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A note on the q-factorization of power series

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Abstract

We consider the Andrews factorization of a power series into a q-product, and we show how to obtain the corresponding sequences in terms of one another.

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1 Introduction

In [1] it is commented that many major results in number theory, analysis, and combinatorics take the form of "a series equals a product". Thus, Andrews [2] considers the factorization of an ordinary power series with unit constant term into a q-product

$$1+\sum_{n=1}^{\infty} r(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{a_n}}, \tag{1}$$

which generates the recurrence relation

$$nr(n) = \sum_{j=1}^{n} A_j r(n-j), \tag{2}$$

where

$$A_j = \sum_{d/j} d a_d. \tag{3}$$

Schneider-Sills-Waldron [1] mention that the following question is natural:

Can explicit formulas be given, to express the sequences a_n and $r(n)$ in terms of one another? (4)

In Sec. 2 we show that (4) has an affirmative answer.

2 The answer is Yes for the question (4)

The solution of (2) is given by [3, 4, 5]:

$$r(n) = \frac{1}{n!} B_n \left(0!A_1, 1!A_2, \ldots, (n-1)!A_n\right), \tag{5}$$

in terms of complete Bell polynomials [5, 6]. Then, if we know the sequence $\{a_n\}$, this relation (5) allows determine $r(n)$.

The inversion of (5) is immediate [7]

$$(n-1)!A_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)! B_{n,k}(r(1), r(2), \ldots, r(n-k+1)), \quad n \geq 1 \tag{6}$$

in terms of partial Bell polynomials [6, 7, 8]. Hence if we know $r(n)$, then (6) gives the quantities A_m, and finally the sequence $\{a_m\}$ is determined via the Möbius inversion of (3) [9, 10]:

$$na_n = \sum_{d/n} \mu\left(\frac{n}{d}\right) A_d. \tag{7}$$

Therefore, the question (4) has an affirmative answer.

Remark 1. Important arithmetic functions satisfy recurrence relations with the structure (2) [5, 11].

Remark 2. For the case $a_n = 1$, the relation (3) gives $A_j = \sum_{d/j} d = \sigma(j)$, that is, the sum of divisors function [9] participates in (2) and $r(n)$ is the partition function [2, 12]:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{(q;q)_\infty}, \quad np(n) = \sum_{j=1}^{n} \sigma(j)p(n-j) \tag{8}$$

Remark 3. If now $a_n = (-1)^{n+1}k$ then $A_j = k\sum_{d/j}(-1)^{d+1}d$ and $r(n)$ is the number of representations of n as the sum of k triangular numbers [11]:

$$\sum_{n=0}^{\infty} t_k(n)q^n = \prod_{n=1}^{\infty} (1-q^n)^{(-1)^n k}, \quad nt_k(n) = \sum_{j=1}^{n} A_j t_k(n-j) \tag{9}$$

Remark 4. The work [13] has interesting applications of the q-factorization (1) and the corresponding recurrence relation (2). It is clear that the Fine theorem [14, 15, 16] is applicable to (1), hence $r(n)$ is determined by the partitions of n, in fact:

$$r(n) = \sum_{\lambda \vdash n} C_1\left(k_1\right) C_2\left(k_2\right) \cdots, \quad C_j(m) = \binom{a_j + m - 1}{m}, \tag{10}$$

where $\lambda \vdash n$ means all partitions of n, and k_j is the multiplicity of j in a given partition. This relation (10) is an alternative to (5).

Remark 5. The change $a_n \to \lambda a_n$ is connected with the following expressions

$$\left(\sum_{n=0}^{\infty} r(n) q^n\right)^{\lambda} = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{\lambda a_m}} = \sum_{n=0}^{\infty} \tilde{r}(n) q^n \tag{11}$$

then it is natural to investigate the relationship between $r(n)$ and $\tilde{r}(n)$, in fact [17, 18, 19]

$$\begin{aligned} \tilde{r}(n) &= \frac{1}{n!} \sum_{t=1}^{n} \binom{\lambda}{t} t! B_{n,t}(1!r(1), 2!r(2), \ldots, (n-t+1)!r(n-t+1)) \\ &= \frac{1}{n!} B_n\left(0!\lambda A_1, 1!\lambda A_2, \ldots, (n-1)!\lambda A_n\right) \end{aligned} \tag{12}$$

with the recurrence relation:

$$\sum_{j=0}^{n} ((\lambda+1)j - n) r(j) \tilde{r}(n-j) = 0, \quad n \geq 0. \tag{13}$$

Besides, from (3) the mapping $a_m \to \lambda a_m$ implies the change $A_n \to \lambda A_n$, then (5) gives (12).

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