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A NOTE ON PROBABILISTIC DEGENERATE POLY LAH-BELL POLYNOMIALS

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ABSTRACT. In [1], Liu-Ma-Kim-Kim-Hei considered probabilistic degenerate poly Lah Bell polynomials, we aim to derive new results that are not included in this paper.

1. INTRODUCTION

Stirling numbers have been extensively studied in the field of combinatorics. This paper studies the Lah number, also known as the third kind of string number. The Lah number $L(n,k)$ counts the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets (see [1],[6],[8],[9]). Recently, researchers have been studying the polyexponential function and the degenerate polyexponential function (see [2],[3],[6]). It has been used as a mathematical tool to study various special polynomials. Probabilistic versions of these special polynomials have also been developed recently (see[1,3,4,10,12,14]). In this paper, we aim to investigate properties not covered in [1]. This paper's outline is as follows: we recall the definitions of generating functions, the Stirling numbers of the first kind, the Lah numbers, the Lah-Bell polynomials and the degenerate polyexponential function. We remind of the Poisson random variable, the Bernoulli random variable, the geometric random variable, the gamma random variable and the gamma function. In Section 2, we recall the definition of probabilistic degenerate poly Lah-Bell polynomials. These polynomials are considered by Liu-Ma-Kim-Hei (see [1]). In Theorem 1, we get an expression of $LB_{n,\lambda}^{(k,Y)}$ by using a Poisson random variable. In Theorem 2, we have an expression for $LB_{m,\lambda}^{(k,Y)}$ with Lah numbers. In Theorem 3, from the gamma distribution, we obtain an expression for $LB_{n,\lambda}^{(k,Y)}$. In Theorem 4, we get an identity for $LB_{n,\lambda}^{(k,Y)}(x)$.

The Stirling numbers of the first kind are defined by

$$\frac{1}{k!}(log(1+t))^k=\sum_{n=0}^{\infty}S_1(n,k)\frac{t^n}{n!},(\text{see } [1,9,13]). \tag{1}$$

The Lah numbers, also called the third String number, are defined by

$$\frac{1}{k!}\left(\frac{1}{1-t}-1\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!},(\text{see } [1,6,9]). \tag{2}$$

The generating function of the number of Lah-Bell polynomials closely related to the number of Lah numbers is as follows:

$$e^{x(\frac{1}{1-t}-1)}=\sum_{n=0}^{\infty}B_n^L(x)\frac{t^n}{n!},(\text{see } [1,6,9]). \tag{3}$$

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A degenerate polyexponential function is defined by Kim-Kim

$$Ei_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}}{(n-1)!n^k} x^n, \quad (k \in \mathbb{Z}, |x| < 1), \quad (\text{see } [6,7,8]). \tag{4}$$

Where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ is the degenerate falling factorial sequence.

It is well known that a Poisson random variable with parameter $\alpha > 0$ if the probability mass function of X as follows:

$$P(i) = P\{X = i\} = e^{\alpha}\frac{\alpha^i}{i!}, (i = 0,1,2,\cdots), (\text{see } [3,5,15]). \tag{5}$$

A Bernoulli random variable is one of the simplest types of random variables. It represents an experiment or process that has exactly two possible outcomes: success and failure.

The probability distribution of a Bernoulli random variable X is given by

$$P\{X = 1\} = p, P\{X = 0\} = 1 - p, \quad (\text{see } [3,12,15]). \tag{6}$$

Let X be the number of trials required until the first success, then X is called a Geometric random variable with parameter p. Its probability mass function is defined by

$$p(n) = P\{X = n\} = (1-p)^{n-1}p, (n = 1,2,3,\cdots), \quad (\text{see } [14]). \tag{7}$$

A continuous random variable Y whose density function is given by

$$f(y) = \begin{cases} \beta e^{-\beta y}\frac{(\beta y)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } y > 0, \\ 0, & \text{if } y \leq 0, \end{cases} \quad (\text{see } [3,4,5,15]), \tag{8}$$

for some $\alpha, \beta > 0$ is said to be the gamma random variable with parameters α, β, which is denoted by $Y \sim \Gamma(\alpha, \beta)$.

The gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \quad (\text{see } [11]). \tag{9}$$

Where $x \in \mathbb{C}$, $Re(x) > 0$.

2. Probabilistic Degenerate poly Lah-Bell polynomials

In [1], Liu-Ma-Kim-Kim-Hei considered probabilistic degenerate poly Lah Bell polynomials which are given by

$$Ei_{k,\lambda}\left(x\left(E\left[\left(\frac{1}{1-t}\right)^Y\right] - 1\right)\right) = \sum_{n=1}^{\infty} LB_{n,\lambda}^{(k,Y)}(x)\frac{t^n}{n!}, \quad (\text{see } [1]). \tag{10}$$

Let Y be a Poisson random variable, then we note that

$$E\left[\left(\frac{1}{1-t}\right)^Y\right] = e^{-\alpha(1-\frac{1}{1-t})}. \tag{11}$$

From (10) and (12), we have

$$
\begin{aligned}
(12)\qquad \sum_{n=1}^{\infty} LB_{n,\lambda}^{(k,Y)}(x)\frac{t^n}{n!} &= Ei_{k,\lambda}\left(x\left(E\left[\left(\frac{1}{1-t}\right)^{Y}\right]-1\right)\right)\\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}x^m}{(m-1)!m^k}\left(e^{-\alpha(1-\frac{1}{1-t})}-1\right)^m\\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}x^k}{(m-1)!m^k}\left(\sum_{l=0}^{\infty}B_l^L(\alpha)\frac{t^l}{l!}\right)^m\\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}x^m}{(m-1)!m^k}\left(\sum_{l_1=0}^{\infty}B_{l_1}^L(\alpha)\frac{t^{l_1}}{l_1!}\right)\cdots\left(\sum_{l_m=0}^{\infty}B_{l_m}^L(\alpha)\frac{t^{l_m}}{l_m!}\right)\\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}x^m}{(m-1)!m^k}\sum_{n=m}^{\infty}\sum_{l_1+l_2+\cdots+l_m=n}\binom{n}{l_1,l_2,\cdots,l_m}B_{l_1}^L(\alpha)B_{l_2}^L(\alpha)\cdots B_{l_m}^L(x)\frac{t^n}{n!}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{l_1+l_2+\cdots+l_m=n}\binom{n}{l_1,l_2,\cdots,l_m}\frac{(1)_{m,\lambda}x^m}{(m-1)!m^k}B_{l_1}^L(\alpha)B_{l_2}^L(\alpha)\cdots B_{l_m}^L(\alpha)\frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficient on both sides of (12), we have the following theorem.

Theorem 1. *Let Y be a Poisson random variable. For $n \geq 1$, we have*

$$LB_{n,\lambda}^{(k,Y)}(x) = \sum_{m=1}^{n}\sum_{l_1+l_2+\cdots+l_m=n}\binom{n}{l_1,l_2,\cdots,l_m}\frac{(1)_{m,\lambda}x^m}{(m-1)!m^k}B_{l_1}^L(\alpha)B_{l_2}^L(\alpha)\cdots B_{l_m}^L(\alpha).$$

Let Y be a Bernoulli random variable, then we note that

$$(13)\qquad E\left[\left(\frac{1}{1-t}\right)^{Y}\right] = (1-p)+\frac{p}{1-t}.$$

From (10) and (13), we have

$$
\begin{aligned}
(14)\qquad \sum_{m=1}^{\infty} LB_{m,\lambda}^{(k,Y)}(x)\frac{t^m}{m!} &= Ei_{k,\lambda}\left(x\left(E\left[\left(\frac{1}{1-t}\right)^{Y}\right]-1\right)\right)\\
&= \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}x^n}{(n-1)!n^k}\left((1-p)+\frac{p}{1-t}-1\right)^n\\
&= \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}x^n}{(n-1)!n^k}\left(p\left(\frac{1}{1-t}-1\right)\right)^n\\
&= \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}n!(px)^n}{(n-1)!n^k}\sum_{m=n}^{\infty}L(n,m)\frac{t^m}{m!}\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{m}\frac{(1)_{n,\lambda}(px)^n}{n^{k-1}}L(n,m)\frac{t^m}{m!}.
\end{aligned}
$$

By comparing the coefficients on both sides of (14), we have the following theorem.

Theorem 2. *Let Y be a Bernoulli random variable. For $m \geq 1$, we have*

$$LB_{m,\lambda}^{(k,Y)}(x) = \sum_{n=1}^{m} \frac{(1)_{n,\lambda}(px)^n}{n^{k-1}} L(n,m).$$

By expansion, we note that

$$E\left[\left(\frac{1}{1-t}\right)^{Y}\right] = E[(1-t)^{-Y}] = E[e^{-Ylog(1-t)}] = \sum_{n=0}^{\infty} \frac{(-log(1-t))^n}{n!} E[Y^n]. \tag{15}$$

When Y$\sim \Gamma(\alpha,\beta)$, we consider *n*-th moments of the gamma distribution. Then we note that

$$E[Y^n] = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\beta^n}. \tag{16}$$

From (15) and (16), we have

(17)

$$\begin{aligned}\sum_{n=1}^{\infty} LB_{n,\lambda}^{(k,Y)}(x)\frac{t^n}{n!} &= Ei_{k,\lambda}\left(x\left(E\left[\left(\frac{1}{1-t}\right)^{Y}\right]-1\right)\right)\\ &= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}x^m}{(m-1)!m^k}\left(\sum_{l=1}^{\infty}\frac{\Gamma(\alpha+l)}{\Gamma(\alpha)l!\beta^n}(-log(1-t))^l\right)^m\\ &= \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda}x^m}{(m-1)!m^k}\left(\sum_{l=1}^{\infty}\frac{\Gamma(\alpha+l)(-1)^l}{\Gamma(\alpha)\beta^n}\sum_{n=l}^{\infty}(-1)^n S_1(n,l)\frac{t^n}{n!}\right)^m\\ &= \sum_{m=1}^{\infty} \frac{(1)_{n,\lambda}x^m}{(m-1)!m^k}\sum_{n=m}^{\infty}\sum_{l=1}^{n}\sum_{i_1+i_2+\cdots+i_m=n}\binom{n}{i_1,i_2,\cdots,i_m}\left(\frac{\Gamma(\alpha+l)}{\Gamma(\alpha)\beta^l}\right)^m\\ &\times(-1)^{l+n}S_1(i_1,l)S_1(i_2,l)\cdots S_1(i_m,l)\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{i_1+i_2+\cdots+i_m=n}\frac{(-1)^{l+n}(1)_{m,\lambda}}{(m-1)!m^k}\left(\frac{\Gamma(\alpha+l)x}{\Gamma(\alpha)\beta^l}\right)^m\\ &\times S_1(i_1,l)S_1(i_2,l)\cdots S_1(i_m,l)\frac{t^n}{n!}.\end{aligned}$$

Thus, by comparing the coefficients on both sides of (17), we have the following theorem.

Theorem 3. *Let Y$\sim \Gamma(\alpha,\beta)$. For $n \geq 1$, we have*

$$LB_{n,\lambda}^{(k,Y)}(x) = \sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{i_1+i_2+\cdots+i_m=n}\frac{(-1)^{l+n}(1)_{m,\lambda}}{(m-1)!m^k}\left(\frac{\Gamma(\alpha+l)x}{\Gamma(\alpha)\beta^l}\right)^m S_1(i_1,l)S_1(i_2,l)\cdots S_1(i_m,l).$$

Let Y be a Geometric random variable, then we note that

$$
\begin{aligned}
(18)\qquad E\left[\left(\frac{1}{1-t}\right)^{Y}\right] &= \sum_{k=1}^{\infty}\left(\frac{1}{1-t}\right)^{k} p(1-p)^{k-1} \\
&= p\sum_{l=1}^{\infty}\frac{(1-p)^{l-1}}{(1-t)^{l}} \\
&= \frac{p}{1-t}\sum_{l=0}^{\infty}\left(\frac{1-p}{1-t}\right)^{l} \\
&= \frac{1}{1-\frac{t}{p}},
\end{aligned}
$$

where $|t/p| < 1$.

From (10) and (18), we have

$$
\begin{aligned}
(19)\qquad \sum_{n=1}^{\infty} LB_{n,\lambda}^{(k,Y)}(x)\frac{t^{n}}{n!} &= Ei_{k,\lambda}\left(x\left(E\left[\left(\frac{1}{1-t}\right)^{Y}\right]-1\right)\right) \\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}x^{m}}{(m-1)!m^{k}}\left(\frac{1}{1-\frac{t}{p}}-1\right)^{m} \\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}x^{m}}{(m-1)!m^{k}}\left(\sum_{l=1}^{\infty}\frac{t^{l}}{p^{l}}\right)^{m} \\
&= \sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}x^{m}}{(m-1)!m^{k}}\sum_{n=m}^{\infty}\sum_{l_1+l_2+\cdots+l_m=n}\frac{n!}{p^{n}}\frac{t^{n}}{n!} \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{l_1+l_2+\cdots+l_m=n}\frac{n!(1)_{m,\lambda}x^{m}}{p^{n}(m-1)!m^{k}}\frac{t^{n}}{n!}.
\end{aligned}
$$

Thus, we have the following theorem.

Theorem 4. *Let Y be a Geometric random variable. For $n \geq 1$, we have*

$$
LB_{n,\lambda}^{(k,Y)}(x) = \sum_{m=1}^{n}\sum_{l_1+l_2+\cdots+l_m=n}\frac{n!(1)_{m,\lambda}x^{m}}{p^{n}(m-1)!m^{k}}.
$$

3. Conclusion

Degenerate polyexponential functions have been studied as a tool for observing various polynomials. In this paper, we observed some properties or relations with degenerate poly-Lah Bell polynomials and the definitions of probability theory. Multi polyexponential function, which is used as another mathematical tool, is also widely used as a mathematical tool, but it seems that using the multi version with probabilistic poly Lah Bell polynomials can produce new results.

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