Advanced Studies in Contemporary Mathematics
36 (2026), No. 1, pp. 45 - 52

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2026.36.1.007

AN INFORMAL OVERVIEW OF EULER'S CONSTANT AND THE GAMMA FUNCTION

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ABSTRACT. We recall several known properties of Euler's constant γ (also known as the Euler–Mascheroni constant) and survey its relation to harmonic numbers, logarithmic series, and the Gamma function. Although γ has been studied for centuries, its arithmetic nature remains unknown: it is not known whether γ is rational, irrational, or transcendental.

1. INTRODUCTION

Euler's constant γ is defined as

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right). \tag{1}$$

It was first identified by Euler in 1734 (he used the symbol C), and later Mascheroni introduced the symbol γ. Despite being one of the most common constants in mathematics, the nature of γ remains mysterious. No one knows whether it is irrational or transcendental. Hardy once remarked that he would have offered his Savilian Chair to anyone who could prove γ irrational, and Hilbert called the problem "unapproachable." By continued fraction methods, Brent and McMillan showed that any rational expression $\gamma = p/q$ would require $q > 10^{15000}$.

The famous English mathematician G.H. Hardy is alleged to have offered to give up his Savilian Chair at Oxford to anyone who proved gamma to be irrational, although no written reference for this quote seems to be known. Hilbert mentioned the irrationality of gamma as an unsolved problem that seems "unapproachable" and in front of which mathematicians stand helpless. Conway and Guy (1996) are "prepared to bet that it is transcendental," although they do not expect a proof to be achieved within their lifetimes.

2. EULER'S COMPUTATION AND SERIES REPRESENTATIONS

Theorem 2.1 (Mercator's Formula)**.**

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots = \sum_{k\geq 1}(-1)^{k+1}\frac{x^k}{k}.$$

2010 *Mathematics Subject Classification.* 11J72; 11J81;
Key words and phrases. irrationality, transcendence, algebraic independence.

Euler replaces x by $1/m$ with $m = 1,2,3,4,\cdots$ in Mercator's formula for $\log(1+x)$:

$$\log 2 = \frac{1}{1} - \frac{1}{2}\left(\frac{1}{1}\right)^2 + \frac{1}{3}\left(\frac{1}{1}\right)^2 - \cdots$$
$$\log\frac{3}{2} = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^2 - \cdots$$
$$\log\frac{4}{3} = \frac{1}{3} - \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{3}\left(\frac{1}{3}\right)^2 - \cdots$$
$$\log\frac{5}{4} = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{4}\right)^2 - \cdots$$

Adding the first n terms of this sequence of formulae (telescoping series), Euler finds

$$\log(n+1) = H_n - \frac{1}{2}H_{n,2} + \frac{1}{3}H_{n,3} - \frac{1}{4}H_{n,4} + \cdots.$$

We have

$$\log(n+1) = H_n - \frac{1}{2}H_{n,2} + \frac{1}{3}H_{n,3} - \cdots$$

with

$$H_{n,m} = \sum_{j=1}^{n} \frac{1}{j^m} \qquad \text{for } n \geq 1 \text{ and } m \geq 1.$$

Hence, $H_{n,1} = H_n$ and, for $m \geq 2$,

$$\lim_{n\to\infty} H_{n,m} = \zeta(m).$$

In the formula

$$H_n - \log(n+1) = \frac{1}{2}H_{n,2} - \frac{1}{3}H_{n,3} + \cdots,$$

when n tends to infinity. The right hand side tends to

$$\sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m}$$

which is the sum of an alternating series with a decreasing general term. Hence the left hand side has a limit, which is γ.

Lorenzo Mascheroni (1792) produced 32 decimals the first 19 of them are correct ; the first 15 decimal were already found by Euler in 1755 and then in 1765.
Von Soldner (1809) : 22 decimals

$$\gamma = 0.577\ 215\ 664\ 901\ 532\ 860\ 606\ 5$$

C.F. Gauss, F.G.B Nicolai : 40 decimals

$$\gamma = 0.577\ 215\ 664\ 901\ 532\ 860\ 6\underline{1}8\ 112\ 090\ 082\ 39$$

1872 :	J.W.L. Glaisher	100 decimals
1878 :	J.C. Adams	263 decimals
1952 :	J.W. Wrench Jr	328 decimals
1962 :	D. Knuth	127 decimals
1963 :	D.W. Sweeney	3566 decimals
9164 :	W.A. Beyer and M.S. Waterman	7114 decimals (4879 correct)
1977 :	R.P. Brent	20 7000 decimals
1980 :	R.P. Brent and E.M. McMillan	30 000 decimals
2010 :	Yee	29 844 489 544 decimals

Definition 2.2 (Gamma Function)**.**

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \cdot \frac{dz}{t} = e^{-\gamma z} \frac{1}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.$$

Γ meets γ :

$$\Gamma'(1) = -\gamma = \int_0^\infty e^{-x} \log x dx.$$

Theorem 2.3 (Euler's formuale)**.**

$$\gamma = \int_0^\infty \left(\frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t}}{t} \right) dt, \quad \gamma = \int_0^1 \left(\frac{1}{1-z} + \frac{1}{\log z} \right) dz.$$

$$\gamma = \sum_{n=2}^{\infty} \frac{n-1}{n} (\zeta(n) - 1), \quad \gamma = \frac{3}{4} - \frac{1}{2}\log 2 + \sum_{k=1}^{\infty} \left(1 - \frac{1}{2k+1}\right)(\zeta(2k+1) - 1).$$

Quoting Euler (1768)

"$\mathcal{O}$ = 0.5772156649015325 qui numerus eo maiori attentione dignus videtur, quod eum, cum olim in hac investigatione multum studii consumsissem, nullo modo ad cognitum quantitatum genus reducere valui."

This number seems also the more noteworthy because
even though I have spent much effort in investigating it,
I have not been able to reduce it to a known kind of quantity.

"Manet ergo quaestio magni momenti, cujusdam indolis sit numerus iste $\mathcal{O}$ et ad quodnam genus quantitatum sit referendus"

Therefore the question remains of great moment,
of what character the number $\mathcal{O}$ is and among
what species of quantities it can be classified.

3. Arithmetic and Open Problems

Although γ appears in many analytic contexts, basic number-theoretic questions remain unanswered. We do not know whether γ is rational, irrational, or transcendental. There are also no known linear relations between γ and other constants like π or $\log 2$. Various approaches using continued fractions, integrals, and series have been explored, but without success.

Conjecture 3.1. *Euler constant is irrational.*

If $\gamma = p/q$, then $q > 10^{15\ 000}$.
Contiued fraction expansion : 30 000 first terms have been computed.

$$\gamma = [0,1,1,2,1,2,1,4,3,13,5,1,1,8,1,2,4,1,1,40,1,11,3,\cdots]$$

Theorem 3.2 (Hendrik W. Lenstra (1977))**.** *At least one of the two numbers* γ, e^{γ} *is transcendental.*

Ph.D. thesis, Mathematisch Centrum, Universiteit van Amsterdam, 1977.

Conjecture 3.3. *The number*

$$e^{\gamma} = 1.\ 781\ 072\ 417\ 990\ 197\ 985\ 236\ 504\ 103\ 107\ 179\cdots$$

is irrational.

Also, if $e^{\gamma} = p/q$, then $q > 10^{15\ 000}$.

Conjecture 3.4. *The Euler constant is not a period in the sense of Kontsevich and Zagier*

So, what is a period?
Definition 3.5 (Periods : Maxime Kontsevich and Don Zagier)**.**
*A **period** is a complex number whose real and imaginary parts are values of absolutely convergent integrals of ration functions with rational coefficients, over domains in* $\mathbb{R}^n$ *given by polynomial inequalities with rational coefficients*

Example 1.

$$\sqrt{2} = \int_{2x^2\le 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1<x<2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2+y^2\le 1} dxdy,$$

A product of period is (sub-algebra of $\mathbb{C}$*), but* $1/\pi$ *is expected not to be a period.*

3.1. **Euler Constant and Arithmetic Functions.** The function sum of divisors

$$\sigma(n) = \sum_{d|n} d.$$

Theorem 3.6 (T.H. Grönwall (1913))**.**

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n\log n} = e^{\gamma}$$

Theorem 3.7 (Guy Robin (1984))**.** *Riemann hypothesis is equivalent to*

$$\sigma(n) < e^{\gamma} n \log\log n$$

for all $n \ge 5041$

More recent results

Theorem 3.8 (Jeffrey C. Lagarias (2001))**.** *Riemann hypothesis is equivalent to*

$$\sigma(n) < H_n + \exp(H_n)\log(H_n) \quad \textit{for all } n > 1.$$

The function *number of divisors* $d(n)$ is defined for n a positive integer by

$$d(n) = \sum_{d|n} 1$$

Theorem 3.9 (Dirichlet (1849))**.**

$$\sum_{k=1}^{n} = n\log n + (2\gamma - 1)n + \mathcal{O}(\sqrt{n}).$$

Let θ be the infimum of the exponents β for which

$$\sum_{k=1}^{n} d(k) = n\log n + (2\gamma - 1)n + \mathcal{O}(n^{\beta}).$$

Dirichlet's Theorem yields $\theta \le \frac{1}{2}$.

This estimate was improved by Voronoi in 1903 : $\theta \le \frac{1}{3}$,

and van der Corput in 1992 : $\theta \le \frac{33}{100}$.

In 1915, Hardy and Landau proved $\theta \ge \frac{1}{4}$.

The exact value of θ is not yet known.

3.2. **The Euler-Gompertz Constant.**

Definition 3.10.

$$\delta = \int_0^1 \frac{dt}{1 - \log t} = \int_0^{\infty} e^{-t} \log(t+1)dt = 0! - 1! + 2! - 3! + 4! - 5! + \cdots$$
$$= 0.596\ 347\ 362\ 323\ 194\ 074\ 341\ 078\ 499\ 369\ 279\ 376\ 074\ 177\cdots$$

Theorem 3.11 (A.B. Shidlovskii (1959))**.** *One at least of the two numbers* γ, δ *is irrational.*

4. The sequence T_k

For $k \ge 0$, set

$$T_k = \sum_{n \le 2^k} d(n).$$

Consider the binary expansion

$$T_k = \sum_{i=0}^{v_k} a_i 2^i.$$

If $a_{l+i} = 0$ for $0 \le i \le L - 1$, we say that the *binary expansion of* T_k *has a gap of length at least* L *starting with* l.

Proposition 4.1. *Assume that for infinitely many positive* k*, there exist* l *and* L *satisfying*

$$2 + \frac{3\log k}{\log 2} \le k - l \le L$$

and that the binary expansion of T_k *has a gap of length at least* L *starting at* l*. The Euler's constant is irrational.*
In other terms, one at least of the following two properties is true :

(i) the binary expansion of T_k does not have extremely long gaps ;
(ii) the Euler constant is irrational.
One expects that both properties are true!

Proof. The relation

$$\sum_{j=1}^{n} d(j) = n\log n + (2\gamma - 1)n + \mathcal{O}(n^{\theta})$$

for $n = 2^k$ and $\theta = \frac{1}{2}$ can be written

$$T_k = 2^k k \log 2 + 2^k(2\gamma - 1) + \mathcal{O}(2^{k/2}).$$

To say that the binary expansion of T_k has a gap of length at least l starting at l terms

$$T_k = \sum_{i=l+L}^{\nu_k} a_i 2^i + \sum_{i=0}^{l-1} a_i 2^i.$$

Setting

$$b = 1 + \sum_{i=l+L}^{\nu_k} a_i 2^{i-k}$$

and dividing by 2^k yields

$$|k\log 2 + 2\gamma + b| < 2^{l-k} + c2^{-k/2}$$

with a constant $c > 0$.
Using the irrationality measure for $\log 2$:

$$\left|\log 2 - \frac{p}{q}\right| \geq \frac{1}{q^{3,58}}$$

which is valid for large q, we deduce, under the assumptions of the proposition, that the number γ is irrational. □

5. Some Results

Theorem 5.1 (K. Mahler (1968))**.** *The number*

$$\frac{\pi}{2}\frac{Y_0(2)}{J_0(2)} - \gamma$$

is transcendental.

The Bessel functions of first and second kind

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n},$$

$$Y_0(z) = \frac{\pi}{2}\left(\log\left(\frac{z}{2}\right) + \gamma\right)J_0(z) + \frac{\pi}{2}\left(\sum_{n=0}^{\infty}(-1)^n \frac{H_n}{(n!)^2}\left(\frac{z^2}{4}\right)^n\right).$$

Theorem 5.2 (T. Rivoal, Kh. Pilehrood, T. Pilehrood (2012))**.**
At least one of the two numbers γ, δ are transcendental.

Theorem 5.3 (T. Rivoal (2012))**.** *Simultaneous rational approximations for the Euler constant and for the Euler-Gompertz constant.*

$$\left|\gamma - \frac{p}{q}\right| + \left|\delta - \frac{r}{q}\right| > \frac{C(\varepsilon)}{q^{3+\varepsilon}}.$$

Method of Mahler : Two of the numbers e, γ, δ are algebraically independent.

Theorem 5.4 (Peter Bundschuh)**.** *For $p/q \in \mathbb{Q}/\mathbb{Z}$, the number*

$$\frac{\Gamma'(p/q)}{\Gamma(p/q)} + \gamma$$

is transcendental.

Definition 5.5 (Digamma Function)**.** *For $z \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$*

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right)$$

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}.$$

Special Values of Gamma of the Digamma Function

$$\psi(1) = -\gamma = -04,577\ 215\cdots, \qquad \psi\left(\frac{1}{2}\right) = -2\log 2 - \gamma = -1.963\ 510\cdots$$

$$\psi\left(\frac{1}{4}\right) = -\frac{\pi}{2} - 3\log 2 - \gamma = -4,227\ 453\cdots, \quad \psi\left(\frac{3}{4}\right) = \frac{\pi}{2} - 3\log 2 = \gamma = -1,085\ 860\cdots$$

Hence

$$\psi(1) + \psi(1/4) - 3\psi(1/2) + \psi(3/4) = 0$$

Conjecture 5.6. *Let K be a number field over which the q-th cyclotomic polynomial is irreducible. Then the $\varphi(q)$ numbers $\psi(a/q)$ with $1 \leq a \leq q$ and $(a,q) = 1$ are linearly independent over K.*

Lagrais quotes Kontsevich : the Euler constant is an exponential period :

$$\gamma = \int_0^1 \int_x^1 \frac{e^{-x}}{y} dy dx - \int_1^{\infty} \int_1^x \frac{e^{-x}}{y} dy dy.$$

Rests on

$$-\gamma = \int_0^{\infty} e^{-x} \log x dx.$$

The Euler-Gompertz constant is an exponential period :

$$\delta = \int_0^{\infty} \frac{e^{-t}}{1+t} dt,$$

One conjectures that it is not a period.

6. Conclusion

In summary, Euler's constant γ is defined by the limiting difference between the harmonic series and the natural logarithm. It plays a central role in analysis and number theory, appearing in series, integrals, and the Gamma function. Despite centuries of study, its arithmetic nature remains a mystery: no proof exists that γ is irrational or transcendental. As Hilbert called it, the constant remains "unapproachable."

Euler's constant connects analysis and arithmetic, history and modern research, and continues to inspire investigation. We hope this informal overview provides an accessible starting point for those curious about the properties and mysteries of γ.

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