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TOPOLOGICAL GROUPS WITH REPRESENTATIONS OF BOUNDED DIMENSION

A. I. Shtern

Abstract. The main result of the paper is the proof of a theorem that strengthens the classical 1949–1972 results by Kaplansky, Isaacs–Passman, and C. C. Moore on locally compact groups all of whose irreducible unitary representations are finite-dimensional and the dimensions of these representations are jointly bounded: these groups are finite extensions of commutative locally compact groups. The most important reason to consider representations of locally compact groups is the Gel'fand–Raykov theorem on the existence of a family of continuous irreducible unitary representations, for every locally compact group, that separate the points of the group algebra. It turns out that the latter property is the key one. In this connection, it is proved that, if a separated topological group has a family of continuous irreducible unitary representations separating the points of the group algebra, if all these representations are finite-dimensional, and their dimensions are jointly bounded, then the group contains a commutative normal subgroup of finite index.

§ 1. Introduction

The term "topological group" everywhere means "separated topological group".

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In the class of locally compact groups, the characterization of groups all of whose continuous irreducible unitary representations are finite-dimensional and their dimensions are jointly bounded (the structure theorem) has been known since 1972. A number of results by outstanding authors prepared this result. In 1949, Kaplansky [1] proved, in particular, that a unimodular connected locally compact group all of whose continuous irreducible unitary representations are finite-dimensional and their dimensions are jointly bounded is commutative. In 1961, Amitsur [2] found the possible dimensions of finite-dimensional unitary irreducible representations of groups with a commutative normal subgroup of index two. Isaacs and Passman [3] proved in 1964 that, for discrete groupsm this condition on representations is equivalent to the existence of a commutative normal subgroup of finite index. In 1967, Grosser and Moskowitz [4] got rid of the unimodularity condition in Kaplansky's theorem mentioned above, proving that the connected component of the identity element in a locally compact group all of whose continuous irreducible unitary representations are finite-dimensional and their dimensions are collectively bounded is commutative. In 1968, Thoma [5] completed the study of (not necessarily countable) discrete groups of type I by proving that all such groups have a commutative normal subgroup of finite index (and thus the dimensions of the irreducible representations are jointly bounded). The final result in this direction was obtained by Calvin Moore [6] in 1972, who established that all continuous irreducible unitary representations of a locally compact group are finite-dimensional and their dimensions are jointly bounded if and only if the group has a closed commutative normal subgroup of finite index. In 1979, Schlichting [7] published a short proof using the Freudenthal–Weyl theorem on the existence of a continuous embedding of groups all of whose irreducible unitary representations are finite-dimensional into a compact topological group.

However, the class of locally compact groups considered in all these papers has two undoubted advantages. First, the Gelfand–Raikov theorem [8] ensures the Hilbert representability of groups of this class, that is, the existence of a family of irreducible continuous unitary representations separating the elements of the group algebra. Certainly, this property ensures the meaningfulness of the problem statement. Second, a developed structural theory of locally compact groups may be useful at some stages of the proof. At the same time, for the study of phenomena associated with unitary representations of groups, one should consider the restriction of the problem to the class of locally compact groups as an excessive simplification, since it excludes from consideration (not necessarily locally compact) topological groups that have a family of continuous unitary representations separating the points of the group. It is obvious, for example, that such groups can be embedded in compact groups if the dimensions of all representations of the separating family are finite-dimensional, and such groups are, therefore, isomorphic as abstract groups to subgroups of the direct product of a compact topological group and a finite-dimensional vector group. And in the problem of the structure of groups all of whose irreducible unitary representations are finite-dimensional and have jointly bounded dimensions, it is natural to restrict ourselves to the condition of the existence of a family of continuous unitary representations, separating the points of the group algebra, that have jointly bounded dimensions. Thus, in this connection, it is of interest to answer the following question: what are Hilbert representable groups all of whose continuous irreducible unitary representations are finite-dimensional, and their dimensions are jointly bounded?

The present paper contains the naturally expected answer to this question. The corresponding groups are finite extensions of commutative Hilbert representable groups.

The main theorem is proved in Sec. 2. The result is discussed in Sec. 3.

§ 2. Proof of the Main Theorem

Theorem 1. *Let G be a topological group whose family of irreducible continuous unitary representations separates the points of the group, let all these representations be finite-dimensional, and let the dimensions of these representations be jointly bounded (*$\dim \pi \le n = n(G)$ *for* $\pi \in \widehat{G}$*, where* $\widehat{G}$ *stands for the dual space of* G*). Then the group* G *contains a commutative normal subgroup of finite index.*

Proof. Let the conditions of the theorem be satisfied. For every $\pi \in \widehat{G}$, consider the unitary group U_π of the representation space π and introduce a compact topological group

$$K_1 = \prod_{\pi \in \widehat{G}} U_\pi,$$

equipped with a continuous embedding ρ of G into K defined by

$$\rho(g) = \prod_{\pi \in \widehat{G}} \pi(g), \qquad g \in G.$$

Let K be the closure of the image $\rho(G)$ in K_1; K is a compact topological group. Any continuous irreducible unitary representation τ of K defines a continuous irreducible unitary representation of G by the rule $\sigma = \tau \circ \rho$. By the assumption, all such representations are finite-dimensional, and their dimensions are jointly bounded. Therefore, all continuous irreducible unitary representations of K are finite-dimensional, and their dimensions are jointly bounded. By Moore's theorem, this means that K is a finite extension of a commutative compact normal subgroup C of K. Consider the restriction of the canonical homomorphism of K onto the quotient group of K by a normal subgroup C to the image $\rho(G)$ of G under the embedding ρ of G in K (so that G is isomorphic to $\rho(G)$ in the group-theoretic sense). Under this restriction, the image $\rho(G)$ goes to a subgroup of the finite group K/C. Consequently, the kernel of this mapping, which is the normal subgroup $\rho(G) \cap C$ of $\rho(G)$ lying in the commutative group C, is commutative, and the quotient group of $\rho(G)$ by $\rho(G) \cap C$ is isomorphic to a subgroup of a finite group K/C, which completes the proof.

§ 3. Discussion

It follows from the proof of the main theorem that the condition of existence of a family of irreducible continuous unitary representations separating the points of the group algebra of the group in the discrete topology and having jointly bounded dimensions can be replaced by the condition of existence of a family of (not necessarily irreducible) continuous unitary representations separating the elements of the same group algebra, where the dimensions of these representations are jointly bounded by some positive integer n. This condition is sufficient for the polynomial identity

$$\sum_{\sigma \in S(m)} \operatorname{sgn}(\sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(m)} = 0$$

for $m \geq r(n)$ (where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ and $r(n)$ is defined in 3.6.1 of [9]) as the least positive integer for which the identity holds for all linear operators $X_1, X_2, \dots, X_m$ in all representation spaces of the family under consideration, and the inequality $r(n+1) \geq r(n) + 2$, proved there, ensures that the topological group under consideration does not even have discontinuous irreducible representations of dimension greater than n. Thus, provided that the requirement on separating the elements of the group by representations from the family under consideration is preserved, the condition of irreducibility of unitary representations can be omitted.

The last proof of the theorem given above immediately explains the impossibility of giving up the condition of joint boundedness of the dimensions of the representations in an attempt to obtain a finite extension of a commutative group in the answer. If we omit this condition, the group can still be embedded in the product of a compact group and a finite-dimensional vector group, but there are no additional conditions on the compact group. In particular, connected semisimple compact Lie groups have continuous irreducible representations of arbitrarily large dimension, and they can be included in the identity component of a compact group.

The natural question of whether the result obtained above for continuous representations can be extended to not necessarily continuous locally bounded representations of the group is, albeit somewhat unexpectedly, obviously affirmative. Groups that are finite extensions of commutative groups are perfect (i.e., they coincide with its algebraic commutator) only if the commutative normal subgroup is finite, and thus the entire group is finite, but it follows from the above consideration about the polynomial identity that, if there is a family of representations of a group with dimensions not exceeding a positive integer n that separates the elements of the group algebra of this group regarded in discrete topology, then this group cannot have irreducible representations of dimension exceeding n, and it remains to apply the Isaaks–Passman [3] result on discrete groups: the group contains a commutative normal subgroup of finite index.

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Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991 Russia
Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia
Scientific Research Institute for System Analysis of the National Research Centre "Kurchatov Institute",
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru