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Some notes on Pontryagin's Maximum Principle

D. Dolgy

ABSTRACT. Dynamic optimization plays a central role in many scientific and engineering disciplines, where the objective is to determine a control strategy that optimizes a given performance criterion over time. Pontryagin's Maximum Principle (PMP), developed in the late 1950s by Lev Pontryagin and his collaborators, provides a powerful theoretical framework for addressing such problems. This article presents an overview of the Maximum Principle, outlines its mathematical foundations, and discusses its effectiveness and applicability in solving a wide range of dynamic optimization problems.

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1. Introduction

Dynamic optimization involves making decisions over time in systems governed by differential equations. Applications range from economics and robotics to aerospace engineering and biology. Traditional methods for solving such problems include the calculus of variations and numerical optimization techniques. However, these approaches often fall short when handling complex systems with constraints on control and state variables.

Pontryagin's Maximum Principle emerged as a landmark development in optimal control theory [6]. It transforms the original dynamic optimization problem into a boundary-value problem, thus providing necessary conditions for optimality in the form of a Hamiltonian maximization condition. This principle is now regarded as a cornerstone of modern control theory.

Remember that optimal control theory began to take shape as a mathematical discipline in the 1950s [2,4,5]. The motivation for its development were the actual problems of automatic control, satellite navigation, aircraft control, chemical engineering and a number of other engineering problems.

Optimal control is regarded as a modern branch of the classical calculus of variations, which is the branch of mathematics that emerged about three centuries ago at the junction of mechanics, mathematical analysis and the theory of differential equations. The calculus of variations studies problems of extreme in which it is necessary to find the maximum or the minimum of some numerical characteristic (functional) defined on the set of curves, surfaces, or other mathematical objects of a complex nature.

The development of the calculus of variations is associated with the names of some famous scientists: Bernoulli, Euler, Newton, Lagrange, Weierstrass, Hamilton and others. Optimal control problems differ from variation problems by the additional requirements imposed on sought solution, and these requirements are sometimes difficult and even impossible to fit applying for solving the methods of the calculus of variations. The need for practical methods resulted in further development of variation calculus, which ultimately led to the formation of the modern theory of optimal control. This theory, absorbed all previous achievements in the calculus of variations, and it was enriched with new results and new content.

2. Formulation of the Optimal Control problem

A formulation of the problem of optimal control includes a control objective, a mathematical model of the controlled object, constraints and a description of a class of controls.

The control objective is a request expressed in a formal form for the behavior of a controlled object. An objective of the control can be, for example, a transfer of the controlled object from one position to another in a finite amount of time or to keep the trajectory of motion within given limits, etc. Often the objective of control is to optimize (maximize or minimize) an objective functional, that is, a numerical parameter specified on a set of processes. The values of the objective functional characterize a “quality” of processes. For the optimization of a functional procedure, we allocate the best quality processes from various ones.

A *mathematical model of a controlled object* is some law of transformation of controls into trajectories of an object. It can be set by a system of ordinary differential equations, partial differential equations, integral equations, recurrence relations, or in other ways.

Constraints are additional conditions for processes that arise from the physical meaning of the statement of a control problem. The requirements related with the safe operation of a controlled object lead to *phase constraints* on a state vector or to *mixed constraints* on state vectors and controls simultaneously. In particular, the initial conditions for differential equations can be regarded as the simplest phase constraints.

The class of controls is defined by specifying the analytical properties and the range of control variables. For example, we can use class of controls $K(R \to U)$ consists of piecewise continuous functions $u(t): R \to R^r$ with values in a compact $U \subset R^r$. But optimal control can use more general classes of summarizing or measurable controls that are dictated by the physical meaning of the problem or by the wish to ensure the solvability of the problem. A wider a class of controls allows for greater possibility for the optimal control to exist. However, the expansion of the class of controls requires using a more sophisticated mathematical apparatus and details of the theory of functions, functional analysis and differential equations.

Objective functionals

By [1,3], in optimal control theory, we traditionally consider three types of objective functionals defined on the processes $x(t), u(t)$ of a system of differential equations

$$\dot{x} = f(x,u,t),\ x(t_0) = x_0 . \tag{1}$$

(a) Terminal functional (Mayer functional)

$$J_1 = \Phi(x(t_1), t_1), \tag{2}$$

is defined by a scalar function $\Phi(x,t)$ on the ends $(x(t_1), t_1)$ of the integral curves $(x(t), t)$, where t_1 is fixed or not fixed in advance in a given moment of time, $t_1 > t_0$.

(b) The integral functional (Lagrange functional) is given by a scalar function $F(x,u,t)$ in a form of definite integral

$$J_2 = \int_{t_0}^{t_1} F(x(t),u(t),t)dt. \tag{3}$$

Function $F(x,u,t)$ is assumed to be continuous with respect to its arguments x,u,t and has continuous partial derivatives with respect to state variable x. Then, a complicated function $t \to F(x(t),u(t),t)$ is piecewise continuous on a segment $[t_0,t_1]$, and the existence of the integral (3) is guaranteed by the appropriate theorem of mathematical analysis.

(c) Mayer-Bolts functional

$$J_3 = \Phi(x(t_1),t_1) + \int_{t_0}^{t_1} F(x(t),u(t),t)dt, \tag{4}$$

is the sum of the functionals (2) and (3).

If the function $\Phi(x,t)$ belongs to the class $C_1(R^n \times R \to R)$ and $\Phi(x_0,t_0) = 0$, then the terminal functional can be easily transformed into an integral one. Indeed, according to the Leibniz-Newton formula

$$\Phi(x(t_1),t_1) = \int_{t_0}^{t_1} d\Phi(x(t),t) = \int_{t_0}^{t_1} \left[\Phi_x(x(t),t)'\dot{x}(t) + \Phi_t(x(t),t)\right]dt =$$
$$= \int_{t_0}^{t_1} \left[\Phi_x(x(t),t)'f(x(t),u(t),t) + \Phi_t(x(t),t)\right]dt.$$

The integral in the right-hand side is a Lagrange functional with a generating function

$$F(x,u,t) = \Phi_x(x,t)'f(x,u,t) + \Phi_t(x,t).$$

A reverse transition from the integral functional to the terminal functional is carried out by extending the phase space, that is, by the introducing an additional phase variable x_{n+1} according to the formulas

$$\dot{x}_{n+1} = F(x,u,t),\ x_{n+1}(t_0) = 0.$$

Appending these relations to the conditions for (1), we obtain an extended system of differential equations and initial conditions

$$\dot{x} = f(x,u,t),\ \dot{x}_{n+1} = F(x,u,t),\ x(t_0) = x_0,\ x_{n+1}(t_0) = 0.$$

If $x(t), u(t)$ is a process of the system (1), then a triple $x(t)$, $x_{n+1}(t) = \int_{t_0}^{t} F(x(\tau), u(\tau), \tau) d\tau$, $u(t)$ will be a process of extended system. From here, we have

$$\int_{t_0}^{t_1} F(x(t), u(t), t) dt = x_{n+1}(t_1)$$

when $t = t_1$. Consequently, the integral functional in the system (1) coincides with the terminal functional in an extended system with a generating function $\Phi(x, x_{n+1}) = x_{n+1}$.

The above methods are then used to transform the objective functional and constraints to a terminal or integral form. Thus, it is important that when constructing a theory, we can only apply functional of one type. The results for the other types of functional are then obtained by using the above transformations.

Constraints on the Ends of a Trajectory. Terminology

Let t_0, t_1 be fixed or not in advance moments of time, $t_0 < t_1$, and let $x(t), u(t)$ be an arbitrary process of the system (1). The points $x(t_0)$ and $x(t_1)$ are referred to as the *left and right ends of a trajectory* $x(t)$, and the pairs $(x(t_0), t_0)$, $(x(t_1), t_1)$ are referred to as *the left and right ends of an integral curve* $(x(t), t)$. The most general constraint on the ends of an integral curve has the form

$$(x(t_0), x(t_1), t_0, t_1) \in \Gamma,$$

where Γ is a given set of Cartesian product $R^n \times R^n \times R \times R$. If this inclusion unambiguously (ambiguously) defines the points $x(t_0)$, $x(t_1)$, we speak about *fixed* (*mobile*) ends of a trajectory. We apply the same terms to the ends $(x(t_0), t_0)$, $(x(t_1), t_1)$ of an integral curve or to the moments of time t_0, t_1. The end of the trajectory that does not impose any restrictions is referred to as the *free end of a trajectory*. There may be different combinations of requirements for the ends of integral curves in optimal control problems. For example, the left end of an integral curve can be fixed and the right end can be a free end at the same time when the moments t_0, t_1 are fixed or mobile. There can also be fixed ends of trajectory while the moments of time

t_0, t_1 are mobile, and so on. Several types of problems that will be the subject of our further study are now considered.

We consider several basic optimal control problems introduced in [1,3]. According to classification

The Simplest Problem of optimal control (*S*-problem) consists of minimizing a terminal functional on a set of processes $x(t), u(t)$ of a controlled system with fixed left and right ends of a trajectory with fixed end points of time. This problem has the form

$$J = \Phi(x(t_1)) \to \min,$$

$$\dot{x} = f(x,u,t),\ x(t_0) = x_0,\ u \in U,\ t \in [t_0, t_1],$$

where a scalar function $\Phi(x)$ belongs to the class $C_1(R^n \to R)$. Regarding the function f, the range of control *U* and the class of control, the agreements that we set up earlier remain valid. The objective of control, the mathematical model of the controlled object, the phase constraint in the form of the initial condition and the restrictions on the vector of control are represented in an *S*-problem.

Two-point Minimum Time Problem (*M-problem*) is an optimal control problem with fixed endpoints of a trajectory and mobile moments of time:

$$J = t_1 - t_0 \to \min,$$

$$\dot{x} = f(x,u,t),\ x(t_0) = x_0,\ x(t_1) = x_1,\ u \in U,\ t_0 \le t_1.$$

Here x_0, x_1 are the given points of space R^n. The problem is thus to minimize a transition time from the point x_0 to the point x_1 along the trajectory of a system of differential equations of a controlled object by means of an appropriate control and end points of time t_0, t_1. The solution of the problem is trivial when $x_0 = x_1$. Leaving aside this case, we assume that $x_0 \ne x_1$.

General Optimal Control Problem (*G-problem*) is the problem that has mobile ends of an integral curve:

$$J_0 = \Phi_0(x(t_0), x(t_1), t_0, t_1) \to \min,$$
$$J_i = \Phi_i(x(t_0), x(t_1), t_0, t_1) \begin{cases} \le 0, \ i = 1,\dots,m_0, \\ = 0, \ i = m_0 + 1,\dots,m, \end{cases}$$
$$\dot{x} = f(x,u,t), \ u \in U, \ t_0 \le t_1.$$

Here $\Phi_0,\dots,\Phi_m$ are the given functions of the class $C_1(R^n \times R^n \times R \times R \to R)$, m_0 is an integer nonnegative number, and m is a natural number. If $m_0 = 0$ or $m_0 = m$, then the *G*-problem only has constraints-equalities $J_i = 0, \ i = 1,\dots,m$, or only constraints-inequalities $J_i \le 0, \ i = 1,\dots,m$, respectively. The *process* is said to be a quaternion $x(t), u(t), t_0, t_1$ that satisfies all conditions of the *G*-problem except, possibly, the first condition. A process $x(t), u(t), t_0, t_1$ is regarded to be *optimal* if for any other process $\tilde{x}(t), \tilde{u}(t), \tilde{t}_0, \tilde{t}_1$, the following inequality is true

$$\Phi_0(x(t_0), x(t_1), t_0, t_1) \le \Phi_0(\tilde{x}(\tilde{t}_0), \tilde{x}(\tilde{t}_1), \tilde{t}_0, \tilde{t}_1).$$

The *G*-problem consists of determining the optimal process.

Note that S-problem and M-problem are particular cases of G-problem. We can easily get them specifying objective functional and constraints in G-problem.

Along with above problems of optimal control, there are more special and particular forms that are not considered in this review.

The basic tool for solution of the problems of optimal control is maximum principle. According to [1], we formulate it for optimal control problems with terminal functional.

3. Formulation of the Maximum Principle

Theorem 1 (maximum principle for *G*-problem) *Let* $x(t), u(t), t_0, t_1$ *be an optimal process of the G-problem. Then there exists a vector* $\lambda = (\lambda_0,\dots,\lambda_m)$ *and a continuous solution* $\psi(t)$ *of a conjugate system of differential equations*

$$\dot{\psi} = -H_x(\psi, x(t), u(t), t),$$

satisfying conditions:

1) non-triviality, non-negativity and complementary slackness

$\lambda \ne 0$, $\lambda_i \ge 0, \ i = 0,\dots,m_0$, $\lambda_i \Phi_i(x(\boldsymbol{t}), \boldsymbol{t}) = 0, \ i = 1,\dots,m_0$;

2) transversality

$$\psi(t_0) = L_{x^0}(\lambda, x(\boldsymbol{t}), \boldsymbol{t}),\ \psi(t_1) = -L_{x^1}(\lambda, x(\boldsymbol{t}), \boldsymbol{t}),$$

$$\dot{L}_{t_0}(\lambda, x(\boldsymbol{t}), \boldsymbol{t}) = 0,\ \dot{L}_{t_1}(\lambda, x(\boldsymbol{t}), \boldsymbol{t}) = 0;$$

3) maximum of Hamiltonian

$$H(\psi(t), x(t), u(t), t) = \max_{u \in U} H(\psi(t), x(t), u, t),\ t \in [t_0, t_1]$$

with functions

$$L(\lambda, x^0, x^1, t_0, t_1) = \sum_{i=0}^{m} \lambda_i \Phi_i(x^0, x^1, t_0, t_1),\ H(\psi, x, u, t) = \sum_{j=1}^{n} \psi_j f_j(x, u, t).$$

Theorem 2 (maximum principle for *S*-problem) *If* $x(t), u(t)$ *is an optimal process of the S-problem, then the condition of the maximum of the Hamiltonian*

$$H(\psi(t), x(t), u(t), t) = \max_{u \in U} H(\psi(t), x(t), u, t)$$

holds at every moment $t \in [t_0, t_1]$, *where* $\psi(t)$ *is the corresponding solution of the conjugate Cauchy problem* $\begin{cases} \dot{\psi} = -H(\psi, x(t), u(t), t) \\ \psi(t_1) = -\Phi_x(x(t_1)) \end{cases}$.

Theorem 3 (maximum principle for *M*-problem) *If* $x(t), u(t), t_0, t_1$ *is an optimal process of the M-problem, then there exists a non-trivial continuous solution* $\psi(t)$ *of a conjugate system of differential equations*

$$\dot{\psi} = -H_x(\psi, x(t), u(t), t),$$

such that

$$H(\psi(t), x(t), u(t), t) = \max_{u \in U} H(\psi(t), x(t), u, t),\ t \in [t_0, t_1],$$

$$H(\psi(t_0), x(t_0), u(t_0), t_0) = H(\psi(t_1), x(t_1), u(t_1), t_1) \geq 0,$$

where $H(\psi, x, u, t) = \psi' f(x, u, t)$.

Since *S*- and *M*-problems are particular cases of General problem, we can use only Theorem 1 for solution of them. Nevertheless, it turn out that applying special conditions (Theorem 2 and Theorem 3) are more effective.

Example of solution of *G*-problem. Determine an optimal process in the General problem

$$t_1 \to \min, \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}, \; x_1(0) = \xi_1, \; x_2(0) = \xi_2, \; x_2(t_1) = 0, \; |u| \le 1, \; t_1 \ge 0,$$

where ξ_1 , ξ_2 are some real numbers, $\xi_2 \neq 0$.

Solution. We then construct the Lagrange function and the Hamiltonian

$$L = \lambda_0 t_1 + \lambda_1 [x_1(0) - \xi_1] + \lambda_2 [x_2(0) - \xi_2] + \lambda_3 x_2(t_1), \quad H = \psi_1 x_2 + \psi_2 u$$

and write a conjugate system of differential equations and transversality conditions

$$\begin{cases} \dot{\psi}_1 = 0 \\ \dot{\psi}_2 = -\psi_1 \end{cases}, \quad \begin{cases} \psi_1(0) = \lambda_1 \\ \psi_2(0) = \lambda_2 \end{cases}, \quad \begin{cases} \psi_1(t_1) = 0 \\ \psi_2(t_1) = -\lambda_3 \end{cases}, \quad \lambda_0 + \lambda_3 \dot{x}_2(t_1) = 0 .$$

From here, we obtain

$$\psi_1(t) = \lambda_1 = 0, \; \psi_2(t) = \lambda_2 = -\lambda_3,$$

and consequently, $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (\lambda_0, 0, \lambda_2, -\lambda_2)$. From the condition of the maximum for function *H* by control, we obtain $u(t) = \operatorname{sign} \psi_2(t) = \operatorname{sign} \lambda_2$. Then the last transversality condition has the form

$$\lambda_0 + \lambda_3 \dot{x}_2(t_1) = \lambda_0 - \lambda_2 u(t_1) = \lambda_0 - \lambda_2 \operatorname{sign} \lambda_2 = \lambda_0 - |\lambda_2| = 0.$$

If $\lambda_0 = 0$, then we have $\lambda_2 = 0$ from the above equality, which leads to the triviality of the Lagrange multipliers and contradicts the maximum principle. Therefore, without a loss of generality, we set $\lambda_0 = |\lambda_2| = 1$. As a result, we determine the structure of an extreme control $u(t) = \operatorname{sign} \lambda_2$. This is a constant function that takes the values +1 or -1. Control $u(t) = 1$ generates the trajectory

$$x_1(t) = \frac{t^2}{2} + \xi_2 t + \xi_1, \; x_2(t) = t + \xi_2,$$

that intersects the line $x_2 = 0$ in moments $t_1 = -\xi_2$ for $\xi_2 < 0$. Analogously, the trajectory corresponding to control $u(t) = -1$

$$x_1(t) = -\frac{t^2}{2} + \xi_2 t + \xi_1, \; x_2(t) = -t + \xi_2$$

intersects the line $x_2 = 0$ in moments $t_1 = \xi_2$ for $\xi_2 > 0$.

4. Advantages of the Maximum Principle

Pontryagin's Maximum Principle offers several advantages that make it a powerful tool in dynamic optimization:

- **Generality:** Applicable to systems with constraints on control and state variables, as well as non-linear dynamics.
- **Constructiveness:** Transforms the optimization problem into a system of differential equations, facilitating numerical implementation.
- **Versatility:** Used in both time-optimal and energy-optimal control problems.
- **Insight:** Provides deeper understanding of the structure of optimal trajectories and control laws.

5. Applications

Pontryagin's Maximum Principle has been successfully applied in various domains:

- **Aerospace engineering:** For trajectory optimization of rockets and spacecraft.
- **Economics:** In optimal growth and consumption models.
- **Mechanical systems:** In controlling robotic arms or autonomous vehicles.
- **Biomedicine:** For modeling and optimizing drug administration strategies.

In each of these cases, the Maximum Principle helps reduce complex optimization problems into solvable mathematical models, often enabling analytical insights or efficient numerical solutions.

6. Limitations and Challenges

Despite its strengths, Pontryagin's Maximum Principle also has limitations:

- **Complexity of solving the resulting two-point boundary-value problem (TPBVP).**
- **No guarantee of sufficiency:** The conditions are necessary but not sufficient; solutions must be checked for optimality. It is possible to get sufficient conditions of optimality for the special classes of optimal control problems, in particular, for linearly convex G-problems [1].
- **Sensitivity to initial guesses in numerical methods.**

To overcome these challenges, hybrid approaches combining Pontryagin's Maximum Principle with numerical optimization or dynamic programming are often used.

6. Conclusion

Pontryagin's Maximum Principle remains one of the most influential tools in dynamic optimization. Its ability to handle a wide class of problems with high efficiency and mathematical rigor makes it indispensable in both theoretical and applied contexts. While modern computational methods continue to evolve, the Pontryagin's Maximum Principle continues to serve as a foundational method in optimal control theory.

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KWANGWOON GLOCAL EDUCATION CENTER, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA & INSTITUTE OF MATHEMATICS AND COMPUTER TECHNOLOGIES, DEPARTMENT OF MATHEMATICS, FAR EASTERN FEDERAL UNIVERSITY, VLADIVOSTOK, RUSSIA

E-mail address: d_dol@mail.ru

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