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# Numerical Conductivity of Red Blood Cells to Arbitrary Viscosity Coefficient Functions

A Mathematical Modeling and Computation with Ruffa-Toni's Definite Integral Techniques

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**Abstract**

In this article theoretically, we calculate numerical conductivity of red blood cells in a blood vessel choosing arbitrary viscosity coefficient functions and on applying MATHEMATICA algorithms. For this purpose, first we construct a mathematical model for computation of the conductivity of red blood cell on applying the lubrication theory of a fluid (blood) in a vessel containing red blood cells, white blood cells and plasma. We derive a definite integral formula involving a viscosity coefficient function for conductivity of a red blood cell. On using Ruffa-Toni's definite integral techniques, numerical conductivity of red blood cells is computed. We also construct their matrix representations. This theory may be used in various medical scientific studies.

**2020 Mathematics Subject Classification:** 37M99, 37M20, 65M10.

**Keywords:** Numerical conductivity of red blood cells, viscosity coefficient function, lubrication theory of fluid, Ruffa-Toni's definite integral techniques, matrix representation of conductivity of RBCs.

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# 1 Introduction

Pertaining to the preliminary historical development of study of conductivity of red blood cells, we describe certain models and its importance in medical sciences done so far by various authors. Hoetink et al. [8] developed a model on using the Maxwell-Fricke theory for computing the blood conductivity of a diluted suspension of ellipsoidal particles, which simulates blood in a steady flow in a rigid vessel. On considering $RBC$ s in form of ellipsoidal particles, they showed that, due to high values of shear stress, the RBCs deform and orient such that one of their long axes is parallel to the streamlines of the blood flow. By this configuration a substantial change in the electric current path through the blood [8, 10] happens. In particular, the electrical conductivity of blood in the direction of flow increases due to channel-like paths available between the aligned RBCs (see in [9]). On the other hand, Gaw et al. [6] extended the investigation and reported the effects of pulsatile blood flow on electrical conductivity in a rigid vessel. Theoretically and experimentally, it was shown that, when the velocity increases during systole, there is a robust linear relationship between the average velocity and the conductivity of the blood. Similarly, a decrease in impedance is observed when blood velocity decreases.

Figure 1: Red Blood Cells

The orientation and deformation of RBCs are the cause of anisotropic blood conductivity. Several experimental studies have investigated the behavior of RBCs in Couette and Poiseuille flow fields, including Fischer et al. [5], Goldsmith et al. [7], Keller and Skalak [11], Bitbol [4], Schmid-Schönbein and Wells [15]. Above studies suggest that RBCs exhibit two types of motion:

First, unsteady motions like flipping, tumbling, and rolling, in which the biconcave shape of RBCs remains unchanged [19].

Secondly, a steady motion where RBCs deform into ellipsoidal particles via high shear stress. In the steady motion, the RBCs maintain a steady orientation with their membrane circulating around the interior viscous fluid (cytoplasm) [1]. This motion is called tank-treading, where the rotating motion of the membrane transfers the tangential stresses of the flow to the cytoplasm.

The models developed by Hoetink et al. [8] and Gaw et al. [6] inspired several applications for the simulation of bioimpedance signals [17,18], and in the field of aortic dissection [3,2]. Such type of approaches also focused on the analytical and numerical solutions for one-dimensional computations of the electrical conductivity of blood, and none offered the possibility to model and compute the 3-D anisotropic electrical conductivity of blood as a field variable. For example, in the study of Badeli et al. [2] showed the important application of the ICG method in detecting aortic dissection. However, the isotropic assumption imposed by the one dimensional formulation led to ignored changes in the blood conductivity values due to varying flow direction, local flow hemodynamics, and disturbances due to pathology (see in [9]).

Motivated by above work, in this article we calculate numerical conductivity of red blood cells on choosing arbitrary viscosity coefficient functions and on using MATHEMATICA algorithms. We derive a definite integral formula involving a viscosity coefficient function for conductivity of a red blood cell. We apply Ruffa-Toni's definite integral techniques [14] to compute numerical conductivity of red blood cell.

# 2 Motion in Narrow Capillaries from the Stand Point of Lubrication Theory and Computation of Conductivity of Red Blood Cells

Besides of the theories illustrated in the Section 1, we introduce the theory of Lighthill [13] in which a theoretical study of the motion in narrow capillaries from the stand point of lubrication theory is presented. This lubrication theory is concerned with fluid motions in which there is a direct balance between pressure gradients and viscous forces. The inertia plays a totally insignificant role in the balance. Also, the small blood vessels in general are places where inertial effects are totally negligible so that the Reynolds number $R$ is such that $R \ll 1$. According to this theory, within the fluid a centripetal pressure force on the red cell ensemble may continue to act, because interactions between red cells, at given volume concentration, are greatest where they are spinning fastest, that is, where the fluid vorticity is greatest near the wall of vessel. These forces of interaction between cells are also generated in lubricating layers between spinning deformable cells. Applying this theory of lubrication, for the conductivity, $\sigma$, of a red blood cell in an organ is calculated for the viscosity $\mu$, which varies with distance $r$, presented in the form [13]

$$\sigma = \left(2a^2\right)^{-1} \int_0^a (\mu(r))^{-1} r^3 dr, \tag{2.1}$$

where, $a$ is a radius of red blood cell.

Recently in the Conference,IC-BDA&C-MM-NS-2025, held on July 28-30, 2025 at School of Computational and Integrative Sciences, JNU New Delhi, India in collaboration of Vijnana Parishad of India, Kumar [12] has presented a mathematical model to formulate the conductivity, $\sigma$, of red blood cell in a vessel on application of above lubrication theory and applying the formula such that for a velocity of the non-Newtonian fluid, $v(r)$, and $\mu(r)$, a viscosity coefficient functions of $r$, a radial distance, then the stress function, $\tau(r)$ acting on the vessel wall, is given by $\tau(r) = -\mu(r)\frac{dv(r)}{dr}$ and derive an equality

$$\int_0^a \{\mu(r)\}^{-1} r^3 dr = -\int_0^a \{\tau(r)\}^{-1} r^3 dv(r) = \rho, \tag{2.2}$$

provided that $a$ is the radius of red blood cell, $\rho$ is an arbitrary constant.
Now on setting $\rho = 2a^2\sigma$ in the relation (2.2), we get

$$\sigma = \left(2a^2\right)^{-1} \int_0^a \{\mu(r)\}^{-1} r^3 dr, \tag{2.3}$$

or

$$\sigma = -\left(2a^2\right)^{-1} \int_0^a \{\tau(r)\}^{-1} r^3 dr. \tag{2.4}$$

On application of formula (2.3), near the standard value of conductivity, $\sigma$, of $RBC$, Kumar [12] investigated various viscosity coefficient functions and obtain the values of conductivity of red blood cells in a vessel.

We improve the formula (2.3) as on setting for arbitrary constant $\rho$ due to (2.2), given by $\rho = \frac{a^2}{4.45268}\sigma$, where $a$ is radius of the red blood cell and $\sigma$ is the conductivity of the fluid.

We find the definite integral formula for the conductivity, $\sigma$, of the red blood cell in the form

$$\sigma = \frac{4.45268}{a^2} \int_0^a \{\mu(r)\}^{-1} r^3 dr. \tag{2.5}$$

The integral (2.5) becomes too useful in obtaining numerical values of the conductivity of a red blood cell of the radius of length $a$.

# 3 Ruffa-Toni's Definite Integral and Computation of Conductivity of the Red Blood Cell

In this section, we consider the definite integral formula due to Ruffa-Toni [14] given by

$$\int_{\alpha}^{\beta} G(t)dt = (\beta - \alpha) \sum_{p=1}^{\infty} 2^{-p} \sum_{n=1}^{2^p-1} (-1)^{n+1} G\left(\alpha + n(\beta - \alpha)2^{-p}\right). \tag{3.1}$$

Then making an appeal to the formulae (2.5) and (3.1) we obtain

$$\sigma = \frac{4.45268}{a} \sum_{p=1}^{\infty} 2^{-p} \sum_{n=1}^{2^p-1} (-1)^{n+1} \frac{(na2^{-p})^3}{\mu(na2^{-p})}. \tag{3.2}$$

Now setting viscosity function $\mu(r)$ for different curves for known finite radius of red blood cell in the formula (3.2), we present following observations in Table 3.1

Table 1: Table 3.1: The conductivity of Red Blood Cell for different viscosity functions

| Serial No. | Viscosity function, $\mu(r)$ | Radius of red blood cell, $a$ | Conductivity $\sigma$ , in Ruffa-Toni's definite integral form | Via *MATHEMATICA*, the value of conductivity, $\sigma$ | The standard value of conductivity of human red blood cells (HRBCs) at 25°C calculated approxima -tely in [16] |
|---|---|---|---|---|---|
| 1. | $\mu(r) = r^2$ | $5 \times 10^{-5}$ | $\sigma_1 = \frac{4.45268}{a} \sum_{p=1}^{\infty} 2^{-p} \sum_{n=1}^{2^p-1} (-1)^{n+1} (na2^{-p})$ | $\sigma_1 = 0.556586$ | 0.5566 SAR/m |
| 2. | $\mu(r) = r^3$ | $5 \times 10^{-5}$ | $\sigma_2 = \frac{4.45268}{a} \sum_{p=1}^{\infty} 2^{-p} \sum_{n=1}^{2^p-1} (-1)^{n+1}$ | $\sigma_2 = 8.9054 \times 10^4$ | |
| 3. | $\mu(r) = r^4$ | $5 \times 10^{-5}$ | $\sigma_3 = \frac{4.45268}{a} \sum_{p=1}^{\infty} 2^{-p} \sum_{n=1}^{2^p-1} \frac{(-1)^{n+1}}{(na2^{-p})}$ | $\sigma_3 = 2.57189625129 \times 10^{10}$ | |

# 4 Algorithms to Calculate the Conductivity of Human Red Blood Cells for Different Viscosity Functions via MATHEMATICA

In this section, we introduce MATHEMATICA coding to calculate the conductivity of human red blood cells via Ruffa-Toni's definite integral for different viscosity functions, $\mu(r)$ as given in above Table 3.1:

**Case 4.1.** When $\mu(r) = r^2, \sigma_1 = \frac{4.45268}{a}\sum_{p=1}^{\infty} 2^{-p}\sum_{n=1}^{2^p-1}(-1)^{n+1}\left(na2^{-p}\right)$,
then via *MATHEMATICA* the numerical value of $\sigma_1$ is found by an algorithm given by
$\mathrm{a} = 5^*10^\wedge - 5$
$\sigma 1 = (4.45268/\mathrm{a})^*\,\mathrm{Sum}\left[2^\wedge - \mathrm{p}^*\,\mathrm{Sum}\left[(-1)^\wedge(\mathrm{n}+1)^*\left(\mathrm{n}^*\mathrm{a}^*2^\wedge - \mathrm{p}\right), \{\mathrm{n}, 1, 2^\wedge(\mathrm{p}-1)\}\right], \{\mathrm{p}, 1, 20\}\right]$
$\mathrm{N}[\sigma 1, 20]$
89053.52

**Case 4.2.** When $\mu(r) = r^3, \sigma_2 = \frac{4.45268}{a}\sum_{p=1}^{\infty} 2^{-p}\sum_{n=1}^{2^p-1}(-1)^{n+1}$, then via MATHEMATICA the numerical value of $\sigma_2$ is found by an algorithm given by
$\mathrm{a} = 5^*10^\wedge - 5$
$\sigma 2 = (4.45268/\mathrm{a})^*\,\mathrm{Sum}\left[2^\wedge - \mathrm{p}^*\,\mathrm{Sum}\left[(-1)^\wedge(\mathrm{n}+1), \{\mathrm{n}, 1, 2^\wedge(\mathrm{p}-1)\}\right], \{\mathrm{p}, 1, 20\}\right]$
$\mathrm{N}[\sigma 1, 20]$ 89053.52

**Case 4.3**. When $\mu(r) = r^4, \sigma_3 = \frac{4.45268}{a}\sum_{p=1}^{\infty} 2^{-p}\sum_{n=1}^{2^p-1}\frac{(-1)^{n+1}}{(na2^{-p})}$, then via MATHEMATICA the numerical value of $\sigma_3$ is found by an algorithm given by
$\mathrm{a} = 5^*10^\wedge - 5$
$\sigma 3 = (4.45268/\mathrm{a})^*\,\mathrm{Sum}\left[2^\wedge - \mathrm{p}^*\,\mathrm{Sum}\left[(-1)^\wedge(\mathrm{n}+1)/\left(\mathrm{n}^*\mathrm{a}^*2^\wedge - \mathrm{p}\right), \{\mathrm{n}, 1, 2^\wedge(\mathrm{p}-1)\}\right], \{\mathrm{p}, 1, 20\}\right]$
$N[\sigma 3, 20]$
$2.5718962512921825 \times 10^{10}$

# 5 Matrix Representation of Red Blood Cells

The Table 3.1 shows that the conductivity formula given by

$$\sigma_1 = \frac{4.45268}{a}\sum_{p=1}^{\infty} 2^{-p}\sum_{n=1}^{2^p-1}(-1)^{n+1}\left(na2^{-p}\right) \tag{5.1}$$

gives the value of conductivity near the standard value $0.5566\mathrm{SAR/m}$. Therefore, we generalize this formula (5.1) for matrix representation of RBCs as in the form

$$\sigma_1 = \frac{\alpha_{ij}}{a}\sum_{p=1}^{\infty} 2^{-p}\sum_{n=1}^{2^p-1}(-1)^{n+1}\left(na2^{-p}\right) \forall i = 1, 2, \ldots, m; j = 1, 2, \ldots, q. \tag{5.2}$$

Here, $\alpha_{ij} \forall i = 1, 2, \ldots, m; j = 1, 2, \ldots, q$ are arbitrary constant near the value 4.45268. In (5.2), considering the constants such that

$$\begin{aligned} &\alpha_{11} = 4.45268, \alpha_{12} = 4.452680001, \alpha_{13} = 4.452680002, \alpha_{21} = 4.452680001, \\ &\alpha_{22} = 4.45268, \alpha_{23} = 4.452680002. \end{aligned} \tag{5.3}$$

Using MATHEMATICA algorithm given in case 4.1 for different constants given in (5.3) and on using the formula (5.2), we obtain the conductivity of various red blood cells of similar radius, given by

$$\begin{aligned} &\sigma_{11} = .55658606160163881516, \sigma_{12} = .55658606172663904754, \\ &\sigma_{13} = .55658606185163939095, \sigma_{21} = .55658606172663904754, \\ &\sigma_{22} = .55658606160163881516, \sigma_{23} = .55658606185163939095. \end{aligned} \tag{5.4}$$

Then, the conductivity of six RBCs given in Figure 1.1 is constructed in the matrix representation as

$$\begin{aligned} &\frac{1}{3}\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{bmatrix}\begin{bmatrix} S(a) & S(a) & S(a) \\ S(a) & S(a) & S(a) \\ S(a) & S(a) & S(a) \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 1.6697581851 & 1.6697581851 & 1.6697581851 \\ 1.6697581851 & 1.6697581851 & 1.6697581851 \end{bmatrix} \\ &= \begin{bmatrix} 0.5565860617 & 0.5565860617 & 0.5565860617 \\ 0.5565860617 & 0.5565860617 & 0.5565860617 \end{bmatrix}. \end{aligned} \tag{5.5}$$

Here in (5.5), $S(a) = \frac{1}{a}\sum_{p=1}^{\infty} 2^{-p} \sum_{n=1}^{2^p-1} (-1)^{n+1} (na2^{-p})$.

On making an appeal to the observations and formulae (5.1) to (5.5), we develop a generalized formula

$$\frac{1}{m}\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \cdots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mm} \end{bmatrix}\begin{bmatrix} S(a_{11}) & \ldots & S(a_{1m}) \\ \vdots & \cdots & \vdots \\ S(a_{m1}) & \cdots & S(a_{mm}) \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & \cdots & \vdots \\ \sigma_{m1} & \cdots & \sigma_{mm} \end{bmatrix} \tag{5.6}$$

provided that $4.45286 \leq \alpha_{jj} \leq 4.45299, S(a_{jj}) = \frac{1}{a_{jj}}\sum_{p=1}^{\infty} 2^{-p} \sum_{n=1}^{2^p-1} (-1)^{n+1} (na_{jj}2^{-p})$, then for all $j = 1, 2, \ldots, m$; by the formula (5.6), we find

$$.5565860660 \leq \sigma_{jj} \leq .5566022492. \tag{5.6}$$

## 6 An Interpretation and Conclusion

An application of Ruffa-Toni's definite integral technique converts the formula into the algorithm to measuring the numerical value of conductivity of red blood cells in the form

$$\sigma_1 = \frac{4.45268}{a_{jj}} \sum_{p=1}^{\infty} 2^{-p} \sum_{n=1}^{2^p-1} (-1)^{n+1} \left(n a_{jj} 2^{-p}\right) \forall j = 1, 2, \ldots, m. \tag{6.1}$$

In this case, the viscosity function is $\mu(r) = r^2$ and the calculated value of conductivity of red blood cell matches to its standard value (see in [16]). Therefore in our investigation, if we observe the viscosity function $\mu(r)$ along the radial distance $r$ of the red blood cells and then plot the graph, if it is parabolic, then we get the conductivity of red blood cells in standard form. On the other hand, in other cases 2 and 3, the conductivity is too high. Hence, by case 1, the formula (5.2) is very suitable and applicable in various medical processes. This study is applicable for the simulation of bioimpedance signals [17, 18], in the field of aortic dissection [3, 2], ICG method in detecting aortic dissection [2] and so on.

By our results and values obtained in the Sections 4 and 5, the matrix representation of RBCs of Figure 1.1 is constructed which seems to be interesting.

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