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THE FIRST COHOMOLOGY GROUP OF UNITS OF SOME REAL BIQUADRATIC NUMBER FIELDS

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Abstract. Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with $d_1 > 1$ and $d_2 > 1$ be two coprime positive squarefree integers. Denote by E_K the unit group of K and by G_K the Galois groups of $K/\mathbb{Q}$. The purpose of this paper is to investigate $H^1(G_K, E_K)$, the first cohomology group of G_K with coefficients in E_K, when the prime 2 is not totally ramified in K.

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Group of units, Biquadratic fields, First cohomology group.

1. Introduction and Notations

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be a real biquadratic number field such that $d_1 > 1$ and $d_2 > 1$ and $d_3 = d_1d_2$ are square-free integers. Let $k_1 = \mathbb{Q}(\sqrt{d_1})$, $k_2 = \mathbb{Q}(\sqrt{d_2})$ and $k_3 = \mathbb{Q}(\sqrt{d_3})$ be the three quadratic subfields of K. Let $\epsilon_i = x_i + y_i\sqrt{d_i}$ be the fundamental unit of k_i, where x_i and y_i are rational numbers, for $i = 1, 2, 3$. Put $N(m_i) = Norm_{k_i/\mathbb{Q}}(m_i)$ where $m_i \in k_i$, for $i = 1, 2, 3$. Let $H := H^1(G_K, E_K)$ be the first cohomology group of G_K, the Galois group of $K/\mathbb{Q}$, with coefficients in E_K, the unit group of K. Define $a_i = N(\epsilon_i + 1) = 2(x_i + 1)$ if $N\epsilon_i = 1$ and $a_i = 1$ otherwise, for $i = 1, 2, 3$. Let $\widetilde{H}$ be the subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ generated by $[a_1]$, $[a_2]$, $[a_3]$, $[d_1], [d_2]$ and $[d_3]$, where $[a_i]$ and $[d_i]$ denote the classes of a_i and d_i respectively in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, for $i = 1, 2, 3$.

In 1980, Setzer [9] gave the general form of the first cohomology group of units of the real biquadratic number fields (see Theorem 2.1 below). In 1982, Zantema gave the first cohomology group of units of both the cyclic number fields (see [11, p. 10]) and the imaginary biquadratic number fields (see [11, p. 10, Lemma 4.3]). The determination of first cohomology group of units may have several applications in Class Field Theory and Pólya Theory (see [11] and [5]). In this work, we use Setzer's result (see Theorem 2.1) to give explicitly the first cohomology group of units of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $d_1 > 1$ and $d_2 > 1$ are square-free integers with $(d_1, d_2) = 1$ and the prime 2 is not totally ramified in $K/\mathbb{Q}$.

2. Preliminary results

Let e_2 be the ramification index of the prime number 2 in $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ over $\mathbb{Q}$. The prime number 2 is the only prime that can be totally ramified in $K/\mathbb{Q}$. When the prime 2 is totally ramified in $K/\mathbb{Q}$, i.e., $e_2 = 4 = [K : \mathbb{Q}]$, we have $(d_1, d_2) \equiv (3, 2)$ or $(2, 3) \pmod 4$. Therefore, either $N\epsilon_1 \neq N\epsilon_2 = N\epsilon_3 = 1$, $N\epsilon_2 \neq N\epsilon_1 = N\epsilon_3 = 1$ or $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$. When $e_2 \neq 4$, i.e., the prime 2 is not totally ramified in $K/\mathbb{Q}$, we have either $e_2 = 1$ or $e_2 = 2$.

Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be a real biquadratic number field and $H := H^1(G_K, E_K)$. Recall that $\widetilde{H}$ is the group generated by $[d_1]$, $[d_2]$, $[d_3]$, $[a_1]$, $[a_2]$ and $[a_3]$. The following theorem was given by Setzer in [9, Theorem 4], and Zantema mentioned it later in [11, pp. 14 - 15].

Theorem 2.1. [9, Theorem 4] *We have $\widetilde{H} \simeq H$, except for the next two cases in which $\widetilde{H}$ is canonically isomorphic to a subgroup of index 2 in H:*

(1) *the prime 2 is totally ramified in $K/\mathbb{Q}$, and there exists integral $z_i \in k_i$, $i \in \{1,2,3\}$ such that $N_1(z_1) = N_2(z_2) = N_3(z_3) = \pm 2$,*
(2) *all the quadratic subfields k_i contain units of norm -1 and $E_K = E_{k_1}E_{k_2}E_{k_3}$.*

Proposition 2.2. ([6] *or* [1, p. 385]) *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, such that d_1 and d_2 are two square-free integers. Then, we have the following possibilities for a system of fundamental units of K :*

(1) $\epsilon_i, \epsilon_j, \epsilon_k$.
(2) $\sqrt{\epsilon_i}, \epsilon_j, \epsilon_k$ *with* $N\epsilon_i = 1$.
(3) $\sqrt{\epsilon_i}, \sqrt{\epsilon_j}, \epsilon_k$ *such that* $N\epsilon_i = N\epsilon_j = 1$.
(4) $\sqrt{\epsilon_i\epsilon_j}, \epsilon_j, \epsilon_k$ *such that* $N\epsilon_i = N\epsilon_j = 1$.
(5) $\sqrt{\epsilon_i\epsilon_j}, \sqrt{\epsilon_k}, \epsilon_j$ *where* $N\epsilon_i = N\epsilon_j = N\epsilon_k = 1$.
(6) $\sqrt{\epsilon_i\epsilon_j}, \sqrt{\epsilon_j\epsilon_k}, \sqrt{\epsilon_k\epsilon_i}$ *where* $N\epsilon_i = N\epsilon_j = N\epsilon_k = 1$.
(7) $\sqrt{\epsilon_i\epsilon_j\epsilon_k}, \epsilon_j, \epsilon_k$ *where* $N\epsilon_i = N\epsilon_j = N\epsilon_k = 1$.
(8) $\sqrt{\epsilon_i\epsilon_j\epsilon_k}, \epsilon_j, \epsilon_k$ *with* $N\epsilon_i = N\epsilon_j = N\epsilon_k = -1$.
where $\{\epsilon_i, \epsilon_j, \epsilon_k\} = \{\epsilon_3, \epsilon_1, \epsilon_2\}$.

Lemma 2.3. [3, Lemma 2] *Let $d \equiv 1 \pmod 4$ be a positive square-free integer, and let $\epsilon = x + y\sqrt{d}$ be the fundamental unit of $k = \mathbb{Q}(\sqrt{d})$. Assume $N(\epsilon) = 1$, then:*

(1) *$x + 1$ and $x - 1$ are not squares in $\mathbb{N}$, i.e., 2ϵ is not a square in $k = \mathbb{Q}(\sqrt{d})$.*
(2) *for all prime p dividing d, $p(x+1)$ and $p(x-1)$ are not squares in $\mathbb{N}$.*

Remark 2.4. *Let $d > 1$ be a square-free integer, and let $\epsilon = x + y\sqrt{d}$ be the fundamental unit of $k = \mathbb{Q}(\sqrt{d})$ such that $N\epsilon = 1$. When we say that $(x \pm 1)$ is a square in $\mathbb{N}$, this means that $(x+1)$ or $(x-1)$ is a square in $\mathbb{N}$.*

Lemma 2.5. [1, Lemma 5] *Let $d > 1$ be a square-free integer, and let $\epsilon = x + y\sqrt{d}$ be the fundamental unit of $k = \mathbb{Q}(\sqrt{d})$, where x and y are integers or semi-integers. If $N(\epsilon) = 1$, then $2(x+1)$, $2(x-1)$, $2d(x+1)$, and $2d(x-1)$ are not squares in $\mathbb{Q}$.*

By means of the above lemma, we have :

Lemma 2.6. *Let $k = \mathbb{Q}(\sqrt{d})$, where d is a positive square-free integer and $\epsilon = x + y\sqrt{d}$ is the fundamental unit of k, and x and y are integers or semi-integers. If $N(\epsilon) = 1$, then there exists a unique pair of positive integers (λ, λ') with $\lambda\lambda' = d$ such that either:*

(1) $2\lambda(x \pm 1)$ *is a square in* $\mathbb{N}$ *where* $\lambda \neq (1$ *and* $d)$, *or*

(2) $\lambda(x \pm 1)$ *is a square in* $\mathbb{N}$ *where* $\lambda \neq 2$.

Proof. As $N(\epsilon) = 1$, then $x^2 - 1 = (x \pm 1)(x \mp 1) = y^2 d$. Notice that $\gcd(x+1, x-1)$ is dividing 2, so we distinguish the following two cases:
Case 1: Assume that $\gcd(x+1, x-1) = 2$. By the unique factorization in $\mathbb{Z}$, there exist two unique positive divisors λ and λ' of d such that $\lambda\lambda' = d$ and there exist two numbers $z_1, z_2 \in \mathbb{Z}$ such that $2z_1z_2 = y$ and $(x \pm 1) = 2\lambda z_1^2$. This means that $2\lambda(x \pm 1)$ is a square in $\mathbb{N}$. By Lemma 2.5, we get that $\lambda \neq 1$ and d.
Case 2: Assume that $\gcd(x+1, x-1) = 1$. By the unique factorization in $\mathbb{Z}$, there exist two unique positive divisors λ and λ' of d such that $\lambda\lambda' = d$ and there exist two numbers $z_1, z_2 \in \mathbb{Z}$ such that $z_1z_2 = y$ and $(x \pm 1) = \lambda z_1^2$. Hence, we get that $\lambda(x \pm 1)$ is a square in $\mathbb{N}$. By Lemma 2.5 we get that $\lambda \neq 2$.
The preceding proof addresses the case where x and y are integers. We now consider the case where x and y are semi-integers.
We mention that the only case where we can have x and y are semi-integers is when $k = \mathbb{Q}(\sqrt{d})$ and $d \equiv 1 \pmod 4$, since $\mathcal{O}_k$, the ring of integers of k is $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$. In this case, we have $x = \frac{a}{2}$ and $y = \frac{b}{2}$ with a and b having the same parity, which means that either a and b are both odd or even numbers at the same time. We refer the reader to [8, p. 43, Theorem 1]. Note that when both a and b are even numbers, x and y become integers, which is not the case we are considering.
We have $\epsilon = x + y\sqrt{d}$ with $N\epsilon = 1$, so $x^2 - 1 = (x \pm 1)(x \mp 1) = y^2 d$, therefore $(2x)^2 - 4 = (2x \pm 2)(2x \mp 2) = (2y)^2 d$. Notice that $\gcd(2x+2, 2x-2) = 1$, taking into account that $2x$ is an odd integer. By the unique factorization in $\mathbb{Z}$, there exist two unique positive divisors λ and λ' of d such that $\lambda\lambda' = d$ and there exist two numbers $z_1, z_2 \in \mathbb{Z}$ such that $z_1z_2 = 2y$ and $(2x \pm 2) = \lambda z_1^2$. Therefore, $\lambda(2x \pm 2) = 2\lambda(x \pm 1)$ is a square with $\lambda \neq 1$ and d. Hence, the lemma is proved. □

We refer the reader to [4] to see similar cases of the lemma above.

Remark 2.7. *Keep the hypothesis of the above lemma.*

(1) *Notice that if d is an odd prime number, then $d(x \pm 1)$ or $(x \pm 1)$ is a square in* $\mathbb{N}$.
(2) *Assume that* $d \equiv 1 \pmod 4$ *is a positive square-free integer. Note that if* $d \equiv 1 \pmod 8$, *then x and y are integers, whereas if* $d \equiv 5 \pmod 8$, *then x and y can be integers as well as semi-integers* (*see* [2, Lemma 4.2]).

Example 2.8. *Let* $k = \mathbb{Q}(\sqrt{105})$ *such that* $d = 3 \cdot 5 \cdot 7 = 105$, *and let* $\epsilon = 41 + 4\sqrt{105}$ *be the fundamental unit of k with* $N(\epsilon) = 1$. *Note that here we have $x = 41$ and $y = 4$ are integers, and thus we find that* $\gcd(x+1, x-1) = \gcd(42, 40) = 2$. *Therefore, we are in case* 1 *of the above proof.*
We have $2(x+1) = 2 \cdot 40 = 2^4 \cdot 5$. *Hence,* $2\lambda(x+1) = 2 \cdot 5(x+1) = (2^2 \cdot 5)^2$ *is a square, where $\lambda = 5$ is distinct from* 1, *and* $d = 105$, *and* $\lambda' = 3 \cdot 7$ *with* $\lambda\lambda' = d = 105$. *It is worth mentioning that the pair* (λ, λ') *with* $\lambda = 5$ *and* $\lambda' = 3 \cdot 7$ *is the unique pair that satisfies* $2\lambda(x+1)$ *being a square and* $\lambda\lambda' = 5 \cdot 3 \cdot 7 = 105$.

Example 2.9. *Let* $k = \mathbb{Q}(\sqrt{805})$ *such that* $d = 5 \cdot 7 \cdot 23 = 805$*, and let* $\epsilon = \frac{1447}{2} + \frac{51}{2}\sqrt{805}$ *be the fundamental unit of* k *with* $N(\epsilon) = 1$*. Here we have* $x = \frac{1447}{2}$ *and* $y = \frac{51}{2}$*. So,* $(\frac{1447}{2})^2 - 1 = (\frac{1447}{2} \pm 1)(\frac{1447}{2} \mp 1) = (\frac{51}{2})^2 \cdot 805$*, therefore* $1447^2 - 4 = (1447 \pm 2)(1447 \mp 2) = 51^2 \cdot 805$*. Thus, we have* $\gcd(2x+2, 2x-2) = \gcd(1449, 1445) = 1$ *where* $2x + 2 = 1449 = 3^2 \cdot 7 \cdot 23$ *and* $2x - 2 = 1445 = 5 \cdot 17^2$*.*
We have $2(x+1) = 2(\frac{1447}{2} + 1) = 2(\frac{1447+2}{2}) = 1449 = 3^2 \cdot 7 \cdot 23$ *and hence we get that* $2\lambda(x+1) = 3^2 \cdot 7^2 \cdot 23^2$ *is a square with* $\lambda = 7 \cdot 23$*.*

Remark 2.10. *Let* $k = \mathbb{Q}(\sqrt{d})$*, where* d *is a positive square-free integer. Let* $\epsilon = x + y\sqrt{d}$ *be the fundamental unit of* k*. Throughout this work,* x *and* y *are either integers or semi-integers. Additionally,* λ *and* λ' *are positive integers dividing* d *with* $\lambda\lambda' = d$*.*

The following result is based on the lemma above and we establishe the equivalence of two statements concerning squares in the rational numbers $\mathbb{Q}$ and elements within the quadratic field $\mathbb{Q}(\sqrt{d})$.

Lemma 2.11. *Let* $k = \mathbb{Q}(\sqrt{d})$*, where* d *is a square-free integer and* $\epsilon = x + y\sqrt{d}$ *is the fundamental unit of* k*. Let* $N(\epsilon) = 1$ *and let* (λ, λ') *be the unique pair,* $\lambda\lambda' = d$ *satisfying the two items in the lemma above. Then, we have the two following equivalences:*

(1) $2\lambda(x \pm 1)$ *is a square in* $\mathbb{N}$ *if and only if* $\sqrt{\lambda\epsilon}$ *is in* $\mathbb{Q}(\sqrt{d})$*, where* $\lambda \neq (1$ *and* $d)$*.*
(2) $\lambda(x \pm 1)$ *is a square in* $\mathbb{N}$ *if and only if* $\sqrt{2\lambda\epsilon}$ *is in* $\mathbb{Q}(\sqrt{d})$*, where* $\lambda \neq 2$*.*

Proof. (1) Let us start by proving the direct implication. Assume that $2\lambda(x \pm 1)$ is a square in $\mathbb{N}$ such that $\lambda \neq 1$ and d. So, $\begin{cases} x \pm 1 = 2\lambda z_1^2 \\ x \mp 1 = 2\lambda' z_2^2 \end{cases}$, where $2z_1z_2 = y$, $x = \lambda z_1^2 + \lambda' z_2^2$ and $\lambda\lambda' = d$, then $\sqrt{\epsilon} = z_1\sqrt{\lambda} + z_2\sqrt{\lambda'}$. Thus, we have $\sqrt{\lambda\epsilon} = z_1\lambda + z_2\sqrt{\lambda\lambda'} \in \mathbb{Q}(\sqrt{d})$.

Conversely, assume that $\sqrt{\lambda\epsilon} \in \mathbb{Q}(\sqrt{d})$ where $\lambda \neq 1$ and d. Then, $\sqrt{\lambda\epsilon} = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$. By squaring we obtain $\lambda\epsilon = a^2 + b^2 d + 2ab\sqrt{d}$. On the other hand, we have $\epsilon = x + y\sqrt{d}$, and thus we get that $\lambda\epsilon = \lambda x + \lambda y\sqrt{d} = a^2 + b^2 d + 2ab\sqrt{d}$, implying that $\lambda x = a^2 + b^2 d = a^2 + b^2\lambda\lambda'$ and $\lambda y = 2ab$. So, we have $x = \frac{a}{\lambda}^2 + b^2\lambda' = \lambda a_1^2 + \lambda' b^2$ with $a_1 = \frac{a}{\lambda}$ and $y = \frac{2ab}{\lambda} = 2a_1 b$.
Since $x^2 - 1 = (x \pm 1)(x \mp 1) = y^2 d = (2a_1 b)^2 d$, we get that $x \pm 1 = 2a_1^2\lambda$ (also $x \mp 1 = 2b^2\lambda'$), which means $2\lambda(x \pm 1)$ is a square in $\mathbb{N}$ with $\lambda \neq 1$ and d.

(2) For the second equivalent, let us start by the direct implication. Suppose that $\lambda(x \pm 1)$ is a square in $\mathbb{N}$ such that $\lambda \neq 2$. So, we have $\begin{cases} x \pm 1 = \lambda z_1^2 \\ x \mp 1 = \lambda' z_2^2 \end{cases}$, where $z_1z_2 = y$, $2x = \lambda z_1^2 + \lambda' z_2^2$ and $\lambda\lambda' = d$. Then, $\sqrt{2\epsilon} = z_1\sqrt{\lambda} + z_2\sqrt{\lambda'}$. So, $\sqrt{2\lambda\epsilon} = z_1\lambda + z_2\sqrt{\lambda\lambda'} \in \mathbb{Q}(\sqrt{d})$.

Conversely, assume that $\sqrt{2\lambda\epsilon} \in \mathbb{Q}(\sqrt{d})$ such that $\lambda \neq 2$, then $\sqrt{2\lambda\epsilon} = a' + b'\sqrt{d} \in \mathbb{Q}(\sqrt{d})$. So, $2\lambda\epsilon = a'^2 + b'^2 d + 2a'b'\sqrt{d}$. We have

$\epsilon = x + y\sqrt{d}$. Therefore, $2\lambda\epsilon = 2\lambda x + 2\lambda y\sqrt{d} = a'^2 + b'^2 d + 2a'b'\sqrt{d}$. Thence, $2\lambda x = a'^2 + b'^2 d = a'^2 + b'^2\lambda\lambda'$ and $\lambda y = a'b'$. Thus, we get $2x = \frac{a'}{\lambda}^2 + b'^2\lambda' = \lambda a_1'^2 + \lambda' b'^2$ with $a_1' = \frac{a'}{\lambda}$ and $y = \frac{a'b'}{\lambda} = a_1' b'$. As $x^2 - 1 = (x \pm 1)(x \mp 1) = y^2 d = (a_1' b')^2 d$, we have $x \pm 1 = a_1'^2 \lambda$ (as well $x \mp 1 = b'^2\lambda$), which means $\lambda(x \pm 1)$ is a square in $\mathbb{N}$ with $\lambda \neq 2$. □

In particular, by setting $\lambda = 1$ in the second assertion of Lemma 2.11, we get that: $(x \pm 1)$ is a square in $\mathbb{N}$ if and only if $\sqrt{2\epsilon}$ is in $\mathbb{Q}(\sqrt{d})$.

In Propositions 2.12 and 2.13, we give the conditions in which the units of K are squares. We provide another way of expressing what was given by Louboutin in [7, Corollary 3.2].

Proposition 2.12. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where $d_1 > 1$ and $d_2 > 1$ are two square-free integers and $(d_1, d_2) = 1$ and $d_3 = d_1 d_2$. Let $\epsilon_i = x_i + y_i\sqrt{d_i}$ be the fundamental unit of $k_i = \mathbb{Q}(\sqrt{d_i})$ where $N\epsilon_i = 1$ for $i = 1, 2, 3$. Then:*

(1) *$\sqrt{\epsilon_3} \in K$ if and only if $2d_1(x_3 \pm 1)$ is a square in $\mathbb{N}$.*
(2) *$\sqrt{\epsilon_1\epsilon_2} \in K$ if and only if $(x_1 \pm 1)$ and $(x_2 \pm 1)$ are squares in $\mathbb{N}$.*
(3) *$\sqrt{\epsilon_j\epsilon_3} \in K$ for $j = 1, 2$ if and only if one of the following cases holds:*
   (a) *$\lambda_j(x_j \pm 1)$ and $\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_j \neq 2$ and $\lambda_3 \neq 2$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$.*
   (b) *$2\lambda_j(x_j \pm 1)$ and $2\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_j \neq (1, d_j)$ and $\lambda_3 \neq (1, d_3)$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$.*

Proof. We start by the first equivalent.

(1) We apply Lemma 2.11 and we get that: $2d_1(x_3 \pm 1)$ is a square in $\mathbb{N}$ if and only if $\sqrt{d_1\epsilon_3} \in k_3 = \mathbb{Q}(\sqrt{d_1 d_2})$, which means that $\sqrt{\epsilon_3} \in K$.
(2) For the second equivalent:
   $(x_1 \pm 1)$ and $(x_2 \pm 1)$ are squares in $\mathbb{N}$ if and only if $\sqrt{2\epsilon_1} \in k_1 = \mathbb{Q}(\sqrt{d_1})$ and $\sqrt{2\epsilon_2} \in k_2 = \mathbb{Q}(\sqrt{d_2})$ (see Lemma 2.11). As a result $\sqrt{2\epsilon_1}\sqrt{2\epsilon_2} = 2\sqrt{\epsilon_1\epsilon_2} \in K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.
(3) For the third equivalent:
   Let us start conversely. We have the following items.
   (a) For the first item we have $\lambda_j(x_j \pm 1)$ and $\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_j \neq 2$ and $\lambda_3 \neq 2$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$ for $j \in \{1, 2\}$, then $\sqrt{2\lambda_j\epsilon_j} \in k_j = \mathbb{Q}(\sqrt{d_j})$ and $\sqrt{2\lambda_3\epsilon_3} \in k_3 = \mathbb{Q}(\sqrt{d_3})$ for $j = 1, 2$. So, we obtain $\sqrt{2\lambda_j\epsilon_j}\sqrt{2\lambda_3\epsilon_3} = 2\sqrt{\lambda_j\lambda_3\epsilon_j\epsilon_3} \in K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $\lambda_j \neq 2$ and $\lambda_3 \neq 2$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$ for $j = 1, 2$. So, we get that $\sqrt{\epsilon_j\epsilon_3} \in K$ for $j = 1, 2$.
   (b) Now, $2\lambda_j(x_j \pm 1)$ and $2\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_j \neq (1, d_j)$ and $\lambda_3 \neq (1, d_3)$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$, therefore $\sqrt{\lambda_j\epsilon_j} \in \mathbb{Q}(\sqrt{d_j})$ and $\sqrt{\lambda_3\epsilon_3} \in \mathbb{Q}(\sqrt{d_3})$ for $j = 1, 2$. Then, we get that $\sqrt{\lambda_j\epsilon_j}\sqrt{\lambda_3\epsilon_3} = \sqrt{\lambda_j\lambda_3\epsilon_j\epsilon_3} \in K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, such that $\lambda_j \neq (1, d_j)$ and $\lambda_3 \neq (1, d_3)$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$ with $j = 1, 2$. So, $\sqrt{\epsilon_j\epsilon_3} \in K$ for $j = 1, 2$.

   To prove the direct implication, we use the contrapositive. We know that: P implies (Q or R), equivalent to Not (Q or R) implies Not

P, equivalent to (Not Q and Not R) implies Not P, equivalent to Not Q implies Not P and Not R implies Not P.
Let P be $\sqrt{\epsilon_j\epsilon_3} \in K$ for $j = 1, 2$. Let Q be (a) in the proposition. Let R be (b) in the proposition.
We have Not Q as follow: $\lambda_j(x_j \pm 1)$ or $\lambda_3(x_3 \pm 1)$ is not a square in $\mathbb{N}$, where $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$ with $j = 1, 2$.
By the above lemma we get that:
$\sqrt{2\lambda_j\epsilon_j} \notin k_j$ or $\sqrt{2\lambda_3\epsilon_3} \notin k_3$, such that $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$ with $j = 1, 2$.
Therefore, we get that $\sqrt{\epsilon_j\epsilon_3} \notin K$, with $j = 1, 2$. Hence, we proved that Not Q implies Not P.
Similarly to the above, we prove that Not R implies Not P, therefore we complete the proof of the proposition.

□

Proposition 2.13. *Let* $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ *where* $(d_1, d_2) = 1$ *and* $d_3 = d_1d_2$. *Let* $\epsilon_i = x_i + y_i\sqrt{d_i}$ *be the fundamental unit of* $k_i = \mathbb{Q}(\sqrt{d_i})$ *where* $N\epsilon_i = 1$ *for* $i = 1, 2, 3$. *Then,* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ *if and only if one of the following assertions is satisfied:*

(1) $(x_j \pm 1)$ *and* $\lambda_k(x_k \pm 1)$ *and* $2\lambda_3(x_3 \pm 1)$ *are squares in* $\mathbb{N}$, *where* $\lambda_k \neq (2, d_k)$ *and* $\lambda_3 \neq (1, d_3)$ *and either* $[\lambda_k\lambda_3] = [d_k]$ *or* $[d_3]$, *or* $\lambda_k = \lambda_3$ *with* $j \neq k = 1, 2$.
(2) $2\lambda_j(x_j \pm 1)$ *and* $(x_k \pm 1)$ *and* $\lambda_3(x_3 \pm 1)$ *are squares in* $\mathbb{N}$, *such that* $\lambda_j \neq (1, d_j)$ *and* $\lambda_3 \neq (2, d_3)$ *and either* $[\lambda_j\lambda_3] = [d_j]$ *or* $[d_3]$, *or* $\lambda_j = \lambda_3$ *with* $j \neq k = 1, 2$.
(3) $2\lambda_1(x_1 \pm 1)$ *and* $2\lambda_2(x_2 \pm 1)$ *and* $2\lambda_3(x_3 \pm 1)$ *are squares in* $\mathbb{N}$, *where* $\lambda_i \neq (1, d_i)$ *for* $i = 1, 2, 3$ *and either* $[\lambda_1\lambda_2\lambda_3] = [d_3]$, *or* $\lambda_1\lambda_2 = \lambda_3$.

Proof. Let's start conversely.

(1) When $(x_j \pm 1)$ and $\lambda_k(x_k \pm 1)$ and $2\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_k \neq (2, d_k)$ and $\lambda_3 \neq (1, d_3)$ and either $[\lambda_k\lambda_3] = [d_k]$ or $[d_3]$, or $\lambda_k = \lambda_3$ with $j \neq k = 1, 2$, so $\sqrt{2\epsilon_j} \in \mathbb{Q}(\sqrt{d_j})$ and $\sqrt{2\lambda_k\epsilon_k} \in \mathbb{Q}(\sqrt{d_k})$ and $\sqrt{\lambda_3\epsilon_3} \in \mathbb{Q}(\sqrt{d_3})$ (see Lemma 2.11), then $\sqrt{2\epsilon_j} \times \sqrt{2\lambda_k\epsilon_k} \times \sqrt{\lambda_3\epsilon_3} = 2\sqrt{\lambda_k\lambda_3}\sqrt{\epsilon_j\epsilon_k\epsilon_3} \in K$ such that $j \neq k = 1, 2$. As we have either $[\lambda_k\lambda_3] = [d_k]$ or $[d_3]$, or $\lambda_k = \lambda_3$ with $j \neq k\{1, 2\}$, then $\sqrt{\epsilon_j\epsilon_k\epsilon_3} = \sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$.
(2) When $2\lambda_j(x_j \pm 1)$ and $(x_k \pm 1)$ and $\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, such that $\lambda_j \neq (1, d_j)$ and $\lambda_3 \neq (2, d_3)$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$ with $j \neq k = 1, 2$, therefore $\sqrt{\lambda_j\epsilon_j} \in \mathbb{Q}(\sqrt{d_j})$ and $\sqrt{2\epsilon_k} \in \mathbb{Q}(\sqrt{d_k})$ and $\sqrt{2\lambda_3\epsilon_3} \in \mathbb{Q}(\sqrt{d_3})$ so we get that $2\sqrt{\lambda_j\lambda_3}\sqrt{\epsilon_j\epsilon_k\epsilon_3} \in K$. Since either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$, then $\sqrt{\epsilon_j\epsilon_k\epsilon_3} = \sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$.
(3) When $2\lambda_1(x_1 \pm 1)$ and $2\lambda_2(x_2 \pm 1)$ and $2\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_i \neq (1, d_i)$ for $i = 1, 2, 3$ and either $[\lambda_1\lambda_2\lambda_3] = [d_3]$, or $\lambda_1\lambda_2 = \lambda_3$, then $\sqrt{\lambda_i\epsilon_i} \in \mathbb{Q}(\sqrt{d_i})$ for $i = 1, 2, 3$. Thence, we obtain $\sqrt{\lambda_1\epsilon_1}\sqrt{\lambda_2\epsilon_2}\sqrt{\lambda_3\epsilon_3} = \sqrt{\lambda_1\lambda_2\lambda_3}\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ and thus $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$.

Now for proving the direct implication, we use the same process of Proposition 2.12. Let P be $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$. Let Q, R, and S be (1), (2), and (3) in

the proposition stated above, respectively. It is well known that P implies (Q or R or S), is equivalent to Not Q implies Not P, and Not R implies Not P, and Not S implies Not P.
We have Not Q as follow: $(x_j \pm 1)$ or $\lambda_i(x_i \pm 1)$ or $2\lambda_3(x_3 \pm 1)$ is not a square in $\mathbb{N}$, where $[\lambda_i\lambda_3] = [d_i]$ or $[d_3]$, or $\lambda_i = \lambda_3$ with $j \neq i = 1, 2$, therefore we get that $\sqrt{2\epsilon_j} \notin k_j$ or $\sqrt{2\lambda_i\epsilon_i} \notin k_i$ or $\sqrt{\lambda_3\epsilon_3} \notin k_3$, and thus we find that $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$. Then, we have Not Q implies Not P. Similarly, we get that Not R implies Not P, and Not S implies Not P.

□

Example 2.14. *Let $K = \mathbb{Q}(\sqrt{23}, \sqrt{35})$ such that $d_1 = 23$, $d_2 = 5 \cdot 7 = 35$ and $d_3 = 35 \cdot 23 = 805$. So, we have $\epsilon_1 = 24 + 5\sqrt{23}$, $\epsilon_2 = 6 + \sqrt{35}$ and $\epsilon_3 = \frac{1}{2}(1447 + 51\sqrt{805})$.*
We have $(24 + 1) = 5^2$, and $5(6 - 1) = 5^2$ (here we have $\lambda_2 = 5$), and then $2 \cdot 7 \cdot 23(\frac{1447+2}{2}) = 3^2 \cdot 7^2 \cdot 23^2$ are squares (note that here we have $\lambda_3 = 7 \cdot 23$). Since $5 \cdot 7 \cdot 23 = \lambda_2\lambda_3 = d_3$, we get that $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$.
Furthermore, by the Proof of Lemma 2.11, we get that $\sqrt{2\epsilon_1} = 5 + \sqrt{23} \in \mathbb{Q}(\sqrt{23})$ and $\sqrt{2 \cdot 5\epsilon_2} = 5 + \sqrt{35} \in \mathbb{Q}(\sqrt{35})$ and then $\sqrt{7 \cdot 23\epsilon_3} = \frac{3}{2} \cdot 7 \cdot 23 + \frac{17}{2}\sqrt{805} \in \mathbb{Q}(\sqrt{805})$.

Remark 2.15. *Let $\epsilon_i = x_i + y_i\sqrt{d_i}$ be the fundamental unit of $k_i = \mathbb{Q}(\sqrt{d_i})$ and we let $a_i = N(\epsilon_i + 1) = 2(x_i + 1)$ for $i = 1, 2, 3$ where $N\epsilon_i = 1$ for $i = 1, 2, 3$. Recall that $[a_i]$ is the class of a_i in $\mathbb{Q}^*/\mathbb{Q}^{*2}$, and $[d_i]$ is the class of d_i in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ for $i = 1, 2, 3$.*

1. *When $(x_i \pm 1)$ is a square in $\mathbb{N}$, then $[a_i] = [2]$ or $[2d_i]$ with $i = 1, 2, 3$.*
2. *When $\lambda_i(x_i \pm 1)$ such that $\lambda_i \neq (1, 2$ and $d_i)$ is a square in $\mathbb{Q}$, then $[a_i] = [2\lambda_i]$ or $[2\lambda_i']$ with $\lambda_i\lambda_i' = d_i$ for $i = 1, 2, 3$.*
3. *When $2\lambda_i(x_i \pm 1)$ is a square in $\mathbb{N}$ such that $\lambda_i \neq (1$ and $d_i)$, then $[a_i] = [\lambda_i]$ or $[\lambda_i']$ where $\lambda_i\lambda_i' = d_i$ for $i = 1, 2, 3$.*

We now use Propositions 2.12 and 2.13 in the following proposition.
Define $\langle[c_1], [c_2], [c_3], [c_4], [c_5]\rangle$ to be the group generated by $[c_1]$ $[c_2]$, $[c_3]$, $[c_4]$, and $[c_5]$.

Proposition 2.16. *Let $K = \mathbb{Q}(\sqrt{d}_1, \sqrt{d}_2)$ where $(d_1, d_2) = 1$, and let $\epsilon_i = x_i + y_i\sqrt{d_i}$ be the fundamental unit of $k_i = \mathbb{Q}(\sqrt{d_i})$ where $N\epsilon_i = 1$ and we let $a_i = 2(x_i + 1)$ for $i = 1, 2, 3$.*

1. *When $\sqrt{\epsilon_3} \in K$, then $[a_3] \in \langle[d_1], [d_2]\rangle$.*
2. *Let $\sqrt{\epsilon_1\epsilon_2} \in K$, so $[a_1]$ and $[a_2] \in \langle[d_1], [d_2], [2]\rangle$.*
3. *Assuming that $\sqrt{\epsilon_j\epsilon_3} \in K$ for $j \in \{1, 2\}$, then $[a_3] \in \langle[d_1], [d_2], [a_j]\rangle$.*
4. *When $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, we get that $[a_3] \in \langle[d_1], [d_2], [a_1], [a_2]\rangle$.*

Proof. (1) As $\epsilon_3 = x_3 + y_3\sqrt{d_3}$ is the fundamental unit of $k_3 = \mathbb{Q}(\sqrt{d_3})$ and $N\epsilon_3 = 1$, then $(x_3 \pm 1)(x_3 \mp 1) = y_3^2 d_3$.
By Proposition 2.12, when $\sqrt{\epsilon_3} \in K$ so $2d_1(x_3 \pm 1)$ is a square in $\mathbb{N}$, i.e.,

$$\begin{cases} x_3 + 1 = 2d_1t_3^2 \\ x_3 - 1 = 2d_2t_3'^2 \end{cases} \text{or} \begin{cases} x_3 - 1 = 2d_1t_3^2 \\ x_3 + 1 = 2d_2t_3'^2 \end{cases}$$

where $2t_3t_3' = y_3$ and $d_1d_2 = d_3$, implies that $[a_3] = [2(x_3 + 1)] = [d_1]$ or $[d_2] \in \langle[d_1], [d_2]\rangle$.

(2) According to Proposition 2.12, we get that when $\sqrt{\epsilon_1\epsilon_2} \in K$ then $(x_1 \pm 1)$ and $(x_2 \pm 1)$ are squares in $\mathbb{N}$. Then, $[a_1] = [2(x_1+1)] = [2]$ or $[2d_1]$ and $[a_2] = [2(x_2+1)] = [2]$ or $[2d_2]$ (see Remark 2.15). Thus, $[a_1]$ and $[a_2] \in \langle [d_1], [d_2], [2] \rangle$.

(3) Now for the third assertion. We have when $\sqrt{\epsilon_j\epsilon_3} \in K$ for $j \in \{1,2\}$, by Proposition 2.12, we get that one of the following items is satisfied:

(a) $\lambda_j(x_j \pm 1)$ and $\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_j \neq 2$ and $\lambda_3 \neq 2$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$, then $[a_j] = [2\lambda_j] \notin \langle [d_1], [d_2] \rangle$ and $[a_3] = [2\lambda_3] \notin \langle [d_1], [d_2] \rangle$. Therefore, either $[a_3] = [2\lambda_3] = [2\lambda_j][d_i] = [a_j][d_i]$, or $[a_3] = [a_j]$ with $i \in \{1,2,3\}$ and $j \in \{1,2\}$. Thus, we get that $[a_3] \in \langle [d_1], [d_2], [a_j] \rangle$ with $j \in \{1,2\}$.

(b) $2\lambda_j(x_j \pm 1)$ and $2\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_j \neq (1, d_j)$ and $\lambda_3 \neq (1, d_3)$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$, therefore we get that $[a_j] = [\lambda_j] \notin \langle [d_1], [d_2] \rangle$ and $[a_3] = [\lambda_3] \notin \langle [d_1], [d_2] \rangle$. So, either $[a_3] = [a_j][d_i]$ with $i = 1,2,3$, or $[a_3] = [a_j]$ with $j \in \{1,2\}$. Consequently, we obtain $[a_3] \in \langle [d_1], [d_2], [a_j] \rangle$ with $j \in \{1,2\}$.

(4) According to Proposition 2.13, we get that when $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, then one of the following cases holds.

(a) $(x_j \pm 1)$ and $\lambda_k(x_k \pm 1)$ and $2\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_k \neq (2, d_k)$ and $\lambda_3 \neq (1, d_3)$ and either $[\lambda_k\lambda_3] = [d_k]$ or $[d_3]$, or $\lambda_k = \lambda_3$ with $j \neq k = 1,2$.
Then, we find that $[a_j] = [2(x_j+1)] = [2]$ or $[2d_j]$, and $[a_k] = [2\lambda_k]$ and $[a_3] = [\lambda_3]$. So, we have $[a_j] \in \langle [d_1], [d_2], [2] \rangle$ and we get either $[a_3] = [a_j][a_k][d_i]$, or $[a_3] = [a_j][a_k]$ with $i \in \{1,2,3\}$ and $j \neq k \in \{1,2\}$. Consequently, we have $[a_3] \in \langle [d_1], [d_2], [a_1], [a_2] \rangle$.

(b) $2\lambda_j(x_j \pm 1)$ and $(x_k \pm 1)$ and $\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, such that $\lambda_j \neq (1, d_j)$ and $\lambda_3 \neq (2, d_3)$ and either $[\lambda_j\lambda_3] = [d_j]$ or $[d_3]$, or $\lambda_j = \lambda_3$ with $j \neq k = 1,2$.
So, we get that $[a_j] = [2(x_j+1)] = [\lambda_j]$ and $[a_k] = [2]$ or $[2d_k]$ and $[a_3] = [2\lambda_3]$. Thus, we get that $[a_k] \in \langle [d_1], [d_2], [2] \rangle$ and either $[a_3] = [2\lambda_3] = [a_k][a_j][d_i]$, or $[a_3] = [a_k][a_j]$ with $i \in \{1,2,3\}$ and $j \neq k \in \{1,2\}$. As a result, $[a_3] \in \langle [d_1], [d_2], [a_1], [a_2] \rangle$.

(c) $2\lambda_1(x_1 \pm 1)$ and $2\lambda_2(x_2 \pm 1)$ and $2\lambda_3(x_3 \pm 1)$ are squares in $\mathbb{N}$, where $\lambda_i \neq (1, d_i)$ for $i = 1,2,3$ and either $[\lambda_1\lambda_2\lambda_3] = [d_3]$, or $\lambda_1\lambda_2 = \lambda_3$. Then, we get that $[a_i] = [2(x_i+1)] = [\lambda_i] \notin \langle [d_1], [d_2] \rangle$ for $i = 1,2,3$. Since we have $[\lambda_1\lambda_2\lambda_3] = [d_3]$ or $\lambda_1\lambda_2 = \lambda_3$, then we obtain either: $[a_3] = [a_1][a_2][d_3]$, or $[a_3] = [a_1][a_2]$. So, $[a_3] \in \langle [d_1], [d_2], [a_1], [a_2] \rangle$.

□

3. The first cohomology group of units of some fields of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $(d_1, d_2) = 1$ and $e_2 \neq 4$

When we say that the prime 2 is not totally ramified in $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ over $\mathbb{Q}$, then we have $(d_1, d_2) \equiv (1,1), (1,2), (2,1), (1,3), (3,1)$ or $(3,3)$

(mod 4), in other words $(d_1, d_2) \not\equiv (2, 3)$ and $(3, 2) \pmod 4$. It is well-known that when we have either $N\epsilon_1 \neq N\epsilon_2 = N\epsilon_3 = 1$, $N\epsilon_2 \neq N\epsilon_1 = N\epsilon_3 = 1$ or $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$, then we get either $e_2 = 4$ or $e_2 \neq 4$. In the theorem below, we give the first cohomology group of units of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $(d_1, d_2) = 1$ and $e_2 \neq 4$.

Theorem 3.1. *Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where $d_1 > 1$ and $d_2 > 1$ are two square-free integers with $(d_1, d_2) = 1$. Then*

(1) $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^2$. *If*
 (a) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ *or*
 (b) $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_3} \in K$.
(2) $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^3$. *When*
 (a) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$
 (b) $N\epsilon_1 = N\epsilon_2 = -1$, $N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_3} \notin K$
 (c) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = 1$ *and either* $\sqrt{\epsilon_3} \in K$ *or* $\sqrt{\epsilon_k\epsilon_3} \in K$, *where* $e_2 \neq 4$ *and* $j \neq k\{1, 2\}$
 (d) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = -1$ *with* $j \neq k\{1, 2\}$
 (e) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and* $\sqrt{\epsilon_1\epsilon_2} \in K$ *and* $\sqrt{\epsilon_1\epsilon_3} \in K$ *and* $\sqrt{\epsilon_2\epsilon_3} \in K$ *where* $e_2 \neq 4$
(3) $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^4$. *If*
 (a) $N\epsilon_1 = N\epsilon_2 = 1$, $N\epsilon_3 = -1$
 (b) $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = 1$, $\sqrt{\epsilon_3} \notin K$ *and* $\sqrt{\epsilon_k\epsilon_3} \notin K$, *where* $e_2 \neq 4$ $j \neq k\{1, 2\}$ *or*
 (c) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ *and either* $\sqrt{\epsilon_3} \in K$, $\sqrt{\epsilon_1\epsilon_2} \in K$, $\sqrt{\epsilon_1\epsilon_3} \in K$, $\sqrt{\epsilon_2\epsilon_3} \in K$ *or* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ *where* $e_2 \neq 4$.
(4) $H^1(G_K, E_K) \simeq (\mathbb{Z}/2\mathbb{Z})^5$. *When*
 (a) $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$, $\sqrt{\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$, $\sqrt{\epsilon_2\epsilon_3} \notin K$ *and* $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ *where* $e_2 \neq 4$.

Proof. Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where $(d_1, d_2) = 1$. Recall that $\widetilde{H}$ is the subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ generated by $[d_1], [d_2], [d_2], [a_1], [a_2]$ and $[a_3]$ such that $d_3 = d_1d_2$ and $a_i = N(\epsilon_i + 1) = 2(x_i + 1)$ for $i = 1, 2, 3$ when $N\epsilon_i = 1$ otherwise $a_i = 1$ for $i = 1, 2, 3$. We study whether $[d_1], [d_2], [d_3], [a_1], [a_2]$ and $[a_3]$ are linearly independent. Since we have $(d_1, d_2) = 1$, and $d_1d_2 = d_3$ so $[d_3] \in \langle [d_1], [d_2] \rangle$, then $\langle [d_1], [d_2] \rangle$ is the group generated by $[d_1]$ and $[d_2]$.

(1) When $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, then $[a_1] = [a_2] = [a_3] = 1$. So, $\widetilde{H} = \langle [d_1], [d_2] \rangle$ i.e., $\widetilde{H} \simeq (\mathbb{Z}/2\mathbb{Z})^2$. As $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = -1$, then we have to distinguish the two following cases:
 (a) when $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$, by Theorem 2.1, we get that $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^2$.
 (b) otherwise, i.e., $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$, by Theorem 2.1, we get that $H \simeq (\mathbb{Z}/2\mathbb{Z})^3$.
(2) If $N\epsilon_1 = N\epsilon_2 = -1$ and $N\epsilon_3 = 1$, then $[a_1] = [a_2] = 1$. As $N\epsilon_3 = 1$, then we have either $\sqrt{\epsilon_3} \in K$, or $\sqrt{\epsilon_3} \notin K$.
 (a) When $\sqrt{\epsilon_3} \in K$, according to Proposition 2.16, we have $[a_3] \in \langle [d_1], [d_2] \rangle$. Therefore, $\widetilde{H} = \langle [d_1], [d_2] \rangle$ and thus we get that $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

(b) Otherwise, then $[a_3] \notin \langle [d_1], [d_2] \rangle$, so $\widetilde{H} = \langle [d_1], [d_2], [a_3] \rangle$. And thus, we get that $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$.

(3) When $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = 1$ where $j \neq k \in \{1,2\}$ and $e_2 \neq 4$, then $[a_j] = 1$. As $N\epsilon_k = N\epsilon_3 = 1$ with $k \in \{1,2\}$, then we have to distinguish the three following cases.

(a) If $\sqrt{\epsilon_3} \in K$, by Proposition 2.16, we have $[a_3] \in \langle [d_1], [d_2] \rangle$. On the other hand, we have $[a_k] \notin \langle [d_1], [d_2] \rangle$ (see Remark 2.15). Thence, $\widetilde{H} = \langle [d_1], [d_2], [a_k] \rangle$. As a result, we get that $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$.

(b) When $\sqrt{\epsilon_k \epsilon_3} \in K$, such that $k \in \{1,2\}$, then according to Proposition 2.16, we have $[a_3] \in \langle [d_1], [d_2], [a_k] \rangle$ with $k \in \{1,2\}$. So, $\widetilde{H} = \langle [d_1], [d_2], [a_k] \rangle$ and thus we have $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$.

(c) Otherwise, i.e., $\sqrt{\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_3} \notin K$ and $\sqrt{\epsilon_2\epsilon_3} \notin K$, then $[a_3] \notin \langle [d_1], [d_2], [a_k] \rangle$ with $k \in \{1,2\}$. Thus, $\widetilde{H} = \langle [d_1], [d_2], [a_k], [a_3] \rangle$ such that $k \in \{1,2\}$. Consequently, we get that $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

(4) When $N\epsilon_j \neq N\epsilon_k = N\epsilon_3 = -1$ such that $j \neq k = 1, 2$, then $[a_k] = [a_3] = 1$ and $[a_j] \notin \langle [d_1], [d_2] \rangle$ (note that we can have either $[a_j] = [2]$ or $[2d_j]$, or $[a_j] = [2\lambda_j]$ such that $\lambda_j \neq (1, 2$ and $d_j)$ or $[a_j] = [\lambda_j]$ where $\lambda_j \neq (1$ and $d_j)$ with $\lambda_j \lambda_j' = d_j$ for $j \in \{1,2\}$, see Remark 2.15) and thus $\widetilde{H} = \langle [d_1], [d_2], [a_j] \rangle$. So, $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^3$.

(5) Suppose $N\epsilon_1 = N\epsilon_2 = 1$ and $N\epsilon_3 = -1$. Then, $[a_3] = 1$ and d_1 or $d_2 \equiv 1 \pmod 4$ such that all prime divisors of d_1, d_2 and d_3 are not congruent to $3 \pmod 4$. By using Lemma 2.3, we get that $[(x_1+1), (x_1-1), p(x_1+1)$, and $p(x_1-1)]$ are not squares for all prime p dividing d_1 or $[(x_2+1), (x_2-1), q(x_2+1)$, and $q(x_2-1)]$ are not squares for all prime q dividing d_2. And thus we have $[a_1] = [\lambda_1]$, with $\lambda_1 \neq (1, d_1)$ or $[a_2] = [\lambda_2]$, with $\lambda_2 \neq (1, d_2)$ (see Remark 2.15). Hence, $[a_j] \notin \langle [d_1], [d_2], [a_k] \rangle$ with $j \neq k \in \{1,2\}$, therefore we get that $\widetilde{H} = \langle [d_1], [d_2], [a_1], [a_2] \rangle$. Thus, $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

(6) If $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ where $e_2 \neq 4$, then we need to distinguish the following cases.

(a) If $\sqrt{\epsilon_3} \in K$, according to Proposition 2.16, we have $[a_3] \in \langle [d_1], [d_2] \rangle$. We mention that $\sqrt{\epsilon_1\epsilon_2} \notin K$ since $\{\epsilon_1, \epsilon_2, \sqrt{\epsilon_3}\}$ is the system of fundamental units of E_K (see Proposition 2.2). So, $[a_k] \notin \langle [d_1], [d_2], [a_j] \rangle$, $j \neq k \in \{1,2\}$, therefore $\widetilde{H} = \langle [d_1], [d_2], [a_1], [a_2] \rangle$. By Theorem 2.1, we have $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

(b) When $\sqrt{\epsilon_1\epsilon_2} \in K$, by using Proposition 2.16, we get $[a_j] \in \langle [d_1], [d_2], [2] \rangle$ with $j = 1, 2$ (taking into account that $\{\sqrt{\epsilon_1\epsilon_2}, \epsilon_2, \epsilon_3\}$ is the system of fundamental units of E_K). Note that $\sqrt{\epsilon_j\epsilon_3} \notin K$ for $j = 1, 2$ which means that $[a_3] \notin \langle [d_1], [d_2], [a_j] \rangle$. Therefore, $\widetilde{H} = \langle [d_1], [d_2], [a_j], [a_3] \rangle$ where $j = 1, 2$. Thus, we get that $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

(c) If $\sqrt{\epsilon_j\epsilon_3} \in K$, $j = 1, 2$. By Proposition 2.16, we find that $[a_3] \in \langle [d_1], [d_2], [a_j] \rangle$ with $j = 1, 2$. On the other hand, we have $\sqrt{\epsilon_k\epsilon_j} \notin K$ with $j \neq k = 1, 2$, (note that $\{\epsilon_1, \epsilon_2, \sqrt{\epsilon_j\epsilon_3}\}$ is the system of fundamental units of E_K (see Proposition 2.2)).

Hence $[a_k] \notin \langle [d_1],[d_2],[a_j]\rangle$, $j \neq k \in \{1,2\}$. Then, $\widetilde{H} = \langle [d_1],[d_2],[a_k],[a_j]\rangle$, $j \neq k \in \{1,2\}$. Thence, we have $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

(d) If $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \in K$ (here we have $\{\epsilon_1, \epsilon_2, \sqrt{\epsilon_1\epsilon_2\epsilon_3}\}$ is the system of fundamental units of E_K), we get that $[a_3] \in \langle [d_1],[d_2],[a_1],[a_2]\rangle$. Thus, $\widetilde{H} = \langle [d_1],[d_2],[a_1],[a_2]\rangle$. By Theorem 2.1, we get that $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^4$.

(e) Otherwise, i.e., $\sqrt{\epsilon_3} \notin K$, $\sqrt{\epsilon_1\epsilon_2} \notin K$, $\sqrt{\epsilon_1\epsilon_3} \notin K$, $\sqrt{\epsilon_2\epsilon_3} \notin K$ and $\sqrt{\epsilon_1\epsilon_2\epsilon_3} \notin K$ (here we have $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ is the system of fundamental units of E_K), then we get that $\widetilde{H} \simeq H \simeq (\mathbb{Z}/2\mathbb{Z})^5$.

(7) When $N\epsilon_1 = N\epsilon_2 = N\epsilon_3 = 1$ and $\sqrt{\epsilon_1\epsilon_2} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$ and $\sqrt{\epsilon_2\epsilon_3} \in K$ where $e_2 \neq 4$, (Note that in this case $\{\sqrt{\epsilon_1\epsilon_2}, \sqrt{\epsilon_2\epsilon_3}, \sqrt{\epsilon_1\epsilon_3}\}$ is the system of fundamental units of E_K), we have to study the three following cases; $\sqrt{\epsilon_1\epsilon_2} \in K$, $\sqrt{\epsilon_2\epsilon_3} \in K$ and $\sqrt{\epsilon_1\epsilon_3} \in K$. By Proposition 2.16, we know that when $\sqrt{\epsilon_1\epsilon_2} \in K$, then $[a_k] \in \langle [d_1],[d_2],[a_j]\rangle$ with $j \neq k = 1, 2$ and when $\sqrt{\epsilon_j\epsilon_3} \in K$ so $[a_3] \in \langle [d_1],[d_2],[a_j]\rangle$, $j = 1, 2$. As a result, $H^1(G_K, E_K) \simeq \widetilde{H} \simeq E_3$.

□

Now we give some examples of the first cohomology group of units of $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where $(d_1, d_2) = 1$ and the prime 2 is not totally ramified in $K/\mathbb{Q}$.

Example 3.2. *Let $K = \mathbb{Q}(\sqrt{10}, \sqrt{77})$ such that $d_1 = 2 \cdot 5 = 10$ and $d_2 = 7 \cdot 11 = 77$ and then $d_3 = 2 \cdot 5 \cdot 7 \cdot 11 = 770$. We have $\epsilon_1 = 3 + \sqrt{10}$ and $\epsilon_2 = \frac{1}{2}(9 + \sqrt{77})$, and then $\epsilon_3 = 111 + 4\sqrt{770}$ such that $N\epsilon_1 \neq N\epsilon_2 = N\epsilon_3 = 1$. So, $a_1 = 1$ and $a_2 = 2(\frac{9}{2} + 1) = 11$, and then $a_3 = 2(111 + 1) = 2^5 \cdot 7$. Hence, $\widetilde{H} \simeq H \simeq \langle [2 \cdot 5], [7 \cdot 11], [11], [2 \cdot 7]\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^4$.*

Example 3.3. *Let $K = \mathbb{Q}(\sqrt{78}, \sqrt{145})$ where $d_1 = 2 \cdot 3 \cdot 13 = 78$ and $d_2 = 5 \cdot 29 = 145$, and then $d_3 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 29 = 11310$. We have $\epsilon_1 = 53 + 6\sqrt{78}$ and $\epsilon_2 = 12 + \sqrt{145}$, and then $\epsilon_3 = 7019 + 66\sqrt{11310}$ such that $N\epsilon_1 = N\epsilon_3 = 1$ and $N\epsilon_2 = -1$. So, $a_1 = 2(53 + 1) = 2^2 \cdot 3^3$ and $a_2 = 1$, and then $a_3 = 2(7019 + 1) = 2^3 \cdot 3^3 \cdot 5 \cdot 13$. Therefore, $\widetilde{H} \simeq H \simeq \langle [2 \cdot 3 \cdot 13], [5 \cdot 29], [3], [2 \cdot 3 \cdot 5 \cdot 13]\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^4$.*

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