^k N_i$ but $sab \notin \left(\bigcap_{i=1}^k N_i : M\right)$ for some nonunit elements $a, b \in R$ and $m \in M$. Then $sab \notin (N_j : M)$ for some $j \in \{1, \ldots, k\}$. Since N_j is an S-weakly 1-absorbing primary submodule and $abm \in N_j$, we get $s_j m \in M - \text{rad}\,(N_j)$, and hence $sm \in M - \text{rad}\,(N_j) = M - \text{rad}\left(\bigcap_{i=1}^k N_i\right)$. Thus, the proof is complete. □

Proposition 3.10. *Let $f : M_1 \to M_2$ be an R-module epimorphism, and let S be a multiplicative subset of R. Then the following statements hold:*

(1) Suppose that $(f(N_1):_R M_2)\cap S=\emptyset$*. If* N_1 *is an* S*-weakly* 1*-absorbing primary submodule of* M_1 *containing* $\operatorname{Ker} f$*, then* $f(N_1)$ *is an* S*-weakly* 1*-absorbing primary submodule of* M_2*.*

(2) If N_2 *is an* S*-weakly* 1*-absorbing primary submodule of* M_2 *and* $f^{-1}(N_2)\neq M_1$*, then* $f^{-1}(N_2)$ *is an* S*-weakly* 1*-absorbing primary submodule of* M_1*.*

Proof. (1) Let $a,b\in R$ be nonunit elements and $m_2\in M_2$ with $0\neq abm_2\in f(N_1)$. Since f is onto, there exists $m_1\in M_1$ such that $f(m_1)=m_2$. Since $\operatorname{Ker} f\subseteq N_1$ and $0\neq abm_1\in N_1$, we have either $sab\in(N_1:_R M_1)$ or $sm_1\in M_1-\operatorname{rad}(N_1)$. It follows that either $sab\in(f(N_1):_R M_2)$ or $sm_2=sf(m_1)\in M_2-\operatorname{rad}(f(N_1))$. Hence, $f(N_1)$ is an S-weakly 1-absorbing primary submodule of M_2.

(2) First, we show that $\left(f^{-1}(N_2):_R M_1\right)\cap S=\emptyset$. Assume that $r\in\left(f^{-1}(N_2):_R M_1\right)\cap S$. Then $rM_1\in f^{-1}(N_2)$, which implies $rM_2=rf(M_1)\subseteq f(f^{-1}(N_2))\subseteq N_2$, and hence $r\in(N_2:_R M_2)$, a contradiction. Now, let $a,b\in R$ be nonunit elements, $m_1\in M_1$, and suppose $0\neq abm_1\in f^{-1}(N_2)$. Since f is a homomorphism, we have $0\neq abf(m_1)\in N_2$. Since N_2 is S-weakly 1-absorbing primary, there exists $s\in S$ such that $sab\in(N_2:_R M_2)$ or $sf(m_1)\in M_2-\operatorname{rad}(N_2)$. Hence, either $sab\in\left(f^{-1}(N_2):_R M_1\right)$ or $sm_1\in f^{-1}(M_2-\operatorname{rad}(N_2))\subseteq M_1-\operatorname{rad}\left(f^{-1}(N_2)\right)$. Thus, $f^{-1}(N_2)$ is an S-weakly 1-absorbing primary submodule of M_1. □

Based on Proposition 3.10, we have the following result.

Corollary 3.11. *Let* N_1 *and* N_2 *be proper submodules of an* R*-module* M *with* $N_2\subseteq N_1$*, and let* S *be a multiplicative subset of* R*.*

(1) If N_1 *is an* S*-weakly* 1*-absorbing primary submodule of* M*, then* N_1/N_2 *is a* $\bar{S}$*-weakly* 1*-absorbing primary submodule of* M/N_2*.*

(2) If N *is an* S*-weakly* 1*-absorbing primary submodule of* M *and* K *is a submodule of* M *with* $K\nsubseteq N$*, then* $N\cap K$ *is an* S*-weakly* 1*-absorbing primary submodule of* K*.*

(3) If N_2 *is an* S*-*1*-absorbing primary submodule of* M*, and* N_1/N_2 *is a* $\bar{S}$*-weakly* 1*-absorbing primary submodule of the* R*-module* M/N_2*, then* N_1 *is an* S*-*1*-absorbing primary submodule of* M*.*

(4) If N_2 *is an* S*-weakly* 1*-absorbing primary submodule of* M*, and* N_1/N_2 *is a* $\bar{S}$*-weakly* 1*-absorbing primary submodule of the* R*-module* M/N_2*, then* N_1 *is an* S*-weakly* 1*-absorbing primary submodule of* M*.*

Proof. (1) This follows by applying Proposition 3.10 (1).

(2) It suffices to apply Proposition 3.10 (2).

(3) Suppose that N_2 is an S-1-absorbing primary submodule of M, and N_1/N_2 is a $\bar{S}$-weakly 1-absorbing primary submodule of the R-module M/N_2 with respect to s_1 and s_2, respectively. Let a,b be nonunit elements of R, and let $m\in M$ with $abm\in N_1$ and $sab\notin(N_1:_R M)$. If $abm\in N_2$, then since N_2 is S-1-absorbing primary and $s_1ab\notin(N_2:_R M)$, we have $s_1m\in M-\operatorname{rad}(N_2)\subseteq M-\operatorname{rad}(N_1)$, and thus $sm\in M-\operatorname{rad}(N_1)$. Now, suppose that $abm\notin N_2$. Hence, $0\neq ab(m+N_2)\in N_1/N_2$. Since clearly $s_2ab\notin(N_1/N_2:_R M/N_2)$, we conclude that $s_2m+N_2\in M/N_2-\operatorname{rad}(N_1/N_2)$. Therefore, $sm\in M-\operatorname{rad}(N_1)$, and thus N_1 is an S-1-absorbing primary submodule of M.

(4) The proof is similar to (3). □

We close this section with the following result.

Proposition 3.12. *Let S be a multiplicative subset of a ring R, and let M be an R-module. If N is an S-weakly 1-absorbing primary submodule of M and $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a weakly 1-absorbing primary submodule of the $S^{-1}R$-module $S^{-1}M$.*

Proof. Let $\frac{a}{s_1}, \frac{b}{s_2}$ be nonunit elements of $S^{-1}R$ and $\frac{m}{s_3} \in S^{-1}M$ with $0 \neq \frac{a}{s_1}\frac{b}{s_2}\frac{m}{s_3} \in S^{-1}N$. Then there exists $u \in S$ such that $0 \neq uabm \in N$. Since N is an S-weakly 1-absorbing primary submodule, we have either $suab \in (N :_R M)$ or $sm \in \mathrm{rad}_M(N)$. Thus, we conclude that either

$$\frac{a}{s_1}\frac{b}{s_2} = \frac{uab}{us_1s_2} \in S^{-1}(N :_R M) \subseteq \left(S^{-1}N :_{S^{-1}R} S^{-1}M\right),$$

or

$$\frac{m}{s_3} = \frac{sm}{ss_3} \in S^{-1}(\mathrm{rad}_M(N)) \subseteq S^{-1}M - \mathrm{rad}\left(S^{-1}N\right),$$

as required. □

4. S-weakly 1-absorbing primary in trivial ring extensions and amalgamations

Let M be a unitary R-module. The idealization of M in R, denoted by $R \ltimes M = R \oplus M$, is a commutative ring with componentwise addition and multiplication defined by

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$$

for each $r_1, r_2 \in R$ and $m_1, m_2 \in M$ [2]. For an ideal I of R and a submodule N of M, the set $I \ltimes N = I \oplus N$ is not always an ideal of $R \ltimes M$. It is an ideal if and only if $IM \subseteq N$ [2, Theorem 3.1]. Moreover, [2, Theorem 3.2(3)] characterizes the radical of $I \ltimes N$ as $\sqrt{I \ltimes N} = \sqrt{I} \ltimes M$. If S is a multiplicative subset of R, then the sets $S \ltimes M = \{(s, m) : s \in S, m \in M\}$ and $S \ltimes 0 = \{(s, 0) : s \in S\}$ are multiplicative subsets of the ring $R \ltimes M$. Now, we present a property of S-weakly 1-absorbing primary ideals in the trivial extension.

Lemma 4.1. *Let S be a multiplicative subset of a ring R, I a proper ideal of R, and N a proper submodule of an R-module M. If $I \ltimes N$ is an $(S \ltimes M)$-weakly 1-absorbing primary ideal of $R \ltimes M$, then I is an S-weakly 1-absorbing primary ideal of R.*

Proof. Let (s, m) be an $(S \ltimes M)$-element of $I \ltimes N$. Assume that a, b, c are nonunit elements of R such that $0 \neq abc \in I$ and $sc \notin \sqrt{I}$. Then $(0, 0) \neq (a, 0)(b, 0)(c, 0) \in I \ltimes N$ and $(c, 0) \notin \sqrt{I \ltimes N} = \sqrt{I} \ltimes M$. Since $I \ltimes N$ is $(S \ltimes M)$-weakly 1-absorbing primary, we have $(s, m)(a, 0)(b, 0) = (sab, abm) \in I \ltimes N$. Thus, $sab \in I$, as needed. □

Theorem 4.2. *Let M be an R-module, S be a multiplicative subset of R, and I a proper ideal of R. The following assertions are equivalent:*

(1) $I \ltimes M$ is an $(S \ltimes 0)$-weakly 1-absorbing primary ideal of $R \ltimes M$.
(2) $I \ltimes M$ is an $(S \ltimes M)$-weakly 1-absorbing primary ideal of $R \ltimes M$.
(3) I is an S-weakly 1-absorbing primary ideal of R associated with $s \in S$, and if $xyz = 0$ for some nonunit elements $x, y, z \in R$ but $sxy \notin I$ and $sz \notin \sqrt{I}$, then $xy, xz, yz \in \mathrm{ann}_R(M)$.

Proof. (1) $\Rightarrow$ (2) This follows from Remark 3.4, since $S \ltimes 0 \subseteq S \ltimes M$.

(2) $\Rightarrow$ (3) Suppose that $I \ltimes M$ is an $(S \ltimes M)$-weakly 1-absorbing primary ideal of $R \ltimes M$. Then, by Lemma 4.1, I is an S-weakly 1-absorbing primary ideal of R. Now, suppose $xyz = 0$ for some nonunit elements $x, y, z \in R$ but $sxy \notin I$ and $sz \notin \sqrt{I}$. We will show that $xy \in \mathrm{ann}_R(M)$. Assume $xy \notin \mathrm{ann}_R(M)$. Then there exists $m \in M$ such that $xym \neq 0$, and so we have $(0,0) \neq (x,0)(y,0)(z,m) \in I \ltimes M$. As $I \ltimes M$ is an $(S \ltimes M)$-weakly 1-absorbing primary ideal and $(x,0), (y,0), (z,m)$ are nonunit elements in $R \ltimes M$, there exists (s, m_1), an $(S \ltimes M)$-element of $I \ltimes M$, such that either $(s, m_1)(x,0)(y,0) \in I \ltimes M$ or $(s, m_1)(z,m) \in \sqrt{I \ltimes M}$. This implies that either $sxy \in I$ or $sz \in \sqrt{I}$, which is a contradiction. Thus, $xy \in \mathrm{ann}_R(M)$. Similarly, we have $xz, yz \in \mathrm{ann}_R(M)$.

(3) $\Rightarrow$ (1) Let $(0,0) \neq (x, m_1)(y, m_2)(z, m_3) \in I \ltimes M$, where $(x, m_1), (y, m_2), (z, m_3)$ are nonunit elements of $R \ltimes M$. If $xyz = 0$, then $sxy \in I$ and $sz \in \sqrt{I}$, and hence $(s,0)(x,m_1)(y,m_2) \in I \ltimes M$ or $(s,0)(z,m_3) \in \sqrt{I} \ltimes M = \sqrt{I \ltimes M}$. Now, suppose $xyz = 0$, but $sxy \notin I$ and $sz \notin \sqrt{I}$. Then $xy, xz, yz \in \mathrm{ann}_R(M)$. Consequently, we have $(x, m_1)(y, m_2)(z, m_3) = (0,0)$, a contradiction. □

Let R_1 and R_2 be commutative rings, J an ideal of R_2, and $f : R_1 \to R_2$ a ring homomorphism. We can define the following subring of $R_1 \times R_2$:

$$R_1 \bowtie^f J = \{(a, f(a) + j) \mid a \in R_1, j \in J\}.$$

$R_1 \bowtie^f J$ is called the amalgamation of R_1 with R_2 along J with respect to f. This construction generalizes the amalgamated duplication of a ring along an ideal (cf., for instance, [3, 4, 8, 9, 15]). Let I be an ideal of R_1, S a multiplicative set of R_1, and K an ideal of $f(R_1) + J$. Define the following:

$$S' := \{(s, f(s)) \mid s \in S\},$$
$$I \bowtie^f J := \{(i, f(i) + j) \mid i \in I, j \in J\},$$
$$S \bowtie^f J := \{(s, f(s) + j) \mid s \in S, j \in J\}.$$

It is easy to verify that $S \bowtie^f J$ and S' are multiplicative subsets of $R_1 \bowtie^f J$, and $I \bowtie^f J$ is an ideal of $R_1 \bowtie^f J$.

Lemma 4.3. *[16] Let R_1, R_2, I, J, K, and f be as above. Then, we have:*

$$\sqrt{I \bowtie^f J} = \sqrt{I} \bowtie^f J \quad \text{and} \quad \sqrt{\overline{K}^f} = \overline{\sqrt{K}}^f.$$

Next, we determine when the ideal $I \bowtie^f J$ is an $(S \bowtie^f J)$-weakly 1-absorbing primary ideal in $R_1 \bowtie^f J$.

Theorem 4.4. *Consider the amalgamation of rings R_1, R_2 along the ideal J of R_2 with respect to a homomorphism f. Let S be a multiplicative set of R_1 and I an ideal of R_1 disjoint from S. Then the following assertions are equivalent:*

(1) *$I \bowtie^f J$ is an S'-weakly 1-absorbing primary ideal of $R_1 \bowtie^f J$.*
(2) *$I \bowtie^f J$ is an $(S \bowtie^f J)$-weakly 1-absorbing primary ideal of $R_1 \bowtie^f J$.*
(3) *I is an S-weakly 1-absorbing primary ideal of R_1, and for $x, y, z \in R_1$ with $xyz = 0$ but $sxy \notin I$ and $sz \notin \sqrt{I}$, we have:*

$$f(xy)k + f(yz)i + f(x)kj + f(y)ik + f(z)ij + ijk = 0 \quad \text{for all } i, j, k \in J.$$

Proof. First, note that $(S \bowtie^f J) \cap (I \bowtie^f J) = \emptyset$ if and only if $S' \cap (I \bowtie^f J) = \emptyset$, which holds if and only if $S \cap I = \emptyset$.

(1) $\Rightarrow$ (2) This follows from Remark 3.4, since $S' \subseteq (S \bowtie^f J)$.

(2) $\Rightarrow$ (3) Suppose that $I \bowtie^f J$ is an $(S \bowtie^f J)$-weakly 1-absorbing primary ideal of $R_1 \bowtie^f J$ and choose an $(S \bowtie^f J)$-element $(s, f(s)+j)$ of $I \bowtie^f J$. Let $x, y, z \in R_1$ such that $0 \neq xyz \in I$ and $sz \notin \sqrt{I}$. Then $(0,0) \neq (x, f(x))(y, f(y))(z, f(z)) \in I \bowtie^f J$, and by Lemma 4.3, we have $(s, f(s)+j)(z, f(z)) \notin \sqrt{I \bowtie^f J}$. Hence, $(s, f(s)+j)(x, f(x))(y, f(y)) \in I \bowtie^f J$, and so $sxy \in I$. Therefore, I is an S-weakly 1-absorbing primary ideal of R_1. Now, let s be an S-element of I, and let $x, y, z \in R_1$ with $xyz = 0$, $sxy \notin I$, and $sz \notin \sqrt{I}$. Then for every $i, j, k \in J$, we have $(x, f(x)+i)(y, f(y)+j)(z, f(z)+k) \in I \bowtie^f J$, and since $(s, f(s)+i)(z, f(z)+k) \notin \sqrt{I \bowtie^f J}$, we get:

$$f(xy)k + f(yz)i + f(x)kj + f(y)ik + f(z)ij + ijk = 0.$$

(3) $\Rightarrow$ (1) Let $(x, f(x)+i), (y, f(y)+j), (z, f(z)+k) \in R_1 \bowtie^f J$ such that:

$$(0,0) \neq (x, f(x)+i)(y, f(y)+j)(z, f(z)+k) = (xyz, (f(x)+i)(f(y)+j)(f(z)+k)) \in I \bowtie^f J.$$

If $0 \neq xyz$, then $sxy \in I$ or $sz \in \sqrt{I}$ for some S-element s of I. Hence, $(s, f(s))(x, f(x)+i)(y, f(y)+j) \in I \bowtie^f J$ or $(s, f(s))(z, f(z)+k) \in \sqrt{I} \bowtie^f J = \sqrt{I \bowtie^f J}$ as required. Now, suppose $xyz = 0$. Then $f(xy)k + f(yz)i + f(x)kj + f(y)ik + f(z)ij + ijk = 0$, and by assumption, either $sxy \in I$ or $sz \in \sqrt{I}$. Therefore, $(s, f(s))(x, f(x)+i)(y, f(y)+j) \in I \bowtie^f J$ or $(s, f(s))(z, f(z)+k) \in \sqrt{I \bowtie^f J}$. Thus, $I \bowtie^f J$ is an S'-weakly 1-absorbing primary ideal of $R_1 \bowtie^f J$. □

Let I be a proper ideal of R_1. The (amalgamated) duplication of R_1 along I is a special amalgamation given by

$$R_1 \bowtie I = \{(a, a+i) \mid a \in R_1, i \in I\}.$$

The next corollary is an immediate consequence of the previous theorem on the transfer of the S-weakly 1-absorbing primary ideal property to duplications.

Corollary 4.5. *Let R_1 be a ring and I, K be ideals of R_1 with $K \cap S = \emptyset$. Then the following assertions are equivalent:*

(1) *K is an S-weakly 1-absorbing primary ideal of R_1.*
(2) *$K \bowtie I$ is an $(S \bowtie I)$-weakly 1-absorbing primary ideal of $R_1 \bowtie I$.*

Proof. First of all, notice that $(K \bowtie I) \cap (S \bowtie I) = \emptyset$. The rest of the proof follows from the previous theorem. □

Let S be a multiplicative set of R_1 and K an ideal of $f(R_1) + J$. Then clearly, the set

$$\overline{T}^f := \{(s, f(s)+j) \mid s \in R_1, j \in J, f(s)+j \in T\}$$

is a multiplicative subset of $R_1 \bowtie^f J$, and the set

$$\overline{K}^f := \{(a, f(a)+j) \mid a \in R_1, j \in J, f(a)+j \in K\}$$

is an ideal of $R_1 \bowtie^f J$.

Theorem 4.6. *Consider the amalgamation of rings R_1 and R_2 along the ideal J of R_2 with respect to an epimorphism f. Let K be an ideal of R_2, and let T be a multiplicative subset of R_2 disjoint from K. Then the following statements are equivalent:*

(1) $\overline{K}^f$ *is a* $\overline{T}^f$*-weakly* 1*-absorbing primary ideal of* $R_1 \bowtie^f J$.

(2) K *is a* T*-weakly* 1*-absorbing primary ideal of* R_2*, and for* $f(x), f(y), f(z) \in R_2$ *and a* T*-element* $f(s)$ *of* K*, if* $(f(x)+i)(f(y)+j)(f(z)+k) = 0$ *for every* $i, j, k \in J$*, and* $f(s)(f(x)+i)(f(y)+j) \notin K$ *and* $f(s)(f(z)+k) \notin \sqrt{K}$*, then* $xyz = 0$.

Proof. One can easily verify that $T \cap S = \emptyset$ if and only if $\overline{T}^f \cap \overline{K}^f = \emptyset$.

(1) $\Rightarrow$ (2) Suppose that $\overline{K}^f$ is a $\overline{T}^f$-weakly 1-absorbing primary ideal of $R_1 \bowtie^f J$ and choose a $\overline{T}^f$-element $(s, f(s)+j)$ of $\overline{K}^f$. Let $x', y', z' \in R_2$ such that $0 \neq x'y'z' \in K$, say, $x' = f(x), y' = f(y), z' = f(z)$ for $x, y, z \in R_1$. Hence, $(x, f(x)), (y, f(y)), (z, f(z)) \in R_1 \bowtie^f J$ with $(0,0) \neq (x, f(x))(y, f(y))(z, f(z)) = (xyz, f(xyz)) \in \overline{K}^f$. So, we have either

$$(s, f(s)+j)(x, f(x))(y, f(y)) = (sxy, (f(s)+j)f(xy)) \in \overline{K}^f$$

or

$$(s, f(s)+j)(z, f(z)) = (sz, (f(s)+j)f(z)) \in \sqrt{\overline{K}^f} = \overline{\sqrt{K}}^f.$$

Thus, $f(s)+j \in T$ and $(f(s)+j)f(xy) \in K$ or $(f(s)+j)f(z) \in \sqrt{K}$. It follows that K is a T-weakly 1-absorbing primary ideal of R_2. Now, let $f(s)$ be a T-element of K, and $f(x), f(y), f(z) \in R_2$ such that $(f(x)+i)(f(y)+j)(f(z)+k) = 0$, $f(s)(f(x)+i)(f(y)+j) \notin K$ and $f(s)(f(z)+k) \notin \sqrt{K}$. Since $\overline{K}^f$ is a $\overline{T}^f$-weakly 1-absorbing primary ideal, for every $i, j, k \in J$, $(x, f(x)+i)(y, f(y)+j)(z, f(z)+k) \in \overline{K}^f$, then we must have $xyz = 0$ and we are done.

(2) $\Rightarrow$ (1) Choose a T-element $f(s)$ of K. Let $(0,0) \neq (x, f(x)+i)(y, f(y)+j)(z, f(z)+k) = (xyz, (f(x)+i)(f(y)+j)(f(z)+k)) \in \overline{K}^f$ for $(x, f(x)+i), (y, f(y)+j), (z, f(z)+k) \in R_1 \bowtie^f J$. Then $(f(x)+i)(f(y)+j)(f(z)+k) \in K$. Thus,

$$(s, f(s))(x, f(x)+i)(y, f(y)+j) = (sxy, f(s)(f(x)+i)(f(y)+j)) \in \overline{K}^f$$

or

$$(s, f(s))(z, f(z)+k) = (sz, f(sz)+f(s)k) \in \sqrt{\overline{K}^f} = \overline{\sqrt{K}}^f.$$

The result follows. Suppose $(f(x)+i)(f(y)+j)(f(z)+k) = 0$. Then $xyz \neq 0$, and so by our assumption, we conclude either $f(s)(f(x)+i)(f(y)+j) \in K$ or $f(s)(f(z)+k) \in \sqrt{K}$. Hence, again

$$(s, f(s))(x, f(x)+i)(y, f(y)+j) \in \overline{K}^f$$

or

$$(s, f(s))(z, f(z)+k) \in \sqrt{\overline{K}^f},$$

and so $\overline{K}^f$ is a $\overline{T}^f$-weakly 1-absorbing primary ideal of $R_1 \bowtie^f J$. □

In particular, if S is a multiplicative subset of R_1, then $S \times f(S)$ is also a multiplicative subset of $R_1 \bowtie^f J$. Thus, we get the following corollary of Theorem 4.6.

Corollary 4.7. *Let* $R_1, R_2, J, S,$ *and* f *be as in Theorem 4.6. Let* K *be an ideal of* R_2 *and* $T = f(S)$*. Then the following assertions are equivalent:*

(1) $\overline{K}^f$ *is a* $(S \times T)$*-weakly* 1*-absorbing primary ideal of* $R_1 \bowtie^f J$.

(2) $\overline{K}^f$ *is a* $\overline{T}^f$*-weakly* 1*-absorbing primary ideal of* $R_1 \bowtie^f J$.

(3) *K is a T-weakly 1-absorbing primary ideal of R_2 and for $f(x), f(y), f(z) \in R_2$ and a T-element $f(s)$ of K, if $(f(x)+i)(f(y)+j)(f(z)+k) = 0$ for every $i, j, k \in J$, $f(s)(f(x)+i)(f(y)+j) \notin K$ and $f(s)(f(z)+k) \notin \sqrt{K}$, then $xyz = 0$.*

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