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ON $r-$ DYNAMIC COLORING OF EDGE CORONATION OF GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The edge corona of G_1 and G_2 is denoted by $G_1 \diamond G_2$ is the graph obtained by taking m_1 copies of G_2 and then joining the end vertices of i-th edge of G_1 to every vertex in the i-th copy of G_2. An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has a r-dynamic coloring with k colors. In this paper, we obtain the r-dynamic chromatic number of the edge corona of the path and several graphs.

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Keywords and Phrases: cycle, complete graph, complete bipartite graph, double star graph, edge corona, fan graph, path, prism graph, r-dynamic coloring.

1. Introduction

The r-dynamic chromatic number was first introduced by Montgomery [3]. It is also studied under the name r- hued [16], [17]. An r-dynamic coloring of a graph G is a proper coloring and it maps c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$, and (ii) for each vertex $v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v, $d(v)$ its degree and r is a positive integer. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has a r-dynamic proper k-coloring. In this paper, we consider only the graphs which are simple, finite, loopless, and connected. For all terms and definitions that are not specifically defined in this paper, we refer to [2]. The r-dynamic chromatic number has been studied by many authors, for example in [1, 2, 5–10, 12–15].

2. Preliminaries

Let G_1 be a graph with n_1 vertices and m_1 edges. Let G_2 be a graph with n_2 vertices and m_2 edges. The edge corona [4] of G_1 and G_2 is denoted by $G_1 \diamond G_2$ is the graph obtained by taking m_1 copies of G_2 and then joining the end vertices of i-th edge of G_1 to every vertex in the i-th copy of G_2.

Lemma 2.1. *[11]* $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$

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3. Results

In this section, we find the r-dynamic chromatic number of Edge Corona of Path with Path, Cycle, Complete graph, Complete Bipartite graph, Fan graph, Double star graph and Prism graph. Firstly we will show the lower bounds of r-dynamic chromatic number of the graphs and we prove our theorems.

Lemma 3.1. *Let $P_n \diamond P_m$ be the Edge Corona of a path graph P_n with P_m. The lower bound of r-dynamic chromatic number of $P_n \diamond P_m$ is*

$$\chi_r(P_n \diamond P_m) \geq \begin{cases} 4, \ 1 \leq r \leq \delta(P_n \diamond P_m) - 1, \\ r+1, \ \delta(P_n \diamond P_m) \leq r \leq \Delta(P_n \diamond P_m) - 1, \\ \Delta(P_n \diamond P_m) + 1, \ r \geq \Delta(P_n \diamond P_m). \end{cases}$$

Proof. Let $V(P_n \diamond P_m) = \{y_i : 1 \leq i \leq n\} \cup \{x_{ij} : 1 \leq i \leq n-1, 1 \leq j \leq m\}$ and $E(P_n \diamond P_m) = \{y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \left\{x_{ij} x_{i(j+1)} : 1 \leq i \leq n-1, 1 \leq j \leq m-1\right\} \cup \{x_{ij} y_i, x_{ij} y_{i+1} : 1 \leq i \leq n-1, 1 \leq j \leq m\}$.

For $1 \leq r \leq \delta(P_n \diamond P_m) - 1$, by the definition of the Edge Corona graph, the vertices $V = \{x_{ij}, x_{i(j+1)}, y_i, y_{i+1}\}(1 \leq i \leq n-1, 1 \leq j \leq m-1)$ induce a clique of order K_4 in $P_n \diamond P_m$. Thus $\chi_r(P_n \diamond P_m) \geq 4$.

For $\delta(P_n \diamond P_m) \leq r \leq \Delta(P_n \diamond P_m) - 1$, based on Lemma 2.1, we have $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$ such that $\chi_r(P_n \diamond P_m) \geq min\{r, \Delta(P_n \diamond P_m)\} + 1 = r+1$.

For $r \geq \Delta(P_n \diamond P_m)$,we obtain $\chi_r(P_n \diamond P_m) \geq min\{r, \Delta(P_n \diamond P_m)\} + 1 = \Delta(P_n \diamond P_m) + 1$. It concludes the proof. □

Theorem 3.2. *Let $n, m \geq 3$, the r-dynamic chromatic number of $P_n \diamond P_m$ is*

$$\chi_r(P_n \diamond P_m) = \begin{cases} 4, 1 \leq r \leq 3, \\ r+1, 4 \leq r \leq \Delta - 1, \\ 2m+3, r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of $P_n \diamond P_m$ are $\Delta(P_n \diamond P_m) = 2m+2$ and $\delta(P_n \diamond P_m) = 4$, respectively.

Case 1: $1 \leq r \leq 3$

Based on the Lemma 3.1 , the lower bound is $\chi_r(P_n \diamond P_m) \geq 4$. To show the upper bound, we define a map $C_1 : V(P_n \diamond P_m) \rightarrow \{1, 2, 3, 4\}$ by the following.

$C_1(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$

$C_1(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, 3, 4, 3, 4, \ldots\}$.

Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond P_m) \leq 4$. Thus, $\chi_{1 \leq r \leq 3}(P_n \diamond P_m) = 4$.

Case 2: $4 \leq r \leq \Delta - 1$

Based on the Lemma 3.1, the lower bound is $\chi_r(P_n \diamond P_m) \geq r+1$. To show the upper bound, we define a map $C_2 : V(P_n \diamond P_m) \rightarrow \{1, 2, \ldots, r+1\}$ by the following.

$C_2(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, \ldots\}$

Subcase (i): $r \leq m+1$

$C_2(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, \ldots, r+1, 3, 4, \ldots\}$

Subcase (ii): $r > m+1$
If n is even
For $1 \le i \le (n-2)/2$,
$C_2(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{3, 4, \ldots, r+1, 3, 4, \ldots\}$
$C_2(x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}) = \{3, 4, \ldots, m+2\}$
If n is odd
For $1 \le i \le (n-1)/2$,
$C_2(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{3, 4, \ldots, r+1, 3, 4, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond P_m) \le r+1$. Thus, $\chi_{4 \le r \le \Delta-1}(P_n \diamond P_m) = r+1$.
Case 3: $r \ge \Delta$
Based on the Lemma 3.1, the lower bound is $\chi_r(P_n \diamond P_m) \ge \Delta(P_n \diamond P_m) + 1 = 2m+2+1 = 2m+3$. To show the upper bound, we define a map $C_3 : V(P_n \diamond P_m) \to \{1, 2, \ldots, 2m+3\}$ by the following.
$C_3(y_1, y_2, \ldots, y_n) = \{1, 2, 3, 1, 2, 3, \ldots\}$
For $1 \le i \le (n-2)/2$,
$C_3(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{4, 5, \ldots, 2m+3\}$
$C_3(x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}) = \{4, 5, \ldots, m+3\}$
If n is odd
For $1 \le i \le (n-1)/2$,
$C_3(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{4, 5, \ldots, 2m+3\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond P_m) \le 2m+3$. Thus, $\chi_{r \ge \Delta}(P_n \diamond P_m) = 2m+3$. It completes the proof. □

Lemma 3.3. *Let $P_n \diamond C_m$ be the Edge Corona of a path graph P_n with C_m. The lower bound of r-dynamic chromatic number of $P_n \diamond C_m$ is*

$$\chi_r(P_n \diamond C_m) \ge \begin{cases} 4, \; 1 \le r \le \delta(P_n \diamond P_m) - 1, \\ r+1, \; \delta(P_n \diamond P_m) \le r \le \Delta(P_n \diamond C_m) - 1, \\ \Delta(P_n \diamond C_m) + 1, \; r \ge \Delta(P_n \diamond C_m). \end{cases}$$

Proof. Let $V(P_n \diamond C_m) = \{y_i : 1 \le i \le n\} \cup \{x_{ij} : 1 \le i \le n-1, 1 \le j \le m\}$ and $E(P_n \diamond C_m) = \{y_i y_{i+1} : 1 \le i \le n-1\} \cup \left\{x_{ij} x_{i(j+1)} : 1 \le i \le n-1, 1 \le j \le m-1\right\} \cup \{x_{i1} x_{im} : 1 \le i \le n-1\} \cup \{x_{ij} y_i, x_{ij} y_{i+1} : 1 \le i \le n-1, 1 \le j \le m\}$.

For $1 \le r \le \delta(P_n \diamond C_m) - 1$, by the definition of the Edge Corona graph, the vertices $V = \{x_{ij}, x_{i(j+1)}, y_i, y_{i+1}\} (1 \le i \le n-1, 1 \le j \le m-1)$ induce a clique of order K_4 in $P_n \diamond C_m$. Thus $\chi_r(P_n \diamond C_m) \ge 4$.

For $\delta(P_n \diamond C_m) \le r \le \Delta(P_n \diamond C_m) - 1$, based on Lemma 2.1, we have $\chi_r(G) \ge min\{r, \Delta(G)\} + 1$ such that $\chi_r(P_n \diamond C_m) \ge min\{r, \Delta(P_n \diamond C_m)\} + 1 = r+1$.

For $r \ge \Delta(P_n \diamond C_m)$,we obtain $\chi_r(P_n \diamond C_m) \ge min\{r, \Delta(P_n \diamond C_m)\} + 1 = \Delta(P_n \diamond C_m) + 1$. It concludes the proof. □

Theorem 3.4. *Let $n, m \geq 3$, the r-dynamic chromatic number of $P_n \diamond C_m$ is*

$$\chi_r(P_n \diamond C_m) = \begin{cases} 4, 1 \leq r \leq 3, \ m \ is \ even, \\ 5, 1 \leq r \leq 3, \ m \ is \ odd, \\ 5, r = 4, m \equiv 0 (mod 3), \\ 7, r = 4, 5, \ m = 5, \\ 6, r = 4, \ m \ otherwise, \\ 6, r = 5, \ m \neq 5, \\ r+1, 6 \leq r \leq \Delta - 1, \\ 2m+3, r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of $P_n \diamond C_m$ are $\Delta(P_n \diamond C_m) = 2m+2$ and $\delta(P_n \diamond C_m) = 4$, respectively.
Case 1: $1 \leq r \leq 3$
Sub case 1: m is even
Based on the Lemma 3.3 , the lower bound is $\chi_r(P_n \diamond C_m) \geq 4$. To show the upper bound, we define a map $C_1 : V(P_n \diamond C_m) \to \{1, 2, 3, 4\}$ by the following.
$C_1(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
$C_1(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, 3, 4, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond C_m) \leq 4$. Thus, $\chi_{1 \leq r \leq 3}(P_n \diamond C_m) = 4$, m is even .
Sub case 2: m is odd
Based on the Lemma 3.3 , the lower bound is $\chi_r(P_n \diamond C_m) \geq 4$. However, we can not attain the best lower bound. Suppose $\chi_r(P_n \diamond C_m) = 4$. Define a map $C_2 : V(P_n \diamond C_m) \to \{1, 2, 3, 4\}$ by the following.
$C_2(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
$C_2(x_{i1}, x_{i2}, \ldots, x_{i(m-1)}) = \{3, 4, 3, 4, \ldots, 3, 4\}$

If we choose any of the colors $\{c_1, c_2, c_3, c_4\}$ to color the vertex x_{im} , it contradicts to the adjacency condition. Thus, it concludes that $\chi_r(P_n \diamond C_m) \geq 5$.

To show the upper bound, we define a map $C_2 : V(P_n \diamond C_m) \to \{1, 2, 3, 4, 5\}$ by the following.
$C_2(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
$C_2(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, 3, 4, \ldots, 3, 4, 5\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond C_m) \leq 5$. Thus, $\chi_{1 \leq r \leq 3}(P_n \diamond C_m) = 5$, m is odd.
Case 2: $r = 4$
Based on the Lemma 3.3, the lower bound is $\chi_r(P_n \diamond C_m) \geq r + 1 = 5$.
Sub case 1: $m \equiv 0 \pmod 3$
To show the upper bound, we define a map $C_3 : V(P_n \diamond C_m) \to \{1, 2, \ldots, 5\}$ by the following.
$C_3(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
$C_3(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, 5, 3, 4, 5, \ldots, \}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond C_m) \leq 5$. Thus, $\chi_{r=4}(P_n \diamond C_m) = 5$, $m \equiv 0 \pmod 3$.
Sub case 2: $m = 5$

However, we can not attain the best lower bound. Suppose $\chi_r(P_n \diamond C_m) = 5$. Define a map $C_4 : V(P_n \diamond C_m) \to \{1, 2, 3, 4, 5\}$ by the following.
$C_4(y_1, y_2, \ldots, y_m) = \{1, 2, 1, 2, 1, 2, \ldots\}$
$C_4(x_{i1}, x_{i2}, \ldots, x_{i5}) = \{3, 4, 5, 3, 4\}$

It implies $|c(N(x_{i1}))| = 3$. It contradicts with the r-dynamic condition, $|c(N(x_{i1}))| \geq 4$. For the same reason $\chi_r(P_n \diamond C_m) \neq 6$. Thus, it concludes that $\chi_r(P_n \diamond C_m) \geq 7$.
To show the upper bound, we define a map $C_4 : V(P_n \diamond C_m) \to \{1, 2, \ldots, 7\}$ by the following.
$C_4(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
$C_4(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, 5, 6, 7\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond C_m) \leq 7$. Thus, $\chi_{r=4,5}(P_n \diamond C_m) = 7$, $m = 5$.
Sub case 3: For m otherwise.
However, we can not attain the best lower bound. To show the upper bound, we define a map $C_5 : V(P_n \diamond C_m) \to \{1, 2, \ldots, 6\}$ by the following.
$C_5(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$

$$C_5(x_{i1}, x_{i2}, \ldots, x_{im}) = \begin{cases} 3, 4, 5, 3, 4, 5, \ldots, 3, 4, 5, 6, & m \equiv 1(mod3) \\ 3, 4, 5, 6, 3, 4, 5, 6, 3, 4, 5, \ldots, 3, 4, 5, & m \equiv 2(mod3) \end{cases}$$

Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond C_m) \leq 6$. Thus, $\chi_{r=4}(P_n \diamond C_m) = 6$, $m \equiv 1, 2 \pmod 3$.
Case 3: $r = 5$, $m \neq 5$.
Based on the Lemma 3.3, the lower bound is $\chi_r(P_n \diamond C_m) \geq r + 1 = 6$. To show the upper bound, we define a map $C_6 : V(P_n \diamond C_m) \to \{1, 2, 3, 4, 5, 6\}$ by the following.
$C_6(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
Sub case 1: $m = 3$
If n is even
For $1 \leq i \leq (n-2)/2$,
$C_6(x_{(2i-1)1}, x_{(2i-1)2}, x_{(2i-1)3}, x_{(2i)1}, x_{(2i)2}, x_{(2i)3}) = \{3, 4, 5, 6, 3, 4\}$
$C_6(x_{(n-1)1}, x_{(n-1)2}, x_{(n-1)3}) = \{3, 4, 5\}$
If n is odd
For $1 \leq i \leq (n-1)/2$,
$C_6(x_{(2i-1)1}, x_{(2i-1)2}, x_{(2i-1)3}, x_{(2i)1}, x_{(2i)2}, x_{(2i)3}) = \{3, 4, 5, 6, 3, 4\}$
Sub case 2: $m \neq 3$

$$C_6(x_{i1}, x_{i2}, \ldots, x_{im}) = \begin{cases} 3, 4, 5, 6, 4, 5, 6, \ldots, 4, 5 & m \equiv 0(mod3), m \neq 3 \\ 3, 4, 5, 6, 3, 4, 5, \ldots, 3, 4, 5 & m \equiv 1(mod3) \\ 3, 4, 5, 6, 3, 4, 5, 6, 4, 5, 6, \ldots, 4, 5, 6 & m \equiv 2(mod3) \end{cases}$$

Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond C_m) \leq 6$. Thus, $\chi_{r=5}(P_n \diamond C_m) = 6$, $m \neq 5$.
Case 4: $6 \leq r \leq \Delta - 1$
Based on the Lemma 3.3, the lower bound is $\chi_r(P_n \diamond C_m) \geq r + 1$. To show the upper bound, we define a map $C_7 : V(P_n \diamond C_m) \to \{1, 2, \ldots, r + 1\}$ by the following.
$C_7(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, \ldots\}$
Sub case 1: $6 \leq r \leq m + 1$
$C_7(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, \ldots, r + 1, 5, 6, 7, 5, 6, 7, \ldots\}$

Sub case 2: $m+2 \le r \le \Delta - 1$
If n is even
For $1 \le i \le (n-2)/2$,
$C_7(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{3, 4, \ldots, r+1, 3, 4, \ldots\}$
$C_7(x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}) = \{3, 4, \ldots, m+2\}$
If n is odd
For $1 \le i \le (n-1)/2$,
$C_7(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{3, 4, 5, \ldots, r+1, 3, 4, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond C_m) \le r+1$. Thus, $\chi_{6 \le r \le \Delta-1}(P_n \diamond C_m) = r+1$.
Case 5: $r \ge \Delta$
Based on the Lemma 3.3 , the lower bound is $\chi_r(P_n \diamond C_m) \ge \Delta(P_n \diamond C_m) + 1 = 2m+2+1 = 2m+3$. To show the upper bound, we define a map $C_8 : V(P_n \diamond C_m) \to \{1, 2, \ldots, 2m+3\}$ by the following.
$C_8(y_1, y_2, \ldots, y_n) = \{1, 2, 3, 1, 2, 3, \ldots\}$
If n is even
For $1 \le i \le (n-2)/2$,
$C_8(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{4, 5, \ldots, 2m+3\}$
$C_8(x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}) = \{4, 5, \ldots, m+3\}$
If n is odd
For $1 \le i \le (n-1)/2$,
$C_8(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{4, 5, \ldots, 2m+3\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond C_m) \le 2m+3$. Thus, $\chi_{r \ge \Delta}(P_n \diamond C_m) = 2m+3$. It completes the proof. □

Lemma 3.5. *Let $P_n \diamond K_m$ be the Edge Corona of a path graph P_n with K_m. The lower bound of r-dynamic chromatic number of $P_n \diamond K_m$ is*

$$\chi_r(P_n \diamond K_m) \ge \begin{cases} m+2, \ 1 \le r \le \delta(P_n \diamond K_m), \\ r+1, \ \delta(P_n \diamond K_m)+1 \le r \le \Delta(P_n \diamond K_m) - 1, \\ \Delta(P_n \diamond K_m)+1, \ r \ge \Delta(P_n \diamond K_m). \end{cases}$$

Proof. Let $V(P_n \diamond K_m) = \{y_i : 1 \le i \le n\} \cup \{x_{ij} : 1 \le i \le (n-1), 1 \le j \le m\}$ and
$E(P_n \diamond K_m) = \{y_i y_{i+1} : 1 \le i \le (n-1)\}$
$\cup \{x_{ij} x_{ik} : 1 \le i \le (n-1), 1 \le j \le (m-1), 2 \le k \le m, k > j\} \cup \{x_{ij} y_i, x_{ij} y_{i+1} : 1 \le i \le (n-1), 1 \le j \le m\}$.
For $1 \le r \le \delta(P_n \diamond K_m)$, by the definition of the Edge Corona graph, the vertices $V = \{x_{ij}, y_i, y_{i+1}\}(1 \le i \le n-1, 1 \le j \le m)$ induce a clique of order K_{m+2} in $P_n \diamond K_m$. Thus $\chi_r(P_n \diamond K_m) \ge m+2$.
For $\delta(P_n \diamond K_m) + 1 \le r \le \Delta(P_n \diamond K_m) - 1$, based on Lemma 2.1, we have $\chi_r(G) \ge min\{r, \Delta(G)\} + 1$ such that $\chi_r(P_n \diamond K_m) \ge min\{r, \Delta(P_n \diamond K_m)\} + 1 = r+1$.

For $r \geq \Delta(P_n \diamond K_m)$,we obtain $\chi_r(P_n \diamond K_m) \geq min\{r, \Delta(P_n \diamond K_m)\} + 1 = \Delta(P_n \diamond K_m) + 1$. It concludes the proof. □

Theorem 3.6. *Let $n, m \geq 3$, the r-dynamic chromatic number of $P_n \diamond K_m$ is*

$$\chi_r(P_n \diamond K_m) = \begin{cases} m+2, 1 \leq r \leq m+1, \\ r+1, m+2 \leq r \leq \Delta - 1, \\ 2m+3, r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of $P_n \diamond K_m$ are $\Delta(P_n \diamond K_m) = 2m+2$ and $\delta(P_n \diamond K_m) = m+1$, respectively.
Case 1: $1 \leq r \leq m+1$
Based on the Lemma 3.5 , the lower bound is $\chi_r(P_n \diamond K_m) \geq m+2$. To show the upper bound, we define a map $C_1 : V(P_n \diamond K_m) \to \{1, 2, \ldots, m+2\}$ by the following.
$C_1(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_1(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, \ldots, m+2\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond K_m) \leq m+2$. Thus, $\chi_{1 \leq r \leq m+1}(P_n \diamond K_m) = m+2$.
Case 2: $m+2 \leq r \leq \Delta - 1$
Based on the Lemma 3.5 , the lower bound is $\chi_r(P_n \diamond K_m) \geq r+1$. To show the upper bound, we define a map $C_2 : V(P_n \diamond K_m) \to \{1, 2, \ldots, r+1\}$ by the following.
$C_2(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, \ldots\}$
If n is even
For $1 \leq i \leq (n-2)/2$,
$C_2(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{3, 4, \ldots, r+1, 3, 4, \ldots\}$
$C_2(x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}) = \{3, 4, \ldots, m+2\}$
If n is odd
For $1 \leq i \leq (n-1)/2$,
$C_2(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{3, 4, 5, \ldots, r+1, 3, 4, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond K_m) \leq r+1$. Thus, $\chi_{m+2 \leq r \leq \Delta-1}(P_n \diamond K_m) = r+1$.
Case 3: $r \geq \Delta$
Based on the Lemma 3.5 , the lower bound is $\chi_r(P_n \diamond K_m) \geq \Delta(P_n \diamond K_m)+1 = 2m+2+1 = 2m+3$. To show the upper bound, we define a map $C_3 : V(P_n \diamond K_m) \to \{1, 2, \ldots, 2m+3\}$ by the following.
$C_3(y_1, y_2, \ldots, y_n) = \{1, 2, 3, 1, 2, 3, \ldots\}$
If n is even
For $1 \leq i \leq (n-2)/2$,
$C_3(x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{4, 5, \ldots, 2m+3\}$
$C_3(x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}) = \{4, 5, \ldots, m+3\}$
If n is odd

For $1 \le i \le (n-1)/2$,
$C_3(x_{(2i-1)1}, x_{(2i-1)2}, \dots, x_{(2i-1)m}, x_{(2i)1}, x_{(2i)2}, \dots, x_{(2i)m}) = \{4, 5, \dots, 2m+3\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond K_m) \le 2m+3$. Thus, $\chi_{r \ge \Delta}(P_n \diamond K_m) = 2m+3$. It completes the proof. □

Theorem 3.7. *Let $P_k \diamond K_{m,n}$ be the Edge Corona of a path P_k with a Complete Bipartite graph $K_{m,n}$. For $k \ge 3, m, n \ge 2, m < n$, the r-dynamic chromatic number of $P_k \diamond K_{m,n}$ is*

$$\chi_r(P_k \diamond K_{m,n}) = \begin{cases} 4, 1 \le r \le 3, \\ 2(r-1), 4 \le r \le m+2, \\ r+m, m+3 \le r \le n+2, \\ m+n+2, n+3 \le r \le m+n+1, \\ r+1, m+n+2 \le r \le \Delta - 1, \\ 2m+2n+3, r \ge \Delta. \end{cases}$$

Proof. The graph $P_k \diamond K_{m,n}$ is connected graph with vertex set $V(P_k \diamond K_{m,n}) = \{v_i : 1 \le i \le k\} \cup \{x_{is}, y_{it} : 1 \le i \le k-1, 1 \le s \le m, 1 \le t \le n\}$ and edge set
$E(P_k \diamond K_{m,n}) = \{v_i v_{i+1} : 1 \le i \le k-1\} \cup \{x_{is} y_{it} : 1 \le i \le k-1, 1 \le s \le m, 1 \le t \le n\} \cup \{v_i x_{is}, v_i y_{it}, v_{i+1} x_{is}, v_{i+1} y_{it} : 1 \le i \le k-1, 1 \le s \le m, 1 \le t \le n\}$.
The maximum and the minimum degrees of $P_k \diamond K_{m,n}$ are $\Delta(P_k \diamond K_{m,n}) = 2(m+n+1)$ and $\delta(P_k \diamond K_{m,n}) = m+1$, respectively.
By the definition of the Edge Corona graph, the vertices $V = \{v_i, v_{i+1}, x_{is}, y_{it}\}(1 \le i \le n-1, 1 \le s \le m, 1 \le t \le n)$ induce a clique of order K_4 in $P_k \diamond K_{m,n}$. Thus $\chi_r(P_k \diamond K_{m,n}) \ge 4$.
Case 1: $1 \le r \le 3$
The upper bound for $\chi_r(P_k \diamond K_{m,n})$ are by explicit construction.
Define a map $C_1 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, k\}$ by the following.
$C_1(v_1, v_2, \dots, v_n) = \{1, 2, 1, 2, 1, 2, \dots\}$
$C_1(x_{is}) = 3$ and $C_1(y_{it}) = 4$
The map $C_1 : V(P_k \diamond K_{m,n}) \to \{1, 2, 3, 4\}$ gives the upper bound
$\chi_{1 \le r \le 3}(P_k \diamond K_{m,n}) \le 4$.
Hence $\chi_{1 \le r \le 3}(P_k \diamond K_{m,n}) = 4$.
Case 2: $4 \le r \le m+2$
Define a map $C_2 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, k\}$ by the following.
$C_2(v_1, v_2, \dots, v_n) = \{1, 2, 1, 2, 1, 2, \dots\}$
$C_2(x_{is}) = \{3, 4, \dots, r, 3, 3, \dots\}$
$C_2(y_{it}) = \{r+1, r+2, \dots, 2(r-1), r+1, r+1, \dots,\}$
It is clear that C_2 is a map $C_2 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, 2(r-1)\}$.
Hence $\chi_{4 \le r \le m+2}(P_k \diamond K_{m,n}) = 2(r-1)$.
Case 3: $m+3 \le r \le n+2$
Define a map $C_3 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, k\}$ by the following.
$C_3(v_1, v_2, \dots, v_n) = \{1, 2, 1, 2, 1, 2, \dots\}$
$C_3(x_{is}) = \{3, 4, \dots, m+2\}$
$C_3(y_{it}) = \{m+3, \dots, r+m, m+3, m+3, \dots,\}$

It is clear that C_3 is a map $C_3 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, 2(r-1)\}$.
Hence $\chi_{m+3 \le r \le n+2}(P_k \diamond K_{m,n}) = r + m$.
Case 4: $n + 3 \le r \le m + n + 1$
Define a map $C_4 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, k\}$ by the following.
$C_4(v_1, v_2, \dots, v_n) = \{1, 2, 1, 2, 1, 2, \dots\}$
$C_4(x_{is}) = \{3, 4, \dots, m + 2\}$
$C_4(y_{it}) = \{m + 3, \dots, m + n + 2 \dots, \}$
It is clear that C_4 is a map $C_4 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, 2(r-1)\}$.
Hence $\chi_{n+3 \le r \le m+n+1}(P_k \diamond K_{m,n}) = r + m$.
Case 5: $m + n + 2 \le r \le \Delta - 1$
Define a map $C_5 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, k\}$ by the following.
$C_5(v_1, v_2, \dots, v_n) = \{1, 2, 1, 2, 1, 2, \dots, \}$
If k is even
For $1 \le i \le (k-2)/2$,
$C_5(x_{(2i-1)1}, \dots, x_{(2i-1)m}, y_{(2i-1)1}, \dots, y_{(2i-1)n}, x_{(2i)1}, \dots, x_{(2i)m}, y_{(2i)1}, \dots,$
$y_{(2i)n}) = \{3, 4, \dots, r + 1, 3, 4, \dots\}$
$C_5(x_{(k-1)1}, \dots, x_{(k-1)m}, y_{(k-1)1}, \dots, y_{(k-1)n}) = \{3, 4, \dots, m + n + 2\}$
If k is odd
For $1 \le i \le (k-1)/2$,
$C_5(x_{(2i-1)1}, \dots, x_{(2i-1)m}, y_{(2i-1)1}, \dots, y_{(2i-1)n}, x_{(2i)1}, \dots, x_{(2i)m}, y_{(2i)1}, \dots,$
$y_{(2i)n}) = \{3, 4, \dots, r + 1, 3, 4, \dots\}$
It is clear that C_5 is a map $C_5 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, r + 1\}$.
Hence $\chi_{m+n+2 \le r \le \Delta - 1}(P_k \diamond K_{m,n}) = r + 1$.
Case 6: $r \ge \Delta$
Define a map $C_6 : V(P_k \diamond K_{m,n}) \to \{1, 2, \dots, 2m + 2n + 3\}$ by the following.
$C_6(v_1, v_2, \dots, v_n) = \{1, 2, 3, 1, 2, 3, \dots\}$
If k is even
For $1 \le i \le (k-2)/2$,
$C_6(x_{(2i-1)1}, \dots, x_{(2i-1)m}, y_{(2i-1)1}, \dots, y_{(2i-1)n}, x_{(2i)1}, \dots, x_{(2i)m}, y_{(2i)1} \dots,$
$y_{(2i)n}) = \{4, 5, \dots, 2m + 2n + 3\}$
$C_6(x_{(k-1)1}, \dots, x_{(k-1)m}, y_{(k-1)1}, \dots, y_{(k-1)n}) = \{4, 5, \dots, m + n + 3\}$
If k is odd
For $1 \le i \le (k-1)/2$,
$C_6(x_{(2i-1)1}, \dots, x_{(2i-1)m}, y_{(2i-1)1}, \dots, y_{(2i-1)n}, x_{(2i)1}, \dots, x_{(2i)m}, y_{(2i)1}, \dots,$
$y_{(2i)n}) = \{4, 5, \dots, 2m + 2n + 3\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_k \diamond K_{m,n}) \le 2m + 2n + 3$. Thus, $\chi_{r \ge \Delta}(P_k \diamond K_{m,n}) = 2m + 2n + 3$. It completes the proof. □

Lemma 3.8. *Let $P_n \diamond F_{1,m}$ be the Edge Corona of a path graph P_n with $F_{1,m}$. The lower bound of r-dynamic chromatic number of $P_n \diamond F_{1,m}$ is*

$$\chi_r(P_n \diamond F_{1,m}) \ge \begin{cases} 5, \ 1 \le r \le \delta(P_n \diamond F_{1,m}), \\ r + 1, \ \delta(P_n \diamond F_{1,m}) + 1 \le r \le \Delta(P_n \diamond F_{1,m}) - 1, \\ \Delta(P_n \diamond F_{1,m}) + 1, \ r \ge \Delta(P_n \diamond F_{1,m}). \end{cases}$$

Proof. Let $V(P_n \diamond F_{1,m}) = \{y_i : 1 \le i \le n\} \cup \{A_i : 1 \le i \le (n-1)\}$
$\cup \{x_{ij} : 1 \le i \le (n-1), 1 \le j \le m\}$ and edge set
$E(P_n \diamond F_{1,m}) = \left\{y_i y_{i+1}, x_{ij} x_{i(j+1)} : 1 \le i \le (n-1)\right\} \cup$

$\{A_i x_{ij}, A_i y_i, A_i y_{i+1}, x_{ij} y_i, x_{ij} y_{i+1} : 1 \leq i \leq (n-1), 1 \leq j \leq m\}$.
For $1 \leq r \leq \delta(P_n \diamond F_{1,m})$, by the definition of the Edge Corona graph, the vertices
$V = \{x_{ij}, x_{i(j+1)}, A_i, y_i, y_{i+1}\}(1 \leq i \leq (n-1))$ induce a clique of order K_5 in $P_n \diamond F_{1,m}$. Thus $\chi_r(P_n \diamond F_{1,m}) \geq 5$.
For $\delta(P_n \diamond F_{1,m}) + 1 \leq r \leq \Delta(P_n \diamond F_{1,m}) - 1$, based on Lemma 2.1 , we have $\chi_r(G) \geq min\{\Delta(G), r\} + 1 = r + 1$. For $r \geq \Delta(P_n \diamond F_{1,m})$, we obtain $\chi_r(P_n \diamond F_{1,m}) \geq min\{r, \Delta(P_n \diamond F_{1,m})\} + 1 = \Delta(P_n \diamond F_{1,m}) + 1$. It concludes the proof. □

Theorem 3.9. *Let $n, m \geq 3$, the r-dynamic chromatic number of $P_n \diamond F_{1,m}$ is*

$$\chi_r(P_n \diamond F_{1,m}) = \begin{cases} 5, 1 \leq r \leq 4, \\ r+1, 5 \leq r \leq \Delta - 1, \\ 2m+5, r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of $P_n \diamond F_{1,m}$ are $\Delta(P_n \diamond F_{1,m}) = 2m + 4$ and $\delta(P_n \diamond F_{1,m}) = 4$, respectively .
Case 1: $1 \leq r \leq 4$
Based on the Lemma 3.8, the lower bound is $\chi_r(P_n \diamond F_{1,m}) \geq 5$. To show the upper bound, we define a map $C_1 : V(P_n \diamond F_{1,m}) \to \{1, 2, 3, 4, 5\}$ by the following.
$C_1(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_1(A_i) = 3$.
$C_1(x_{i1}, x_{i2}, \ldots, x_{im}) = \{4, 5, 4, 5, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond F_{1,m}) \leq 5$. Thus $\chi_{1 \leq r \leq 4}(P_n \diamond F_{1,m}) = 5$.
Case 2: $5 \leq r \leq \Delta - 1$
Based on the Lemma 3.8, the lower bound is $\chi_r(P_n \diamond F_{1,m}) \geq r + 1$. To show the upper bound, we define a map $C_2 : V(P_n \diamond F_{1,m}) \to \{1, 2, \ldots, r+1\}$ by the following.
$C_2(y_1, y_2, \ldots, y_n) = \{1, 2, 1, 2, \ldots\}$
Subcase 1: $5 \leq r \leq m + 2$
For $1 \leq i \leq (n-1)$, $C_2(A_i) = 3$.
$C_2(x_{i1}, x_{i2}, \ldots, x_{im}) = \{4, 5, \ldots, r+1, 4, 5, 6, \ldots\}$
Subcase 2: $m + 3 \leq r \leq \Delta - 1$
If n is even
For $1 \leq i \leq (n-2)/2$,
$C_2(A_{2i-1}, x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, A_{2i}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{3, 4, 5, \ldots, r+1, 4, 5, \ldots\}$
$C_2(A_{n-1}, x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}) = \{3, 4, 5, \ldots, m+3\}$
If n is odd
For $1 \leq i \leq (n-1)/2$,
$C_2(A_{2i-1}, x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, A_{2i}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{4, 5, \ldots, r+1, 4, 5, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond F_{1,m}) \leq r + 1$. Thus $\chi_{5 \leq r \leq \Delta - 1}(P_n \diamond F_{1,m}) = r + 1$.
Subcase 3: $r \geq \Delta$

Based on the Lemma 3.8, the lower bound is $\chi_r(P_n \diamond F_{1,m}) \geq \Delta(P_n \diamond F_{1,m}) + 1 = m + 3 + 1 = m + 4$. To show the upper bound, we define a map $C_3 : V(P_n \diamond F_{1,m}) \to \{1, 2, \ldots, 2m + 5\}$ by the following.
$C_3(y_1, y_2, \ldots, y_n) = \{1, 2, 3, 1, 2, 3, \ldots\}$
If n is even
For $1 \leq i \leq (n-2)/2$,
$C_3(A_{2i-1}, x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, A_{2i}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{4, 5, 6, \ldots, 2m + 5\}$
$C_3(A_{n-1}, x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}) = \{4, 5, 6, \ldots, m + 4\}$
If n is odd
For $1 \leq i \leq (n-1)/2$,
$C_3(A_{2i-1}, x_{(2i-1)1}, x_{(2i-1)2}, \ldots, x_{(2i-1)m}, A_{2i}, x_{(2i)1}, x_{(2i)2}, \ldots, x_{(2i)m}) = \{4, 5, 6, \ldots, 2m + 5\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond F_{1,m}) \leq 2m + 5$. Thus $\chi_{r \geq \Delta}(P_n \diamond F_{1,m}) = 2m + 5$. It completes the proof. □

Lemma 3.10. *Let $P_n \diamond K_{1,m,m}$ be the Edge Corona of a path graph P_n with $K_{1,m,m}$. The lower bound of r-dynamic chromatic number of $P_n \diamond K_{1,m,m}$ is*

$$\chi_r(P_n \diamond K_{1,m,m}) \geq \begin{cases} 4, \ 1 \leq r \leq \delta(P_n \diamond K_{1,m,m}), \\ r + 1, \ \delta(P_n \diamond K_{1,m,m}) + 1 \leq r \leq \Delta(P_n \diamond K_{1,m,m}) - 1, \\ \Delta(P_n \diamond K_{1,m,m}) + 1, \ r \geq \Delta(P_n \diamond K_{1,m,m}). \end{cases}$$

Proof. Let $V(P_n \diamond K_{1,m,m}) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq (n-1)\} \cup \{x_{ij} : 1 \leq i \leq (n-1), 1 \leq j \leq m\} \cup \{y_{ij} : 1 \leq i \leq (n-1), 1 \leq j \leq m\}$.
$E(P_n \diamond K_{1,m,m}) = \{v_i v_{i+1} : 1 \leq i \leq (n-1)\} \cup \{u_i x_{ij}, x_{ij} y_{ij} : 1 \leq i \leq (n-1), 1 \leq j \leq m\} \cup$
$\{u_i v_i, u_i v_{i+1}, x_{ij} v_i, x_{ij} v_{i+1}, y_{ij} v_i, y_{ij} v_{i+1}, : 1 \leq i \leq (n-1), 1 \leq j \leq m\}$.
For $1 \leq r \leq \delta(P_n \diamond K_{1,m,m})$, by the definition of the Edge Corona graph, the vertices $V = \{x_{ij}, u_i, v_i, v_{i+1}\} (1 \leq i \leq n-1)$ induce a clique of order K_4 in $P_n \diamond K_{1,m,m}$.
Thus $\chi_r(P_n \diamond K_{1,m,m}) \geq 4$.
For $\delta(P_n \diamond K_{1,m,m}) + 1 \leq r \leq \Delta(P_n \diamond K_{1,m,m}) - 1$, based on Lemma 2.1, we have $\chi_r(G) \geq min\{\Delta(G), r\} + 1 = r + 1$.
For $r \geq \Delta(P_n \diamond K_{1,m,m})$, we obtain $\chi_r(P_n \diamond K_{1,m,m}) \geq min\{r, \Delta(P_n \diamond K_{1,m,m})\} + 1 = \Delta(P_n \diamond K_{1,m,m}) + 1$. It concludes the proof. □

Theorem 3.11. *Let $n, m \geq 3$, the r-dynamic chromatic number of $P_n \diamond K_{1,m,m}$ is*

$$\chi_r(P_n \diamond K_{1,m,m}) = \begin{cases} 4, 1 \leq r \leq 3, \\ r + 1, 4 \leq r \leq \Delta - 1, \\ 4m + 5, \ r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of $P_n \diamond K_{1,m,m}$ are $\Delta(P_n \diamond K_{1,m,m}) = 4m + 4$ and $\delta(P_n \diamond K_{1,m,m}) = 4$, respectively.
Case 1: $1 \leq r \leq 3$

Based on the Lemma 3.10, the lower bound is $\chi_r(P_n \diamond K_{1,m,m}) \geq 4$. To show the upper bound, we define a map $C_1 : V(P_n \diamond K_{1,m,m}) \to \{1, 2, 3, 4\}$ by the following.
$C_1(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_1(u_i) = C_1(y_{ij}) = 3$ and $C_1(x_{ij}) = 4$.
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_k \diamond K_{1,m,m}) \leq 4$. Thus $\chi_{1\leq r\leq 3}(P_n \diamond K_{1,m,m}) = 4$.
Case 2: $4 \leq r \leq \Delta - 1$
Based on the Lemma 3.10, the lower bound is $\chi_r(P_n \diamond K_{1,m,m}) \geq r + 1$. To show the upper bound, we define a map $C_2 : V(P_n \diamond K_{1,m,m}) \to \{1, 2, \ldots, r+1\}$ by the following.
$C_2(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
Subcase 1: $4 \leq r \leq m + 2$
For $1 \leq i \leq (n-1)$, $C_2(u_i) = 3$.
$C_2(x_{i1}, x_{i2}, \ldots, x_{im}) = \{4, 5, \ldots, r+1, 4, 4, \ldots\}$
$C_2(y_{i1}, y_{i2}, \ldots, y_{im}) = \{5, 4, 5, 5, \ldots\}$
Subcase 2: $m + 3 \leq r \leq 2m + 2$
For $1 \leq i \leq (n-1)$, $C_2(u_i) = 3$.
$C_2(x_{i1}, x_{i2}, \ldots, x_{im}) = \{4, 5, \ldots, m+3\}$
$C_2(y_{i1}, y_{i2}, \ldots, y_{im}) = \{m+4, m+5, \ldots, r+1, 5, 4, 5, 5, \ldots\}$
Subcase 3: $2m + 3 \leq r \leq \Delta - 1$
If n is even
For $1 \leq i \leq (n-2)/2$,
$C_2(u_{2i-1}, x_{(2i-1)j}, y_{(2i-1)j}, u_{2i}, x_{(2i)j}, y_{(2i)j}) = \{3, 4, \ldots, r+1, 3, 4, \ldots\}$
$C_2(u_{n-1}, x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}, y_{(n-1)1}, y_{(n-1)2}, \ldots, y_{(n-1)m}) == \{3, 4, \ldots, 2m+3\}$
If n is odd
For $1 \leq i \leq (n-1)/2$,
$C_2(u_{2i-1}, x_{(2i-1)j}, y_{(2i-1)j}, u_{2i}, x_{(2i)j}, y_{(2i)j}) = \{3, 4, \ldots, r+1, 3, 4, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_k \diamond K_{1,m,m}) \leq r + 1$. Thus $\chi_{4\leq r\leq \Delta-1}(P_n \diamond K_{1,m,m}) = r + 1$.
Case 3: $r \geq \Delta$
Based on the Lemma 3.10, the lower bound is $\chi_r(P_n \diamond K_{1,m,m}) \geq \Delta(P_n \diamond K_{1,m,m}) + 1 = 4m + 4 + 1 = 4m + 5$. To show the upper bound, we define a map $C_3 : V(P_n \diamond K_{1,m,m}) \to \{1, 2, \ldots, 4m+5\}$ by the following.
$C_3(v_1, v_2, \ldots, v_n) = \{1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots\}$
If n is even
For $1 \leq i \leq (n-2)/2$,
$C_3(u_{2i-1}, x_{(2i-1)j}, y_{(2i-1)j}, u_{2i}, x_{(2i)j}, y_{(2i)j}) = \{4, 5, 6, \ldots, 4m+5\}$
$C_3(u_{n-1}, x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)m}, y_{(n-1)1}, y_{(n-1)2}, \ldots, y_{(n-1)m}) == \{4, 5, \ldots, 2m+4\}$
If n is odd
For $1 \leq i \leq (n-1)/2$,
$C_3(u_{2i-1}, x_{(2i-1)j}, y_{(2i-1)j}, u_{2i}, x_{(2i)j}, y_{(2i)j}) = \{4, 5, 6, \ldots, 4m+5\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond K_{1,m,m}) \leq 4m + 5$. Thus $\chi_{r\geq\Delta}(P_n \diamond K_{1,m,m}) = 4m + 5$. It completes the proof. □

Lemma 3.12. *Let $P_n \diamond P_{m,2}$ be the Edge Corona of a path graph P_n with $P_{m,2}$. The lower bound of r-dynamic chromatic number of $P_n \diamond P_{m,2}$ is*

$$\chi_r(P_n \diamond P_{m,2}) \geq \begin{cases} 4, \ 1 \leq r \leq \delta(P_n \diamond P_{m,2}), \\ r+1, \ \delta(P_n \diamond P_{m,2})+1 \leq r \leq \Delta(P_n \diamond P_{m,2})-1, \\ \Delta(P_n \diamond P_{m,2})+1, \ r \geq \Delta(P_n \diamond P_{m,2}). \end{cases}$$

Proof. Let $V(P_n \diamond P_{m,2}) = \{v_i : 1 \leq i \leq n\} \cup \{x_{ij}, y_{ij} : 1 \leq i \leq (n-1), 1 \leq j \leq m\}$ and $E(P_n \diamond P_{m,2}) = \{v_i v_{i+1} : 1 \leq i \leq (n-1)\} \cup \{x_{ij}x_{i(j+1)}, x_{im}x_{i1}, y_{ij}y_{i(j+1)}, y_{im}y_{i1} : 1 \leq i \leq (n-1), 1 \leq j \leq (m-1)\} \cup \{x_{ij}y_{ij} : 1 \leq i \leq (n-1), 1 \leq j \leq m\} \cup \{x_{ij}v_i, y_{ij}v_i, x_{ij}v_{i+1}, y_{ij}v_{i+1} : 1 \leq i \leq (n-1), 1 \leq j \leq m\}$. For $1 \leq r \leq \delta(P_n \diamond P_{m,2})$, by the definition of the Edge Corona graph, the vertices $V = \{v_i, v_{i+1}, x_{ij}, x_{i(j+1)}\}(1 \leq i \leq n-1, 1 \leq j \leq m)$ induce a clique of order K_4 in $P_n \diamond P_{m,2}$.
Thus $\chi_r(P_n \diamond P_{n,2}) \geq 4$.
For $\delta(P_n \diamond P_{m,2})+1 \leq \Delta(P_n \diamond P_{m,2})-1$, based on Lemma 2.1 , we have $\chi_r(G) \geq min\{\Delta(G), r\}+1 = r+1$.
For $r \geq \Delta(P_n \diamond P_{m,2})$, we obtain $\chi_r(P_n \diamond P_{m,2}) \geq min\{r, \Delta(P_n \diamond P_{m,2})\}+1 = \Delta(P_n \diamond P_{m,2})+1$. It concludes the proof. □

Theorem 3.13. *Let $n, m \geq 3$, the r-dynamic chromatic number of $P_n \diamond P_{m,2}$ is*

$$\chi_r(P_n \diamond P_{m,2}) = \begin{cases} 4, 1 \leq r \leq 3, \ m \ is \ even, \\ 5, 1 \leq r \leq 3, \ m \ is \ odd, \\ 5, r=4, m \equiv 0 (mod 3), \\ 6, r=4, \ m \ otherwise, \\ 6, r=5, m \equiv 0 (mod 4), \\ 8, r=5,6,7, \ m=3,6,7,11, \\ 7, r=5,6, \ m \ otherwise, \\ r+1, 6 \leq r \leq \Delta-1, m \equiv 0 (mod 4), \\ r+1, 8 \leq r \leq \Delta-1, \ m=3,6,7,11, \\ r+1, 7 \leq r \leq \Delta-1 \ m \ otherwise, \\ 4m+3, r \geq \Delta. \end{cases}$$

Proof. The maximum and the minimum degrees of $P_n \diamond P_{m,2}$ are $\Delta(P_n \diamond P_{m,2}) = 4m+2$ and $\delta(P_n \diamond P_{m,2}) = 5$, respectively.
Case 1: $1 \leq r \leq 3$
Subcase 1: m is even
Based on the Lemma 3.12, the lower bound is $\chi_r(P_n \diamond P_{m,2}) \geq 4$. To show the upper bound, we define a map $C_1 : V(P_n \diamond P_{m,2}) \to \{1,2,3,4\}$ by the following.
$C_1(v_1, v_2, \ldots, v_n) = \{1,2,1,2,1,2,\ldots\}$
For $1 \leq i \leq (n-1)$, $C_1(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3,4,3,4,\ldots\}$
For $1 \leq i \leq (n-1)$, $C_1(y_{i1}, y_{i2}, \ldots, y_{im}) = \{4,3,4,3,\ldots\}$

Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond P_{m,2}) \leq 4$. Thus $\chi_{1\leq r\leq 3}(P_n \diamond P_{m,2}) = 4$,$m$ is even.
Subcase 2: m is odd
Based on the Lemma 3.12, the lower bound is $\chi_r(P_n \diamond P_{m,2}) \geq 4$. However, we can not attain the best lower bound. Suppose $\chi_r(P_n \diamond P_{m,2}) = 4$. Define a map $C_2 : V(P_n \diamond P_{m,2}) \to \{1, 2, 3, 4\}$ by the following.
$C_2(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_2(x_{i1}, x_{i2}, \ldots, x_{i(m-1)}) = \{3, 4, 3, 4, \ldots, 3, 4\}$
If we choose any of the colors $\{c_1, c_2, c_3, c_4\}$ to color the vertex x_{im}, it contradicts to the adjacency condition.
For $1 \leq i \leq (n-1)$, $C_2(y_{i2}, \ldots, y_{im}) = \{3, 4, 3, 4, \ldots 3, 4\}$
If we choose any of the colors $\{c_1, c_2, c_3, c_4\}$ to color the vertex y_{i1}, it contradicts to the adjacency condition. Thus, it concludes that $\chi_r(P_n \diamond P_{m,2}) \geq 5$. To show the upper bound, we define a map $C_2 : V(P_n \diamond P_{m,2}) \to \{1, 2, 3, 4, 5\}$ by the following.
$C_2(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_2(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, 3, 4, \ldots, 3, 4, 5\}$
For $1 \leq i \leq (n-1)$, $C_2(y_{i1}, y_{i2}, \ldots, y_{im}) = \{5, 3, 4, 3, 4, \ldots 3, 4\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence, $\chi_r(P_n \diamond P_{m,2}) \leq 5$. Thus, $\chi_{1\leq r\leq 3}(P_n \diamond P_{m,2}) = 5$, m is odd.
Case 2: $r = 4$
Based on the Lemma 3.12, the lower bound is $\chi_r(P_n \diamond P_{m,2}) \geq r + 1 = 5$.
Sub case 1: $m \equiv 0 \pmod 3$.
To show the upper bound, we define a map $C_3 : V(P_n \diamond P_{m,2}) \to \{1, 2, \ldots, 5\}$ by the following.
$C_3(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_3(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, 5, 3, 4, 5, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_3(y_{i1}, y_{i2}, \ldots, y_{im}) = \{5, 3, 4, 5, 3, 4, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond P_{m,2}) \leq r + 1$. Thus $\chi_{r=4}(P_n \diamond P_{m,2}) = 5$.
Subcase 2: m otherwise.
We can not attain the best lower bound. To show the upper bound, we define a map $C_4 : V(P_n \diamond P_{m,2}) \to \{1, 2, \ldots, 6\}$ by the following.
$C_4(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_4(x_{i1}, x_{i2}, \ldots, x_{im}) =$
$$\begin{cases} 3, 4, 3, 4, 5, m = 5, \\ 3, 4, 5, 3, 4, 5, \ldots 3, 4, 5, 6, m \equiv (mod 3) \\ 3, 4, 5, 6, 3, 4, 5, 6, 3, 4, 5, \ldots, 3, 4, 5, m \equiv 2(mod 3) \end{cases}$$
For $1 \leq i \leq (n-1)$, $C_4(y_{i1}, y_{i2}, \ldots, y_{im}) =$
$$\begin{cases} 4, 5, 6, 5, 6, m = 5, \\ 6, 3, 4, 5, 3, 4, 5, \ldots, 3, 4, 5, m \equiv 1(mod 3) \\ 4, 5, 6, 3, 4, 5, 6, 3, 4, 5, 6, \ldots, 4, 5, 6, m \equiv 2(mod 3) \end{cases}$$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond P_{m,2}) \leq 6$. Thus $\chi_{r=4}(P_n \diamond P_{m,2}) = 6$, m otherwise.
Case 3: $r = 5$, $m \equiv 0 \pmod 4$.

Based on the Lemma 3.12, the lower bound is $\chi_r(P_n \diamond P_{m,2}) \geq r+1 = 6$. To show the upper bound, we define a map $C_5 : V\,(P_n \diamond P_{m,2}) \to \{1, 2, \ldots, 6\}$ by the following.
$C_5(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_5(x_{i1}, x_{i2}, \ldots, x_{im}) = \{3, 4, 5, 6, 3, 4, 5, 6, \ldots\}$
For $1 \leq i \leq (n-1)$, $C_5(y_{i1}, y_{i2}, \ldots, y_{im}) = \{5, 6, 3, 4, 5, 6, 3, 4, \ldots\}$
Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond P_{m,2}) \leq 6$. Thus $\chi_{r=5}(P_n \diamond P_{m,2}) = 6$, $m \equiv 0 \pmod 4$.
Case 4: $r = 5, 6, 7$, $m = 3, 6, 7, 11$.
Based on the Lemma 3.12, the lower bound is $\chi_r(P_n \diamond P_{m,2}) \geq r+1$. For $r = 5, 6$, We can not attain the best lower bound. To show the upper bound, we define a map $C_6 : V\,(P_n \diamond P_{m,2}) \to \{1, 2, \ldots, 8\}$ by the following.
$C_6(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$,

$$C_6(x_{i1}, x_{i2}, \ldots, x_{im}) = \begin{cases} 3, 4, 5, \quad m = 3, \\ 3, 4, 5, 6, 7, 8, \quad m = 6, \\ 3, 4, 5, 6, 7, 8, 5, \quad m = 7, \\ 3, 4, 5, 6, 7, 3, 4, 5, 6, 7, 8, \;\; m = 11, \end{cases}$$

For $1 \leq i \leq (n-1)$,

$$C_6(y_{i1}, y_{i2}, \ldots, y_{im}) = \{5, 6, 3, 4, 5, 6, 3, 4, \ldots\} \begin{cases} 6, 7, 8, \quad m = 3, \\ 7, 8, 3, 4, 5, 6, \quad m = 6, \\ 8, 6, 3, 4, 5, 6, 7, \quad m = 7, \\ 7, 8, 3, 4, 5, 6, 7, 3, 4, 5, 6, \\ \qquad\qquad m = 11, \end{cases}$$

Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond P_{m,2}) \leq 8$. Thus $\chi_{r=5,6,7}(P_n \diamond P_{m,2}) = 8$, $m = 3, 6, 7, 11$.
Case 5: $r = 5, 6$, m otherwise.
Based on the Lemma 3.12, the lower bound is $\chi_r(P_n \diamond P_{m,2}) \geq r+1$. For $r = 5$, We can not attain the best lower bound. To show the upper bound, we define a map $C_7 : V\,(P_n \diamond P_{m,2}) \to \{1, 2, \ldots, 7\}$ by the following.
Define a map $C_7 : V\,(P_n \diamond P_{m,2}) \to \{1, 2, \ldots, k\}$ by the following.
$C_7(v_1, v_2, \ldots, v_n) = \{1, 2, 1, 2, 1, 2, \ldots\}$
For $1 \leq i \leq (n-1)$,

$$C_7(x_{i1}, x_{i2}, \ldots, x_{im}) = \begin{cases} 3, 4, 5, 6, \ldots, 3, 4, 5, 6, 7, \equiv 1(mod4), \\ 3, 4, 5, 6, \ldots, 3, 4, 5, 6, 7, 3, 4, 5, 6, 7, m \equiv 2(mod4), \\ \qquad\qquad m \geq 10, \\ 3, 4, 5, 6, \ldots, 3, 4, 5, 6, 7, 3, 4, 5, 6, 7, 3, 4, 5, 6, 7, m \equiv \\ \qquad\qquad 3(mod4), m \geq 10, \end{cases}$$

For $1 \le i \le (n-1)$,

$$C_7(y_{i1}, y_{i2}, \ldots, y_{im}) = \begin{cases} 6,7,3,4,5,6,\ldots,3,4,5,6,3,4,5, m \equiv 1(mod4), \\ 6,7,3,4,5,6,7,3,4,5,6,\ldots,3,4,5,6,3,4,5, m \equiv \\ \hfill 2(mod4), m \ge 10, \\ 6,7,3,4,5,6,\ldots,3,4,5,6,7,3,4,5,6,7,3,4,5, m \equiv \\ \hfill 3(mod4), m \ge 10, \end{cases}$$

Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond P_{m,2}) \le 7$. Thus $\chi_{r=5,6}(P_n \diamond P_{m,2}) = 7$, m otherwise.

Case 6: $6 \le r \le \Delta - 1$, $m \equiv 0(mod4)$, $8 \le r \le \Delta - 1$, $m = 3, 6, 7, 11$ and $7 \le r \le \Delta - 1$, m otherwise.

Based on Lemma 3.12, the lower bound is $\chi_r(P_n \diamond P_{m,2}) \ge r + 1$. To show the upper bound, we define a map $C_8 : V(P_n \diamond P_{m,2}) \to \{1, 2, \ldots, r+1\}$ by the following.

$C_8(v_1, v_2, \ldots, v_n) = 1, 2, 1, 2, \ldots$

Sub case 1: $6 \le r \le m + 1$

For $1 \le i \le (n-1)$, $C_8(x_{i1}, x_{i2}, \ldots, x_{i(r-1)}) = \{3, 4, 5, 6, \ldots, r+1\}$

For $m \equiv 0$ (mod 4), Color the remaining vertices x_{ij} and y_{ij} with the colors as we used in case 4 . For $m \equiv 1, 2, 3$ (mod 4), Color the remaining vertices x_{ij} and y_{ij} with the colors as we used in case 5 .

Sub case 2: $m + 2 \le r \le 2m + 1$

For $1 \le i \le (n-1)$, $C_8(x_{i1}, \ldots, x_{im}, y_{i1}, \ldots, y_{i(r-m-1)}) = \{3, 4, 5, 6, \ldots, r+1\}$

For $m \equiv 0$ (mod 4), Color the remaining vertices y_{ij} with the colors as we used in case 4 . For $m \equiv 1, 2, 3$ (mod 4), Color the remaining vertices y_{ij} with the colors as we used in case 5 .

Subcase 3: $2m + 2 \le r \le \Delta - 1$

If n is even

For $1 \le i \le (n-2)/2$,

$C_8(x_{(2i-1)1}, \ldots, x_{(2i-1)m}, y_{(2i-1)1}, \ldots, y_{(2i-1)m}, x_{(2i)1}, \ldots, x_{(2i)m}, y_{(2i)1}, \ldots, y_{(2i)m}) = \{3, 4, 5, 6, \ldots, r+1, 3, 4, \ldots\}$

$C_8(x_{(n-1)1}, \ldots, x_{(n-1)m}, y_{(n-1)1}, \ldots, y_{(n-1)m}) = \{3, 4, 5, 6, \ldots, 2m+2\}$

If n is odd

For $1 \le i \le (n-2)/2$,

$C_8(x_{(2i-1)1}, \ldots, x_{(2i-1)m}, y_{(2i-1)1}, \ldots, y_{(2i-1)m}, x_{(2i)1}, \ldots, x_{(2i)m}, y_{(2i)1}, \ldots, y_{(2i)m}) = \{3, 4, 5, 6, \ldots, r+1, 3, 4, \ldots\}$

Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond P_{m,2}) \le r + 1$. Thus $\chi(P_n \diamond P_{m,2}) = r + 1$.

Case 7: $r \ge \Delta$

Based on the Lemma 3.12, the lower bound is $\chi_r(P_n \diamond P_{m,2}) \ge \Delta(P_n \diamond P_{m,2}) + 1 = 4m + 3$. To show the upper bound, we define a map $C_9 : V(P_n \diamond P_{m,2}) \to \{1, 2, \ldots, 4m+3\}$ by the following.

$C_9(v_1, v_2, \ldots, v_n) = 1, 2, 3, 1, 2, 3, \ldots$

If n is even

For $1 \le i \le (n-2)/2$ and $1 \le j \le m$,

$C_9(x_{(2i-1)j}, y_{(2i-1)j}, x_{(2i)j}, y_{(2i)j}) = \{4, 5, 6, \ldots, 4m+3\}$

$C_9(x_{(n-1)1}, \ldots, x_{(n-1)m}, y_{(n-1)1}, \ldots, y_{(n-1)m}) = \{4, 5, 6, \ldots, 2m+3\}$ If n is odd $C_9(x_{(2i-1)j}, y_{(2i-1)j}, x_{(2i)j}, y_{(2i)j}) = \{4, 5, 6, \ldots, 4m+3\}$ Now, it is easy to check that r-adjacency condition is fulfilled. Hence $\chi_r(P_n \diamond P_{m,2}) \leq 4m+3$. Thus $\chi_{r\geq\Delta}(P_n \diamond P_{m,2}) = 4m+3$. It completes the proof. □

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