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Links between arithmetic functions and integer partitions

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Abstract

We use the Fine and Jameson-Schneider theorems to write certain arithmetic functions in terms of integer partitions via Bell polynomials.

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1 Introduction

Here we consider the functions:

$$\psi_j(q) := (1-q)^{s(j)} = \sum_{n=0}^{\infty} C_j(n) q^n, \tag{1}$$

for any sequence $\{s(n)\}$, and we accept that:

$$\eta(q) := \prod_{j=1}^{\infty} \psi_j\left(q^j\right) = \prod_{j=1}^{\infty} \left(1-q^j\right)^{s(j)} = \sum_{n=0}^{\infty} R(n) q^n, \quad R(0)=1; \tag{2}$$

on the other hand, Jameson-Schneider [1,2] proved the following result:

$$q\frac{d}{dq}\ln\eta = -\sum_{n=1}^{\infty}\left(\sum_{d|n} s(d)\, d\right) q^n. \tag{3}$$

Besides, we know the Fine's theorem [3–5]:

$$\eta = \sum_{n=0}^{\infty} \left(\sum_{\lambda \vdash n} C_1\,(k_1)\, C_2(k_2) \cdots C_n(k_n) \right) q^n, \tag{4}$$

such that $\lambda \vdash n$ means all partitions of n, and k_r is the multiplicity of r in a given partition; therefore, (2) and (4) imply the connection:

$$R(n) = \sum_{\lambda \vdash n} C_1(k_1) C_2(k_2) \cdots C_n(k_n). \tag{5}$$

In Sec. 2 we apply the expressions (1), ..., (5) to several sequences $\{s(n)\}$, in particular, the case $s(j) = \frac{1}{j}$ shows that the partitions of an integer can give the number of divisors of it.

2 Divisor function and integer partitions

From (2) and (3) is immediate the following recurrence relation:

$$nR(n) = \sum_{j=1}^{n} h(j) R(n-j), \quad h(j) = -\sum_{d|j} s(d) d, \quad h(0) = 0, \tag{6}$$

whose solution is given by [6]:

$$R(n) = \frac{1}{n!} B_n \left(0! h(1),\, 1! h(2),\, 2! h(3),\, \ldots,\, (n-1)! h(n) \right), \tag{7}$$

in terms of the complete Bell polynomials [6, 7], with the corresponding inversion [8] for $n \geq 1$:

$$(n-1)! \sum_{d|n} s(d) d = \sum_{k=1}^{n} (-1)^k (k-1)! B_{n,k} \left(1! R(1), 2! R(2), \ldots, (n-k+1)! R(n-k+1) \right), \tag{8}$$

with the presence of the partial Bell polynomials [8, 9]; besides, the coefficients in the expansion (1) can be calculated with the relation:

$$C_j(n) = \frac{(-1)^n}{n!} \prod_{t=0}^{n-1} \left(s(j) - t \right), \quad n \geq 1, \quad C_j(0) = 1. \tag{9}$$

Now we shall apply our expressions to several sequences $\{s(n)\}$:

a) $s(m) = \frac{1}{m} \quad \therefore \quad \sum_{d|n} s(d) d = \sum_{d|n} 1 = d(n) =$ number of divisors of n,

with (5) and (9) obtain the values:

$$C_j(1) = -\frac{1}{j}, \qquad C_j(2) = \frac{1}{2j}\left(\frac{1}{j} - 1\right), \qquad C_j(3) = -\frac{1}{6j}\left(\frac{1}{j} - 1\right)\left(\frac{1}{j} - 2\right), \ldots, \tag{10}$$

$$R(1) = -1, \quad R(2) = -\frac{1}{2}, \quad R(3) = \frac{1}{6}, \quad R(4) = -\frac{1}{24}, \ldots, \tag{11}$$

then (8) generates the property:

$$(n-1)!\,d(n)=\sum_{k=1}^{n}(-1)^k(k-1)!\,B_{n,k}\left(1!R(1),2!R(2),\dots,(n-k+1)!R(n-k+1)\right), \qquad (12)$$

that is, the number of divisors of an integer in terms of its partitions. This expression (12) is an alternative to the following relation deduced in [10]:

$$d(n)=\sum_{r=1}^{n}a_{n-r}\sum_{j=1}^{r}S_{r,j}, \qquad (13)$$

where $S_{n,k}$ is the number of k's in all partitions of n, and [11]:

$$a_j=\begin{cases}0, & j\neq\frac{m}{2}(3m+1),\\ (-1)^m, & j=\frac{m}{2}(3m+1),\end{cases}\qquad m=0,\pm1,\pm2,\dots \qquad (14)$$

that is:

$$a_j=\begin{cases}1, & j=0,5,7,22,26,51,57,92,100,145,155,\dots\\ -1, & j=1,2,12,15,35,40,70,77,117,126,176,\dots\\ 0 & \text{otherwise.}\end{cases} \qquad (15)$$

b) $s(m)=1 \quad \therefore \quad \sum_{d|n}s(d)d=\sigma(n)=$ sum of divisors function,

$$C_j(1)=-1,\quad C_j(t)=0,\quad t\geq 2, \qquad (16)$$

and from (2):

$$\sum_{j=0}^{\infty}R(j)q^j=\prod_{n=1}^{\infty}(1-q^n)=(q;q)_\infty=\sum_{j=0}^{\infty}a_jq^j \quad \therefore \quad R(n)=a_n, \qquad (17)$$

then (5), (16) and (17) imply the identity:

$$a_n=\sum_{\lambda\vdash n}C_1(k_1)C_2(k_2)\cdots C_n(k_n), \qquad (18)$$

over all partitions without repeated parts. Besides, (6) gives the following recurrence relation obtained by Robbins [12] and Osler-Hassen-Chandrupatla [13]:

$$na_n=-\sum_{j=1}^{n}\sigma(j)a_{n-j}, \qquad (19)$$

and from (7) and (8):

$$a_n=\frac{1}{n!}B_n\left(-0!\sigma(1),-1!\sigma(2),-2!\sigma(3),\dots,-(n-1)!\sigma(n)\right), \qquad (20)$$

$$(n-1)!\,\sigma(n)=\sum_{k=1}^{n}(-1)^k(k-1)!\,B_{n,k}\left(1!a_1,2!a_2,\dots,(n-k+1)!a_{n-k+1}\right). \qquad (21)$$

c) $s(m) = \frac{\chi_4(m)}{m}$ where χ_4 is the nontrivial Dirichlet character (mod 4) [14, 15]:

$$\chi_4(n) = \begin{cases} (-1)^{\frac{n-1}{2}}, & n \text{ is odd}, \\ 0, & n \text{ is even}, \end{cases} \tag{22}$$

and we have the Jacobi's expression [16–18]:

$$\sum_{d|n} \chi_4(d) = \frac{1}{4} r_2(n), \tag{23}$$

involving the number of representations of n as the sum of two squares [14, 19, 20]; therefore:

$$C_j(1) = -\frac{\chi_4(j)}{j}, \quad C_2(j) = \frac{1}{2}\frac{\chi_4(j)}{j}\left(\frac{\chi_4(j)}{j} - 1\right), \ldots, \quad R(1) = -1, \quad R(2) = 0, \quad R(3) = -R(4) = \frac{1}{3}, \ldots, \tag{24}$$

such that:

$$(n-1)!\, r_2(n) = 4 \sum_{k=1}^{n} (-1)^k (k-1)!\, B_{n,k}\left(1!R(1), 2!R(2), \ldots, (n-k+1)!R(n-k+1)\right). \tag{25}$$

d) $s(m) = \frac{\mu(m)}{m}$ participating the Möbius function [16–18, 21] $\therefore \sum_{d|n} s(d) d = e_0(n)$ [22], with the Bellman's identity [12, 23–26]:

$$\prod_{j=1}^{\infty} \left(1 - q^j\right)^{\frac{\mu(j)}{j}} = e^{-q} = \sum_{n=0}^{\infty} R(n) q^n \quad \therefore \quad R(j) = \frac{(-1)^j}{j!}, \tag{26}$$

and from (8):

$$\sum_{k=1}^{n} (-1)^k (k-1)!\, B_{n,k}\left(-1, 1, -1, 1, \ldots, (-1)^{n-k+1}\right) = \begin{cases} 1, & n = 1, \\ 0, & n \geq 2. \end{cases} \tag{27}$$

e) $s(m) = \frac{\varphi(m)}{m}$ involving the Euler's totient function [16–18, 22, 27] $\therefore \sum_{d|n} s(d) d = n$ [22], with the property [12, 25, 28]:

$$\prod_{j=1}^{\infty} \left(1 - q^j\right)^{\frac{\varphi(j)}{j}} = e^{-\frac{q}{1-q}} = \sum_{k=0}^{\infty} R(k) q^k \quad \therefore \quad R(n) = \frac{1}{n} \sum_{t=1}^{n} \binom{n}{t} \frac{(-1)^t}{(t-1)!}, \tag{28}$$

thus (8) implies the identity:

$$n! = \sum_{k=1}^{n} (-1)^k (k-1)!\, B_{n,k}\left(1!R(1), 2!R(2), \ldots, (n-k+1)!R(n-k+1)\right), \tag{29}$$

such that $R(1) = -1$, $R(2) = -\frac{1}{2}$, $R(3) = -\frac{1}{6}$, $R(4) = \frac{1}{24}$, $R(5) = \frac{19}{120}, \ldots$ It is possible to write $R(n)$ in terms of Kummer hypergeometric function or an associated Laguerre polynomial:

$$R(n) = -{}_1F_1(1-n; 2; 1) = -\frac{1}{n} L^1_{n-1}(1), \quad n \geq 1. \tag{30}$$

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