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# LOCALLY BOUNDED AUTOMORPHISMS OF CONNECTED LIE GROUPS

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Abstract. Using recent results concerning homomorphisms between connected Lie groups, we obtain some results concerning the automorphisms of connected Lie groups.

## § 1. Introduction

Let $G$ be a connected Lie group and let $f\colon G \to G$ be an automorphism.

Let us apply known results [1, 2] concerning the automatic continuity properties of homomorphisms between Lie groups to obtain continuity conditions for a homomorphism of the form $f$.

## § 2. Preliminaries

Let us recall the results of [1, 2].

**Theorem 1.** [1] *The discontinuity group of every locally bounded homomorphism of a Lie group into a Lie group is commutative.*

**Theorem 2.** [2] *Every locally bounded homomorphism of a Lie group $G$ into a Lie group is continuous on the commutator subgroup $G'$ of the group $G$.*

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## § 3. Main Results

**Theorem 3.** *Let $G$ be a connected Lie group, and let*

$$f\colon G \to G$$

*be a locally bounded automorphism.*

*Let $R$ be the radical and let $L$ be a Levi subgroup of $G$.*

*Then $f$ continuously takes $L$ to a Levi subgroup $\widetilde{L}$ of $G$ (possibly $\widetilde{L} = L$) and takes $R$ into itself.*

The last mapping is continuous on the commutator subgroup $R'$ of $R$ [3, 4].

*Proof.* The continuity of $f$ on $L$ in the intrinsic topology of the Lie group $L$ follows from a general property of continuity of locally bounded homomorphisms of connected semisimple Lie groups into Lie groups (see, e.g., [3, 4]).

A homomorphic continuous image of a semisimple group is a semisimple group.

If the image of $L$ is not a maximal semisimple subgroup, then the image of a maximal semisimple subgroup containing this image under the inverse automorphism is a semisimple group containing a maximal semisimple subgroup, which is impossible.

Thus, $L$ is taken to a Levi subgroup of $G$.

Let us find $f(R)$.

The homomorphic image of $R$ is obviously solvable.

There is a divisible neighborhood of the identity element of $R$ which is a generating set for the connected group $R$.

The $f$-image of this neighborhood is a divisible generating set for the image of $R$. Since the closure of $f(R)$ is a solvable Lie subgroup of $G$, it follows that the index of the identity component of the closure of $f(R)$ is finite.

Then the divisibility of a generating set immediately proves that this generating set is contained in the identity component of the above closure.

Then $f(R)$ contains in this component, and hence the closure of $f(R)$ is connected.

Now it follows from the divisibility of a generating set of $f(R)$ that $f(R)$ itself is connected. Finally, the image of a normal subgroup under an automorphism is a normal subgroup.

Therefore, $f(R)$ is contained in the radical $R$, since this is the greatest connected solvable normal subgroup of $G$.

As above, $f(R)$ cannot be smaller than $R$, since, in this case, the image of $R$ under the inverse automorphism is larger than $R$, which is impossible.

Thus, $f(R) = R$.

An arbitrary locally bounded homomorphism of a solvable Lie group is continuous on the commutator subgroup of this solvable Lie group, which completes the proof of the theorem.

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