Proceedings of the Jangjeon Mathematical Society
29 (2026), No. 1, pp. 1 – 9
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2026.29.1.001

SPIVEY-TYPE RECURRENCE RELATION FOR FULLY DEGENERATE BELL POLYNOMIALS

TAEKYUN KIM, DAE SAN KIM, AND DMITRY V. DOLGY

ABSTRACT. Spivey's combinatorial method revealed an important identity for Bell numbers, involving Stirling numbers of the second kind. This paper extends his work by deriving Spivey-type recurrence relations for fully degenerate Bell polynomials and degenerate Fubini polynomials. Our derivation uses degenerate Stirling numbers of the second kind and two-variable degenerate Fubini polynomials of order α.

1. INTRODUCTION

Spivey derived the following remarkable identity for Bell numbers using a combinatorial method (see (8), [22]):

$$\phi_{n+m}=\sum_{k=0}^{m}\sum_{l=0}^{n}\left\{\begin{matrix} m \\ k \end{matrix}\right\}\binom{n}{l}k^{n-l}\phi_l, \quad (m,n\geq 0), \tag{1}$$

which has since inspired extensive research on Spivey-type relations (see [4,6,8,12-17] and the references therein]).

Spivey's Bell number formula in (1) provides new ways to calculate Bell numbers and their generalizations by relating them to Stirling numbers of the second kind through a combinatorial identity. Its implications include the derivation of new sum formulas for Bell numbers, extensions of the formula to more general mathematical objects like r-Whitney and degenerate Bell numbers, and the development of combinatorial proofs and interpretations for these generalizations. The following are the key implications of Spivey's formula.

New Sum Formulas: The formula provides a fresh way to express Bell numbers as sums involving Stirling numbers of the second kind, which leads to new sum formulas for the Bell numbers themselves (see [22]).

Generalizations: The underlying principles of Spivey's formula have been extended to various other combinatorial quantities, including:

• r-Whitney Numbers: The formula applies to r-Whitney numbers, leading to new identities and a deeper understanding of their structure (see [16]).

• Degenerate Bell Numbers: Spivey-type recurrences have been established for degenerate Bell and Dowling polynomials, providing new insights into these related sequences (see [12,15,16]).

• Lah-Bell Polynomials: Spivey-type realtions have been extended to r-Lah-Bell and λ-analogue of r-Lah-Bell polynomials (see [8]).

• q-Generalizations: The formula has been generalized to the q-analogues of

2010 *Mathematics Subject Classification.* 11B73; 11B83.
Key words and phrases. Spivey-type recurrence relation; fully degenerate Bell polynomials; degenerate Fubini polynomials.

Bell and Stirling numbers, opening up connections to advanced topics in combinatorics and q-calculus (see [6]).

New Proof Techniques: The formula has inspired different approaches to proofs, including:

• Generating Function Proofs: Researchers have provided generating function proofs for Spivey's result, offering an algebraic perspective (see [8,12-14]).

• Use of Boson and Differential Operators: The use of boson and differential operators provides a powerful algebraic framework for deriving Spivey-type relation. This method bypasses traditional combinatorial proofs by leveraging the correspondence between these operators (see [6,8,15]).

• Rook Polynomials: The formula has found a connection with rook polynomials, providing novel, bijective proofs for certain combinatorial identities (see [4]).

Combinatorial Interpretations: The formula provides a basis for developing new combinatorial interpretations, helping to explain the meaning of these number sequences in terms of set partitions and other structures (see [22]).

Connections to Other Fields: Its implications extend to other areas, such as:

• Probability: Bell numbers can be interpreted as moments of a Poisson distribution, and Spivey's formula contributes to this understanding (see [13,14,17]).

• Graphs: The formula can be applied to graph theory for counting partitions where blocks are independent sets (see [4]).

Recently, Kim-Kim showed the following recurrence relation for the degenerate Bell polynomials (see (7)), which is given by

$$(2)\qquad \phi_{n+m,\lambda}(x)=\sum_{k=0}^{m}\sum_{l=0}^{n}\left\{ {m \atop k} \right\}_{\lambda}\binom{n}{l}x^{k}\big(k-m\lambda\big)_{n-l,\lambda}\phi_{l,\lambda}(x),\quad (\text{see [11]}),$$

where m,n are nonnegative integers.
Letting $\lambda\to 0$ in (2) gives Spivey's recurrence relation for Bell polynomials (see (8)):

$$(3)\qquad \phi_{n+m}(x)=\sum_{k=0}^{m}\sum_{l=0}^{n}\left\{ {m \atop k} \right\}\binom{n}{l}x^{k}k^{n-l}\phi_{l}(x),\quad (m,n\geq 0).$$

Moreover, letting $x=1$ yields the Spivey's relation in (1).

In this paper, we show the following two Spivey-type relations:

$$(4)\qquad \mathrm{Bel}_{n+m,\lambda}(x)=\sum_{k=0}^{m}\sum_{l=0}^{n}(1)_{k,\lambda}\left\{ {m \atop k} \right\}_{\lambda}\binom{n}{l}x^{k}F_{n-l,\lambda}^{(k)}(-\lambda x,k-m\lambda)\mathrm{Bel}_{l,\lambda}(x),$$

$$F_{n+m,\lambda}(x)=\sum_{k=0}^{m}\sum_{l=0}^{n}k!\left\{ {m \atop k} \right\}_{\lambda}\binom{n}{l}x^{k}F_{n-l,\lambda}^{(k)}(x,k-m\lambda)F_{l,\lambda}(x),$$

where $\mathrm{Bel}_{n,\lambda}(x)$ are the fully degenerate Bell polynomials (see (9)), $F_{n,\lambda}^{(\alpha)}(x,y)$ are the two variable degenerate Fubini polynomials of order α (see (11)), and $F_{n,\lambda}(x)$ are the degenerate Fubini polynomials (see (14)). Here we note that, by letting $\lambda\to 0$, we get (3) from (4) (see (10), (12)).

For any nonzero $\lambda\in\mathbb{R}$, the degenerate exponentials are defined by

$$(5)\qquad e_{\lambda}^{x}(t)=\sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^{k}}{k!}=(1+\lambda t)^{\frac{x}{\lambda}},\quad e_{\lambda}(t)=e_{\lambda}^{1}(t),\quad (\text{see } [5,9-12,14]),$$

where

$$(x)_{0,\lambda}=1,\quad (x)_{n,\lambda}=x(x-\lambda)\cdots\big(x-(n-1)\lambda\big),\quad (n\geq 1).$$

The degenerate Stirling numbers ${n \brace k}_{\lambda}$ of the second kind are given by (see [12])

$$(x)_{n,\lambda}=\sum_{k=0}^{n}{n \brace k}_{\lambda}(x)_{k},\quad (n\geq 0),\tag{6}$$

$$\frac{1}{k!}\big(e_{\lambda}(t)-1\big)^{k}=\sum_{n=k}^{\infty}{n \brace k}_{\lambda}\frac{t^{n}}{n!},\quad (k\geq 0).$$

Note that

$$\lim_{\lambda\to 0}{n \brace k}_{\lambda}={n \brace k},$$

where ${n \brace k}$ is the ordinary Stirling number of the second kind (see [3,18]).

The degenerate Bell polynomials are defined by (see (6), [11,12])

$$\phi_{n,\lambda}(x)=\sum_{k=0}^{n}{n \brace k}_{\lambda}x^{k},\quad (n\geq 0),\tag{7}$$

$$e^{x(e_{\lambda}(t)-1)}=\sum_{n=0}^{\infty}\phi_{n,\lambda}(x)\frac{t^{n}}{n!}.$$

When $x=1$, $\phi_{n,\lambda}=\phi_{n,\lambda}(1)$ are called the degenerate Bell numbers. Note that $\lim_{\lambda\to 0}\phi_{n,\lambda}(x)=\phi_{n}(x)$, where $\phi_{n}(x)$ are the classical Bell polynomials given by

$$\phi_{n}(x)=\sum_{k=0}^{n}{n \brace k}x^{k},\quad (n\geq 0),\tag{8}$$

$$e^{x(e^{t}-1)}=\sum_{n=0}^{\infty}\phi_{n}(x)\frac{t^{n}}{n!}.$$

The fully degenerate Bell polynomials are introduced by Kim-Kim as (see (6), [5])

$$\mathrm{Bel}_{n,\lambda}(x)=\sum_{k=0}^{n}{n \brace k}_{\lambda}(1)_{k,\lambda}x^{k},\quad (n\geq 0),\tag{9}$$

$$e_{\lambda}\Big(x\big(e_{\lambda}(t)-1\big)\Big)=\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(x)\frac{t^{n}}{n!}.$$

We note here that

$$\lim_{\lambda\to 0}\mathrm{Bel}_{n,\lambda}(x)=\phi_{n}(x),\quad (n\geq 0).\tag{10}$$

When $x=1$, $\mathrm{Bel}_{n,\lambda}=\mathrm{Bel}_{n,\lambda}(1)$, $(n\geq 0)$, are called the fully degenerate Bell numbers.

Two variable Fubini polynomials of order α, $F_{n}^{(\alpha)}(x,y)$, are given by

$$\left(\frac{1}{1-x(e^{t}-1)}\right)^{\alpha}e^{yt}=\sum_{n=0}^{\infty}F_{n}^{(\alpha)}(x,y)\frac{t^{n}}{n!}.$$

As a degenerate version of $F_{n}^{(\alpha)}(x,y)$, the two variable degenerate Fubini polynomials of order α are introduced by Kim-Kim as

$$\left(\frac{1}{1-x(e_{\lambda}(t)-1)}\right)^{\alpha}e_{\lambda}^{y}(t)=\sum_{n=0}^{\infty}F_{n,\lambda}^{(\alpha)}(x,y)\frac{t^{n}}{n!},\quad (\text{see [9]}).\tag{11}$$

When $x=0$, we note that

$$F_{n,\lambda}^{(\alpha)}(0,y)=(y)_{n,\lambda},\quad (n\geq 0). \tag{12}$$

By (11), we easily get (see (6))

$$F_{n,\lambda}^{(\alpha)}(x,0)=\sum_{k=0}^{n}\langle\alpha\rangle_k {n \brace k}_{\lambda} x^k,\quad (n\geq 0),$$

where

$$\langle x\rangle_0=1,\quad \langle x\rangle_n=x(x+1)(x+2)\cdots(x+n-1),\ (n\geq 1).$$

Note that

$$\lim_{\lambda\to 0}F_{n,\lambda}^{(1)}(x,0)=\sum_{k=0}^{n}k!{n \brace k}x^k=F_n(x),$$

where $F_n(x)$ are the ordinary Fubini polynomials given by (see [11])

$$\frac{1}{1-x(e^t-1)}=\sum_{n=0}^{\infty}F_n(x)\frac{t^n}{n!}. \tag{13}$$

The degenerate Fubini polynomials are given by (see (6), [11])

$$F_{n,\lambda}(x)=\sum_{k=0}^{n}k!{n \brace k}_{\lambda}x^k,\quad (n\geq 0), \tag{14}$$

$$\frac{1}{1-x(e_\lambda(t)-1)}=\sum_{n=0}^{\infty}F_{n,\lambda}(x)\frac{t^n}{n!}.$$

Note that

$$\lim_{\lambda\to 0}F_{n,\lambda}(x)=\sum_{k=0}^{n}k!{n \brace k}x^k=F_n(x).$$

The reader may refer to [1,2,5,9-12,14] for the recent developments on various degenerate versions of many special numbers and polynomials, and [3,18,20] as general references for this paper.

2. Spivey-type recurrence relation for fully degenerate Bell polynomials

By (9), we get

$$\begin{aligned}\sum_{m,n=0}^{\infty}\mathrm{Bel}_{n+m,\lambda}\frac{x^n}{n!}\frac{y^m}{m!}&=\sum_{m=0}^{\infty}\left(\frac{d}{dx}\right)^m\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}\frac{x^n}{n!}\frac{y^m}{m!}\\&=\sum_{n=0}^{\infty}\frac{\mathrm{Bel}_{n,\lambda}}{n!}\sum_{m=0}^{\infty}\left(\frac{d}{dx}\right)^m x^n\frac{y^m}{m!}=\sum_{n=0}^{\infty}\frac{\mathrm{Bel}_{n,\lambda}}{n!}\sum_{m=0}^{n}\binom{n}{m}x^{n-m}y^m\\&=\sum_{n=0}^{\infty}\frac{\mathrm{Bel}_{n,\lambda}}{n!}(x+y)^n=e_\lambda\left(e_\lambda(x+y)-1\right)=e_\lambda\left(e_\lambda(x)e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right).\end{aligned} \tag{15}$$

Now, we observe that

(16)

$$
\begin{aligned}
& e_\lambda\left(e_\lambda(x)e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right)=\left(1+\lambda\left(e_\lambda(x)e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right)\right)^{\frac{1}{\lambda}}\\
&=\left(1+\lambda e_\lambda(x)-\lambda e_\lambda(x)+\lambda e_\lambda(x)e_\lambda\left(\frac{y}{1+\lambda x}\right)-\lambda\right)^{\frac{1}{\lambda}}\\
&=\left(1+\lambda\left(e_\lambda(x)-1\right)+\lambda e_\lambda(x)\left(e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right)\right)^{\frac{1}{\lambda}}\\
&=\left(1+\lambda(e_\lambda(x)-1)\right)^{\frac{1}{\lambda}}\left(1+\lambda\frac{e_\lambda(x)\left(e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right)}{1+\lambda(e_\lambda(x)-1)}\right)^{\frac{1}{\lambda}}\\
&=e_\lambda\left(e_\lambda(x)-1\right)e_\lambda\left(\frac{e_\lambda(x)\left(e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right)}{1+\lambda(e_\lambda(x)-1)}\right)\\
&=e_\lambda\left(e_\lambda(x)-1\right)\sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{k!}\left(\frac{e_\lambda(x)}{1+\lambda(e_\lambda(x)-1)}\right)^k\left(e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right)^k\\
&=e_\lambda\left(e_\lambda(x)-1\right)\sum_{k=0}^{\infty}(1)_{k,\lambda}\left(\frac{1}{1+\lambda(e_\lambda(x)-1)}\right)^k e_\lambda^k(x)\sum_{m=k}^{\infty}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda e_\lambda^{-m\lambda}(x)\frac{y^m}{m!}\\
&=\sum_{m=0}^{\infty}\frac{y^m}{m!}\left(\sum_{k=0}^{m}(1)_{k,\lambda}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda\left(\frac{1}{1+\lambda(e_\lambda(x)-1)}\right)^k e^{k-m\lambda}(x)e_\lambda\left(e_\lambda(x)-1\right)\right)\\
&=\sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{j=0}^{\infty}\frac{x^j}{j!}\sum_{k=0}^{m}(1)_{k,\lambda}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda F_{j,\lambda}^{(k)}(-\lambda,k-m\lambda)\sum_{l=0}^{\infty}\mathrm{Bel}_{l,\lambda}\frac{x^l}{l!}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{m}(1)_{k,\lambda}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda\binom{n}{l}\mathrm{Bel}_{l,\lambda}F_{n-l,\lambda}^{(k)}(-\lambda,k-m\lambda)\right)\frac{y^m}{m!}\frac{x^n}{n!}.
\end{aligned}
$$

Therefore, by (15) and (16), we obtain the following theorem.

Theorem 2.1. *For $m,n\geq 0$, we have*

$$
\mathrm{Bel}_{n+m,\lambda}=\sum_{l=0}^{n}\sum_{k=0}^{m}(1)_{k,\lambda}\left\{\begin{matrix}m\\k\end{matrix}\right\}_\lambda\binom{n}{l}\mathrm{Bel}_{l,\lambda}F_{n-l,\lambda}^{(k)}(-\lambda,k-m\lambda).
$$

Letting $\lambda\to 0$ gives Spivey's recurrence relation for Bell numbers (see (10), (12)):

$$
\phi_{n+m}=\lim_{\lambda\to 0}\mathrm{Bel}_{n+m,\lambda}=\sum_{l=0}^{n}\sum_{k=0}^{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}\binom{n}{l}\phi_l k^{n-l}.
$$

From (9), we note that

$$(17)\qquad \sum_{n=0}^{\infty}\mathrm{Bel}_{n+m,\lambda}(t)\frac{x^n}{n!}\frac{y^m}{m!}=\sum_{m=0}^{\infty}\frac{d^m}{dx^m}\sum_{n=0}^{\infty}\mathrm{Bel}_{n,\lambda}(t)\frac{x^n}{n!}\frac{y^m}{m!}$$
$$=\sum_{n=0}^{\infty}\frac{\mathrm{Bel}_{n,\lambda}(t)}{n!}\sum_{m=0}^{\infty}\Big(\frac{d}{dx}\Big)^m x^n\frac{y^m}{m!}=\sum_{n=0}^{\infty}\frac{\mathrm{Bel}_{n,\lambda}(t)}{n!}\sum_{m=0}^{n}\binom{n}{m}x^{n-m}y^m.$$
$$=\sum_{n=0}^{\infty}\frac{\mathrm{Bel}_{n,\lambda}(t)}{n!}(x+y)^n=e_\lambda\Big(t\big(e_\lambda(x+y)-1\big)\Big)$$
$$=e_\lambda\bigg(t\bigg(e_\lambda(x)e_\lambda\Big(\frac{y}{1+\lambda x}\Big)-1\bigg)\bigg).$$

On the other hand, by (5), we get

(18)

$$e_\lambda\bigg(t\bigg(e_\lambda(x)e_\lambda\Big(\frac{y}{1+\lambda x}\Big)-1\bigg)\bigg)=\bigg(1+\lambda t\bigg(e_\lambda(x)e_\lambda\Big(\frac{y}{1+\lambda x}\Big)-1\bigg)\bigg)^{\frac{1}{\lambda}}$$
$$=\bigg(1+\lambda t\big(e_\lambda(x)-1\big)+\lambda te_\lambda(x)\bigg(e_\lambda\Big(\frac{y}{1+\lambda x}\Big)-1\bigg)\bigg)^{\frac{1}{\lambda}}$$
$$=\Big(1+\lambda t(e_\lambda(x)-1)\Big)^{\frac{1}{\lambda}}\bigg(1+\lambda\frac{te_\lambda(x)\big(e_\lambda\big(\frac{y}{1+\lambda x}\big)-1\big)}{1+\lambda t(e_\lambda(x)-1)}\bigg)^{\frac{1}{\lambda}}$$
$$=e_\lambda\Big(t(e_\lambda(x)-1)\Big)e_\lambda\bigg(\frac{te_\lambda(x)\big(e_\lambda\big(\frac{y}{1+\lambda x}\big)-1\big)}{1+\lambda t(e_\lambda(x)-1)}\bigg)$$
$$=e_\lambda\Big(t(e_\lambda(x)-1)\Big)\sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{k!}\frac{t^ke_\lambda^k(x)\big(e_\lambda\big(\frac{y}{1+\lambda x}\big)-1\big)^k}{\big(1+\lambda t(e_\lambda(x)-1)\big)^k}$$
$$=e_\lambda\Big(t(e_\lambda(x)-1)\Big)\sum_{k=0}^{\infty}(1)_{k,\lambda}\bigg(\frac{1}{1+\lambda t(e_\lambda(x)-1)}\bigg)^k t^ke_\lambda^k(x)\sum_{m=k}^{\infty}{m \brace k}_\lambda\frac{y^me_\lambda^{-m\lambda}(x)}{m!}$$
$$=\sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{k=0}^{m}(1)_{k,\lambda}{m \brace k}_\lambda t^k\frac{e_\lambda^{k-m\lambda}(x)}{\big(1+\lambda t(e_\lambda(x)-1)\big)^k}e_\lambda\Big(t(e_\lambda(x)-1)\Big)$$
$$=\sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{k=0}^{m}(1)_{k,\lambda}t^k{m \brace k}_\lambda\sum_{j=0}^{\infty}F_{j,\lambda}^{(k)}(-\lambda t,k-m\lambda)\frac{x^j}{j!}\sum_{l=0}^{\infty}\mathrm{Bel}_{l,\lambda}(t)\frac{x^l}{l!}$$
$$=\sum_{m,n=0}^{\infty}\bigg(\sum_{k=0}^{m}\sum_{l=0}^{n}(1)_{k,\lambda}{m \brace k}_\lambda\binom{n}{l}t^kF_{n-l,\lambda}^{(k)}(-\lambda t,k-m\lambda)\mathrm{Bel}_{l,\lambda}(t)\bigg)\frac{x^n}{n!}\frac{y^m}{m!}.$$

Therefore, by (17) and (18), we obtain the following theorem.

Theorem 2.2. *For $n,m\geq 0$, we have*

$$\mathrm{Bel}_{n+m,\lambda}(t)=\sum_{k=0}^{m}\sum_{l=0}^{n}(1)_{k,\lambda}{m \brace k}_\lambda\binom{n}{l}t^kF_{n-l,\lambda}^{(k)}(-\lambda t,k-m\lambda)\mathrm{Bel}_{l,\lambda}(t).$$

Letting $\lambda\to 0$ gives Spivey's recurrence relation for Bell polynomials (see (10), (12)):

$$\phi_{n+m}(t)=\sum_{k=0}^{m}\sum_{l=0}^{n}{m \brace k}\binom{n}{l}t^kk^{n-l}\phi_l(t),$$

where m,n are nonnegative integers.

From (14), we note that

$$(19)\qquad \sum_{m,n=0}^{\infty} F_{n+m,\lambda}(t)\frac{x^n}{n!}\frac{y^m}{m!} = \sum_{m=0}^{\infty}\left(\frac{d}{dx}\right)^m \sum_{n=0}^{\infty} F_{n,\lambda}(t)\frac{x^n}{n!}\frac{y^m}{m!}$$
$$= \sum_{n=0}^{\infty}\frac{F_{n,\lambda}(t)}{n!}\sum_{m=0}^{\infty}\left(\frac{d}{dx}\right)^m x^n\frac{y^m}{m!} = \sum_{n=0}^{\infty}\frac{F_{n,\lambda}(t)}{n!}\sum_{m=0}^{n}\binom{n}{m}x^{n-m}y^m$$
$$= \sum_{n=0}^{\infty}\frac{F_{n,\lambda}(t)}{n!}(x+y)^n = \frac{1}{1-t(e_\lambda(x+y)-1)}.$$

On the other hand, by (14), we get

$$(20)\qquad \frac{1}{1-t(e_\lambda(x+y)-1)} = \frac{1}{1-t\Big(e_\lambda(x)e_\lambda\big(\frac{y}{1+\lambda x}\big)-1\Big)}$$
$$= \frac{1}{1-t\big(e_\lambda(x)-1\big)-te_\lambda(x)\Big(e_\lambda\big(\frac{y}{1+\lambda x}\big)-1\Big)}$$
$$= \frac{1}{1-t(e_\lambda(x)-1)}\frac{1}{1-\dfrac{te_\lambda(x)\Big(e_\lambda\big(\dfrac{y}{1+\lambda x}\big)-1\Big)}{1-t(e_\lambda(x)-1)}}$$
$$= \frac{1}{1-t(e_\lambda(x)-1)}\sum_{k=0}^{\infty}t^k\left(\frac{1}{1-t(e_\lambda(x)-1)}\right)^k e_\lambda^k(x)\left(e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right)^k$$
$$= \frac{1}{1-t(e_\lambda(x)-1)}\sum_{k=0}^{\infty}\frac{t^k k! e_\lambda^k(x)}{(1-t(e_\lambda(x)-1))^k}\frac{1}{k!}\left(e_\lambda\left(\frac{y}{1+\lambda x}\right)-1\right)^k$$
$$= \frac{1}{1-t(e_\lambda(x)-1)}\sum_{k=0}^{\infty}\frac{t^k k! e_\lambda^k(x)}{(1-t(e_\lambda(x)-1))^k}\sum_{m=k}^{\infty}\left\{\begin{matrix} m \\ k \end{matrix}\right\}_\lambda \frac{y^m e^{-m\lambda}(x)}{m!}$$
$$= \sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{k=0}^{m}t^k k!\left\{\begin{matrix} m \\ k \end{matrix}\right\}_\lambda\left(\frac{1}{1-t(e_\lambda(x)-1)}\right)^k e_\lambda^{k-m\lambda}(x)\frac{1}{1-t(e_\lambda(x)-1)}$$
$$= \sum_{m=0}^{\infty}\frac{y^m}{m!}\sum_{k=0}^{m}t^k k!\left\{\begin{matrix} m \\ k \end{matrix}\right\}_\lambda\sum_{j=0}^{\infty}F_{j,\lambda}^{(k)}(t,k-m\lambda)\frac{x^j}{j!}\sum_{l=0}^{\infty}F_{l,\lambda}(t)\frac{x^l}{l!}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{y^m}{m!}\frac{x^n}{n!}\left(\sum_{l=0}^{n}\sum_{k=0}^{m}t^k k!\left\{\begin{matrix} m \\ k \end{matrix}\right\}_\lambda\binom{n}{l}F_{l,\lambda}(t)F_{n-l,\lambda}^{(k)}(t,k-m\lambda)\right).$$

Therefore, by (19) and (20), we obtain the following theorem.

Theorem 2.3. *For $m,n\geq 0$, we have*

$$F_{n+m,\lambda}(t) = \sum_{k=0}^{m}\sum_{l=0}^{n}k!\left\{\begin{matrix} m \\ k \end{matrix}\right\}_\lambda\binom{n}{l}t^k F_{n-l,\lambda}^{(k)}(t,k-m\lambda)F_{l,\lambda}(t).$$

Letting $\lambda\to 0$, we obtain the following identity (see (13)):

$$F_{n+m}(t) = \sum_{k=0}^{m}\sum_{l=0}^{n}k!\left\{\begin{matrix} m \\ k \end{matrix}\right\}\binom{n}{l}t^k F_{n-l}^{(k)}(t,k)F_l(t).$$

3. Conclusion

Recent research has extensively explored degenerate versions of special polynomials, numbers, and functions, including gamma functions and even umbral calculus. These explorations have utilized a wide range of tools, such as combinatorial methods, generating functions, p-adic analysis, probability theory, quantum mechanics, and operator theory.

In our study, we focused on degenerate versions of the Bell and Fubini polynomials, specifically the fully degenerate Bell polynomials $\mathrm{Bel}_{n,\lambda}(x)$ and the degenerate Fubini polynomials $F_{n,\lambda}(x)$. We successfully derived Spivey-type recurrence relations for these polynomials by employing generating functions.

Our future work will continue to investigate a variety of degenerate versions of special numbers, polynomials, and functions, with the aim of discovering their applications across physics, science, engineering, and mathematics.

Acknowledgements. The work reported in this paper was conducted during the sabbatical year of Kwangwoon University in 2025.

References

[1] Aydin, M. S.; Acikgoz, M.; Araci, S. *A new construction on the degenerate Hurwitz-zeta function associated with certain applications,* Proc. Jangjeon Math. Soc. **25** (2022), no. 2, 195-203.

[2] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51-88.

[3] Comtet, L. *Advanced combinatorics. The art of finite and infinite expansions,* Revised and enlarged edition, D. Reidel Publishing Co., Dordrecht, 1974.

[4] Corcino, R.B.; Celeste, R.O.; Gonzales, K.J.M. *Rook theoretic proofs of some identities related to Spivey's Bell number formula,* arXiv:1410.0742v4 [math.CO], 2016.

[5] Dolgy, D. V.; Kim, D. S.; Kim, T.; Kwon, J. *On fully degenerate Bell numbers and polynomials,* Filomat **34** (2020), no. 2, 507-514.

[6] Katriel, J. *On a generalized recurrence for Bell numbers,* J. Integer Seq. **11** (2008), no. 3, Article 08.3.8, 4 pp.

[7] Kilar, N. *Integral representations and formulas for the unified and modified presentation of Fubini numbers and polynomials,* Proc. Jangjeon Math. Soc. **28** (2025), no. 1, 85-95.

[8] Kim, D. S.; Bu, S.; Lee, H.; Khalil, M.; Kim, T. *Spivey's type recurrence relation for Lah-Bell polynomials,* Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, paper no. 2547873.

[9] Kim, D. S.; Jang, G.-W.; Kwon, H.-I.; Kim, T. *Two variable higher-order degenerate Fubini polynomials,* Proc. Jangjeon Math. Soc. **21** (2018), no. 1, 5-22.

[10] Kim, D. S.; Kim, T. *Moment representations of fully degenerate Bernoulli and degenerate Euler polynomials,* Russ. J. Math. Phys. **31** (2024), no. 4, 682-690.

[11] Kim, T.; Kim, D. S. *Some results on degenerate Fubini and degenerate Bell polynomials,* Appl. Anal. Discrete Math. **17** (2023), no. 2, 548-560.

[12] Kim, T.; Kim, D. S. *Spivey-type recurrence relations for degenerate Bell and Dowling polynomials,* Russ. J. Math. Phys. **32** (2025), no. 2, 288-296.

[13] Kim, T.; Kim, D. S. *Generalization of Spivey's recurrence relation,* Russ. J. Math. Phys. **31** (2024), no. 2, 218-226.

[14] Kim, T.; Kim, D. S. *Probabilistic generalization of Spivey-type relation for degenerate Bell polynomials,* arXiv:2508.17228v1 [math.NT], 2025.

[15] Kim, T.; Kim, D. S. *Recurrence relations for degenerate Bell and Dowling polynomials Dowling polynomials via Boson operators,* Comput. Math. Math. Phys. **65** (2025), no. 9, 2087-2096.

[16] Kim, T.; Kim, D. S. *Spivey's type recurrence relation for degenerate Bell polynomials,* Proc. Jangjeon Math. Soc. **28** (2025), no. 4,393-400.

[17] Privault, N. *Generalized Bell polynomials and the combinatorics of Poisson central moments,* Electron. J. Combin. **18** (2011), # P54.

[18] Roman, S. *The umbral calculus,* Pure and Applied Mathematics 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.

[19] Shablya, Y.; Polyuga, V.; Kruchinin, D. *Combinatorial generation algorithms for discrete structures associated with the Fubini numbers,* Proc. Jangjeon Math. Soc. **28** (2025), no. 1, 53-71.

[20] Simsek, Y. *Identities and relations related to combinatorial numbers and polynomials,* Proc. Jangjeon Math. Soc. **20** (2017), no. 1, 127-135.

[21] Simsek, Y.; Kilar, N. *Generating functions for the Fubini type polynomials and their applications. Exploring mathematical analysis, approximation theory, and optimization-270 years since A.-M. Legendre's birth, 279-380*, Springer Optim. Appl., 207, Springer, Cham, [2023], 2023.

[22] Spivey, M. Z. *A generalized recurrence for Bell numbers,* J. Integer Seq. **11** (2008), no. 2, Article 08.2.5, 3 pp.

[23] Srivastava, H. M.; Srivastava, R.; Muhyi, A.; Yasmin, G.; Islahi, H.; Araci, S. *Construction of a new family of Fubini-type polynomials and its applications,* Adv. Difference Equ. 2021, Paper No. 36, 25 pp.

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA

Email address: `tkkim@kw.ac.kr`

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KOREA

Email address: `dskim@sogang.ac.kr`

KWANGWOON GLOBAL EDUCATION CENTER, KWANGWOON UNIVERSITY, SEOUL 139-7012, REPUBLIC OF KOREA

Email address: `d_dol@mail.ru`