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LAPLACE-RUNGE-LENZ VECTOR VIA NOETHER'S THEOREM

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ABSTRACT. If the action $S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$ is invariant under the infinitesimal transformation $\tilde{t} = t + \epsilon\tau(q, t)$, $\tilde{q_r} = q_r + \epsilon\xi_r(q, t)$, $r = 1, \ldots, n$, with $\epsilon = constant \ll 1$, then the Noether's theorem allows construct the corresponding conserved quantity. The Lanczos technique employs $\epsilon = q_{n+1}$ as a new degree of freedom, thus the Euler-Lagrange equation for this new variable gives the Noether's constant of motion. The Kepler problem has a dynamical symmetry, then here we show that the conservation of the Laplace-Runge-Lenz vector is immediate if we apply the Noether's theorem, in the Lanczos approach, to this symmetry.

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1. INTRODUCTION

In the functional (the concept of action was proposed by Leibnitz [1]) $S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$ we apply the infinitesimal

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transformation ($\epsilon = constant \ll 1$):

$$\tilde{t} = t + \epsilon\tau(q,t), \quad \tilde{q_r} = q_r + \epsilon\xi_r(q,t), \quad r = 1,\ldots,n \tag{1}$$

that is:

$$\tilde{S} = \int_{\tilde{t_1}}^{\tilde{t_2}} L\left(\tilde{q}, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t}\right) d\tilde{t}, \tag{2}$$

then we say that the action is invariant if:

$$\tilde{S} = S + \epsilon \int_{t_1}^{t_2} \frac{d}{dt} Q(q,t) dt, \tag{3}$$

hence the Euler-Lagrange equations [2–5] corresponding to the variational principle $\delta S = 0$:

$$E_r \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tilde{q_r}}}\right) - \frac{\partial L}{\partial q_r} = 0, \quad r = 1,\ldots,n \tag{4}$$

remain intact. Noether [6] studied the case $Q = 0$, and she suggested [7, 8] to Bessel-Hagen [9] the analysis of (3) with $Q \neq 0$ [10].

Therefore, we have a symmetry up to divergence and Noether [6,9–13] proved the existence of the Rund-Trautman identity [11, 12, 14, 15]:

$$\frac{\partial L}{\partial q_r}\xi_r + \frac{\partial L}{\partial \dot{q_r}}\dot{\xi_r} + \frac{\partial L}{\partial t}\tau - \left(\frac{\partial L}{\partial \dot{q_r}}\dot{q_r} - L\right)\dot{\tau} - \frac{dQ}{dt} = 0, \tag{5}$$

which can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_r}}\xi_r - H\tau - Q\right) = \left(\xi_r - \dot{q_r}\tau\right) E_r, \quad H = \frac{\partial L}{\partial \dot{q_c}}\dot{q_c} - L. \tag{6}$$

In (5) and (6) we use the convention of Dedekind [12, 13]-Einstein because we sum over repeated indices. The Rund-Trautman identity offers a more efficient test of invariance [12]. If in (6) we employ the Euler-Lagrange equations (4) we deduce the constant of motion associated to (1):

$$\frac{\partial L}{\partial \dot{q_r}}\xi_r - H\tau - Q = Constant, \tag{7}$$

hence, we have a connection between symmetries and conservation laws [7, 11, 12, 14, 18–21].

In Sec. 2 we exhibit the Lanczos technique [2, 22, 25, 32, 33] to obtain the Noether's conserved quantity (7) as the Euler-Lagrange equation for the parameter $\epsilon(t)$. In Sec. 3 we consider the Kepler problem [26, 27] and we apply this Lanczos method to its dynamical symmetry [[5], pages 99-100] to obtain the conservation of the Laplace-Runge-Lenz vector [28–31].

2. Lanczos approach to Noether's theorem

Lanczos [2, 22] applies the infinitesimal transformation (1) (with $\epsilon = constant$) to the action (2) and uses expansion of Taylor up to first order in ϵ, thus:

$$\tilde{S} = S + \epsilon \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_r} \xi_r + \frac{\partial L}{\partial \dot{q_r}} \dot{\xi_r} + \frac{\partial L}{\partial t} \tau - H \dot{\tau} \right) dt, \tag{8}$$

hence this integrand is equal to $\frac{dQ}{dt}$, in harmony with the Rund-Trautman identity (5).

Now Lanczos proposes to employ (1) into (2) but considering that ϵ is a function, therefore up to 1th order in ϵ:

$$\tilde{S} = \int_{t_1}^{t_2} \left[L + \epsilon \frac{dQ}{dt} + \dot{\epsilon} \left(\frac{\partial L}{\partial \dot{q_r}} \xi_r - H\tau \right) \right] dt = \int_{t_1}^{t_2} \tilde{L} dt, \tag{9}$$

and he accepts that ϵ is a new degree of freedom with its corresponding Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial \dot{\epsilon}} - \frac{\partial \tilde{L}}{\partial \epsilon} \right) = 0 \tag{10}$$

It is clear that:

$$\frac{\partial \tilde{L}}{\partial \dot{\epsilon}} = \frac{\partial L}{\partial \dot{q_r}} \xi_r - H\tau, \quad \frac{\partial \tilde{L}}{\partial \epsilon} = \frac{dQ}{dt},$$

therefore (10) implies (7). In other words, if the parameter of the symmetry is considered as an additional degree of freedom of the variational principle, then its Euler-Lagrange equation gives the Noether's constant of motion. We comment that Neuenschwander [19] obtains the conserved quantity (7) for the case $Q = 0$.

The works [32,33] have applications of this Lanczos technique to some singular Lagrangians employed in [13,34–36]. In [2,37] and [38] the Lanczos method is applied to electromagnetic and gravitational fields, respectively.

3. Dynamical symmetry in the Kepler problem

Here we consider the Lagrangian for the Kepler problem:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{k}{r}, \quad r = \sqrt{x^2 + y^2}, \quad \vec{r} = x\hat{i} + y\hat{j}, \tag{11}$$

with the conservation of the angular momentum whose magnitude is given by:

$$l = m\left(x\dot{y} - y\dot{x}\right) = mr^2\dot{\theta}. \tag{12}$$

Now in L we apply the following transformation [[5], pages 99-100]:

$$\tilde{x} = x + \epsilon_1, \quad \tilde{y} = y + \epsilon_2, \quad \tilde{t} = t, \tag{13}$$

where ϵ_1 and ϵ_2 are constants $\ll 1$, to obtain:

$$\begin{aligned} \tilde{L} - L &= -\epsilon_1 \frac{kx}{r^3} - \epsilon_2 \frac{ky}{r^3} \\ &= -\epsilon_1 \frac{k}{r^2}\cos\theta - \epsilon_2 \frac{k}{r_2}\sin\theta \\ &= -\epsilon_1 \frac{mk}{l}\dot{\theta}\cos\theta - \epsilon_2 \frac{mk}{l}\dot{\theta}\sin\theta, \end{aligned}$$

that is:

$$\tilde{L} = L + \frac{d}{dt}\left[\frac{mk}{l}\left(-\epsilon_1 \sin\theta + \epsilon_2 \cos\theta\right)\right], \tag{14}$$

hence (13) is a dynamical symmetry of L [5], and therefore we shall use the Noether's theorem [in the Lanczos approach] to deduce the conservation law associated with (13).

The Lanczos technique asks to employ in (11) the transformation (13) with $\epsilon_1(t)$ and $\epsilon_2(t)$ as new degrees of freedom, then:

$$\tilde{L} - L = \dot{\epsilon_1} m\dot{x} - \epsilon_1 \frac{kx}{r^3} + \dot{\epsilon_2} m\dot{y} - \epsilon_2 \frac{ky}{r^3}, \tag{15}$$

and the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\epsilon_j}}\right) - \frac{\partial \tilde{L}}{\partial \epsilon_j}, \quad j = 1, 2, \tag{16}$$

imply the expressions:

$$\frac{d}{dt}(m\dot{x}) + \frac{kx}{r^3} = \frac{d}{dt}\left(m\dot{x} + \frac{mk}{l}\sin\theta\right) = 0,$$
$$\frac{d}{dt}(m\dot{y}) + \frac{ky}{r^3} = \frac{d}{dt}\left(m\dot{y} + \frac{mk}{l}\cos\theta\right) = 0,$$

which generate the following two constants of motion:

$$ml\dot{x} + mk\sin\theta = -c_2, \quad ml\dot{y} - mk\cos\theta = c_1. \tag{17}$$

It is easy see that (17) express the conservation of the Laplace-Runge-Lenz vector [4, 26, 39], in fact:

$$\vec{p}x\vec{L} = m\left(\dot{x}\hat{i} + \dot{y}\hat{j}\right) xl\hat{k} = ml\left(\dot{y}\hat{i} - \dot{x}\hat{j}\right),$$
$$-mk\frac{\vec{r}}{r} = -mk\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right),$$

therefore:

$$\begin{aligned} \vec{p}x\vec{L} - mk\frac{\vec{r}}{r} &= (ml\dot{y} - mk\cos\theta)\,\hat{i} - (ml\dot{x} + mk\sin\theta)\,\hat{j} \\ &= c_1\hat{i} + c_2\hat{j}. \end{aligned} \tag{18}$$

Thus, the Lanczos method gives a simple process to apply the Noether's theorem [40–43], in this case, it was possible to determine the conservation law associated with the dynamical symmetry (14) for the Kepler problem.

Remark 1. *If in (8) we use* τ *and* ξ_r *as new degrees of freedom, then the corresponding Euler-Lagrange equations imply the equations of motion (4) and the known relation* $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$.

Remark 2. *It is clear that the process shown here to obtain the conservation of the Laplace-Runge-Lenz vector is based on the premise that (14) does not alter the equations of motion generated by the Lagrangian (11).*

Remark 3. *Regarding the Lanczos method [44]: The introduction of* ϵ *as an added dynamical variable is not an inherent necessity because the equation we obtain by varying with respect to* ϵ *cannot be a new information. It yields something that is a consequence of the equations of motion and is obtainable by the proper manipulations with these equations. But Noether's principle gives the deeper significance for the existence of conservation laws.*

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