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ON RINGS AND MODULES SATISFYING THE ASCENDING CHAIN CONDITION ON DIVISIBILITY

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Abstract. This paper investigates rings and modules satisfying the divisibility condition in ascending chains of ideals and submodules, respectively, denoted by ACC_d. These notions were first introduced by R. Dastanpour and A. Ghorbani as generalizations of the classical ascending chain condition (ACC).

We establish several new and significant results related to these structures. For example, we show that an ACC_d domain satisfies the ACC on absorbing ideals. Furthermore, an ACC_d domain is Noetherian if and only if it satisfies the ACC on principal ideals. This equivalence also holds for ACC_d rings that contain only finitely many non-zero-divisors and satisfy the ACC on principal ideals. We also examine the transfer of the ACC_d condition to amalgamated algebras.

In the context of ACC_d modules, we prove that every free submodule is finitely generated. Additionally, we show that over an ACC_d ring, a module M for which $\mathrm{Ann}(M)$ is not contained in any principal ideal is Noetherian if and only if it is finitely generated.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity, and all modules are unital. If A is a ring, we denote by $Nil(A)$ the set (ideal) of all nilpotent elements of A, and by $tq(A) = A_{A\setminus Z(A)}$ the total ring of quotients of A.

In [6], R. Dastanpour and A. Ghorbani introduced a generalization of the ascending chain condition (ACC) on (right) ideals. A ring A is said to satisfy the *ascending chain condition on divisibility* (ACC_d) on right ideals if in every ascending chain of right ideals of A, each ideal in the chain, except for finitely many, is a left multiple of the next. That is, for every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots,$$

there exists an integer k such that for all $i \geq k$, there exists an element $a_i \in A$ with $I_i = a_i I_{i+1}$. Similarly, in [7], the same authors introduced the notion of ACC_d modules. If all the multiplication factors a_i are units, then the ring satisfies the classical ACC condition and is Noetherian.

While most results in [6] concern associative (possibly noncommutative) rings, the purpose of this paper is to investigate these ideas in the context of *commutative* rings. We present several new results that highlight the behavior of finitely generated and principal ideals in ascending chains.

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We begin by recalling that a proper ideal I of a ring R is called *n-absorbing* if for any $x_1, \ldots, x_{n+1} \in R$ with $x_1 \cdots x_{n+1} \in I$, there exist n of the x_i whose product belongs to I. For simplicity, we refer to such ideals as *absorbing ideals*.

This paper is structured as follows. In Section 2, we study the distribution of finitely generated and principal ideals in ascending chains. In particular, we investigate ascending chains of absorbing ideals in ACC_d rings. We show that in an ACC_d domain, such chains stabilize. This result is further extended to ACC_d rings under the additional assumption that at least one ideal in the chain is prime. As a consequence, we prove that valuation domains satisfying ACC_d contain only one finite chain of absorbing ideals. Therefore, such rings have finite Krull dimension.

We also establish relationships between the classical ACC and the divisibility-based ACC_d. For instance, we show that an ACC_d ring with only finitely many zero-divisors, which also satisfies the ACC on principal ideals, is Noetherian. Moreover, we prove that if A is an ACC_d ring and M is an A-module such that $\mathrm{Ann}(M)$ is not contained in any principal ideal, then M is Noetherian if and only if it is finitely generated.

Additionally, we investigate the transfer of the ACC_d condition to ring extensions, with particular attention to amalgamated algebras.

In Section 3, we demonstrate that every free submodule of an ACC_d module is finitely generated.

2. Rings Satisfying the Ascending Chain Condition on Divisibility

In this section, we study the behavior of ideals in rings satisfying the ascending chain condition on divisibility (ACC_d). We examine how finitely generated, principal, and n-absorbing ideals distribute in such chains and establish conditions under which these chains become stationary. Connections to classical ACC properties and implications for Noetherianity are also discussed.

Definition 2.1. A ring A is said to satisfy the *ascending chain condition on divisibility* (ACC_d) if for every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots,$$

there exists an integer k such that for all $i \geq k$, there exists an element $a_i \in A$ with

$$I_i = a_i I_{i+1}.$$

Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending chain in a ring A satisfying ACC_d. We denote by k_I the smallest integer such that for all $i \geq k_I$, $I_i = a_i I_{i+1}$ for some $a_i \in A$.

Proposition 2.2. *Let A be a ring satisfying the ACC_d condition. Then the following assertions hold:*

(1) If the set of finitely generated ideals in an ascending chain is infinite, then there exists an integer K such that for all $n \geq K$, the ideals I_n are finitely generated.

(2) If the set of principal ideals in an ascending chain is infinite, then there exists an integer K such that for all $n \geq K$, the ideals I_n are principal.

Proof. (1) Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in A, and suppose the number of finitely generated ideals in the chain is infinite. Since A satisfies the ACC_d condition, there exists an integer N such that for all $n \geq N$, $I_n = x_n I_{n+1}$ for some $x_n \in A$. Let $i \geq N$. Because the number of finitely generated ideals is infinite, there exists $K \geq i$ such that I_K is finitely generated. Then $I_i = x_{iK} I_K$ is finitely generated as well. Hence, all ideals I_n with $n \geq K$ are finitely generated.

(2) The argument is similar to (1), replacing "finitely generated" with "principal." □

Theorem 2.3. *Let A be a domain satisfying ACC_d. Then every ascending chain of absorbing ideals in A is stationary; that is, A satisfies the ACC on absorbing ideals.*

To prove this theorem, we first need two lemmas.

Lemma 2.4. *Let A be a domain, and let*

$$P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n \subseteq P_{n+1}$$

be an ascending chain of nonzero ideals such that P_1 is n-absorbing and for each $i \in \{1, \ldots, n\}$, $P_i = x_i P_{i+1}$ for some $x_i \in A$. Then there exists $k < n$ such that $P_k = P_{k+1}$.

Proof. By hypothesis, $P_i = \left(\prod_{j=i}^{m} x_j\right) P_{m+1}$ for all $i \leq m \leq n$. Let $y_{n+1} \in P_{n+1}$. Then

$$x_1 \cdots x_n y_{n+1} \in P_1.$$

Since P_1 is n-absorbing, either $x_1 \cdots x_n \in P_1$ or there exists $j \in \{1, \ldots, n\}$ such that

$$x_1 \cdots \widehat{x_j} \cdots x_n y_{n+1} \in P_1,$$

where $\widehat{x_j}$ indicates omission. In the first case, $x_1 \cdots x_n \in x_1 \cdots x_n P_{n+1}$, which implies $1 \in P_{n+1}$—a contradiction since A is a domain and P_{n+1} is proper. Thus, we must have $y_{n+1} \in x_j P_{n+1}$, and hence $P_{n+1} = x_j P_{n+1}$. This implies $P_{j+1} = P_j$ as desired. □

Lemma 2.5. *Let A be a domain satisfying ACC_d, and let*

$$P_1 \subseteq P_2 \subseteq \cdots$$

be an ascending chain of ideals such that P_n is m-absorbing for some $n \geq k_P$. Then the chain is stationary.

Proof. Assume for contradiction that the chain is not stationary. Then there exists a strictly increasing subchain

$$Q_1 \subset Q_2 \subset \cdots$$

with $Q_i = x_i Q_{i+1}$ for all $i \geq K_P$ and some $x_i \in A$. Apply Lemma 2.4 to the segment

$$Q_n \subseteq Q_{n+1} \subseteq \cdots \subseteq Q_{n+m},$$

where Q_n is m-absorbing. Then there exists k such that $Q_k = Q_{k+1}$, contradicting the strict inclusion $Q_k \subset Q_{k+1}$. Therefore, the original chain must be stationary. □

Proof of Theorem 2.3 Let A be an ACC_d domain and let

$$P_1 \subseteq P_2 \subseteq \cdots$$

be an ascending chain of ideals such that each P_i is k_i-absorbing for some integer k_i. Then the chain satisfies the assumptions of Lemma 2.5, and hence it is stationary. □

Corollary 2.6. *Let A be a ring. If A satisfies the ACC_d condition, then every ascending chain of absorbing ideals that contains a prime ideal is stationary.*

Proof. Let A be an ACC_d ring and

$$P_1 \subseteq P_2 \subseteq \cdots$$

an ascending chain of absorbing ideals such that P_k is a prime ideal for some $k \geq 1$. Then the induced chain

$$P_{k+1}/P_k \subseteq P_{k+2}/P_k \subseteq \cdots$$

is an ascending chain of absorbing ideals in the domain A/P_k, which also satisfies the ACC_d condition. Therefore, the chain stabilizes. Hence, the original chain is stationary. □

Corollary 2.7. *Let A be an ACC_d domain. Then every ascending chain of ideals contains only finitely many absorbing ideals.*

Corollary 2.8. *Let A be an ACC_d ring. Then every ascending chain of ideals that contains a prime ideal contains only finitely many absorbing ideals.*

Corollary 2.9. *Let A be an ACC_d domain, and let*

$$P_1 \subseteq P_2 \subseteq \cdots$$

be an ascending chain of ideals. Suppose there exists an integer K such that for all $i > K$, $P_i = x_i P_{i+1}$ for some $x_i \in A$, and that P_n is absorbing for some $n > K$. Then the chain is stationary.

Proof. By assumption, the chain satisfies the ACC_d condition, and there exists $n > K$ such that P_n is m-absorbing. Then, by Theorem 2.3, the subchain

$$P_n \subseteq P_{n+1} \subseteq \cdots$$

is stationary, and hence the entire chain is stationary. □

It is well known that over a Noetherian ring, a module is Noetherian if and only if it is finitely generated. The following proposition gives a similar result in the ACC_d context.

Proposition 2.10. *Let A be an ACC_d ring and M an A-module such that $\operatorname{Ann}(M)$ is not contained in any principal ideal. Then M is Noetherian if and only if it is finitely generated.*

Proof. Suppose M is a finitely generated A-module, and assume that $\operatorname{Ann}(M)$ is not contained in any principal ideal. Then $A/\operatorname{Ann}(M)$ satisfies the ACC on principal ideals and has only finitely many zero-divisors (since A is ACC_d). By previous results, $A/\operatorname{Ann}(M)$ is Noetherian. Therefore, M is a finitely generated module over a Noetherian ring, hence Noetherian.

Conversely, if M is Noetherian, then it is in particular finitely generated. Hence, the two conditions are equivalent. □

Corollary 2.11. *Let A be an ACC_d ring and let I be an ideal of A such that $\operatorname{Ann}(I)$ is not contained in any principal ideal. Then I is finitely generated if and only if every ideal contained in I is finitely generated.*

Corollary 2.12. *Let A be a local ACC_d ring and M an A-module such that $\operatorname{Ann}(M)$ is maximal and not principal. Then M is Noetherian if and only if it is finitely generated.*

Corollary 2.13. *Let A be an ACC_d ring and I an ideal such that* $\mathrm{Ann}(I)$ *is maximal and not principal. Then I is finitely generated if and only if every ideal contained in I is finitely generated.*

Proposition 2.14. *Let A be a valuation domain satisfying the ACC_d condition. Then:*

(1) There exists only one chain of absorbing ideals, and this chain is finite.
(2) The ring A has finite Krull dimension.

Proof. Let A be a valuation domain satisfying the ACC_d condition.

(1) Let $(P) : P_1 \subseteq P_2 \subseteq \cdots$ and $(Q) : Q_1 \subseteq Q_2 \subseteq \cdots$ be two ascending chains of absorbing ideals in A. By Theorem 2.3, each chain must be finite. Assume that these chains are maximal with respect to inclusion. Then no absorbing ideal exists strictly between any two successive ideals in either chain. Hence, for every i, there exists j such that $Q_i = P_j$. Therefore, the two chains coincide, and there exists only one such chain.

(2) Since every prime ideal is absorbing in a valuation domain, the uniqueness and finiteness of the chain of absorbing ideals implies that there is only one chain of prime ideals. Hence, the Krull dimension of A is finite and equals the length of this chain. □

Theorem 2.15. *Let A be a ring satisfying the ACC_d condition. Assume that A has only finitely many zero-divisors and that every descending chain of principal ideals is stationary. Then A is Noetherian.*

To prove this theorem, we need the following lemma. First, note that for an ACC_d ring and an ascending chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots ,$$

we can associate a descending chain of principal ideals via the relation $I_i = x_i I_{i+1}$ for $i \geq K$. Iterating this relation yields

$$I_k = x_k x_{k+1} \cdots x_{i-1} I_i = a_i I_i.$$

Then the associated descending chain of principal ideals is

$$a_k A \supseteq a_{k+1} A \supseteq \cdots ,$$

which we denote by $\mathrm{DCP}(I)$.

Lemma 2.16. *Let A be an ACC_d ring, and let $I : I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals. Suppose that* $\mathrm{DCP}(I)$ *is stationary and that each I_i contains a non-zero-divisor. Then the chain (I) is stationary.*

Proof. Assume that $\mathrm{DCP}(I)$ is stationary, i.e., $a_n A = a_{n+1} A$ for all $n \geq k$. Then $a_n = x a_{n+1}$ for some $x \in A$, and so

$$a_n = x_k x_{k+1} \cdots x_{n-1} = x x_k x_{k+1} \cdots x_{n-1} x_n.$$

Hence,

$$x_k x_{k+1} \cdots x_{n-1}(1 - x x_n) = 0.$$

Since the product $x_k x_{k+1} \cdots x_{n-1}$ is not a zero-divisor by assumption, we must have $1 - x x_n = 0$, i.e., x_n is a unit. Therefore, $I_n = I_{n+1}$ for all $n \geq k$, and the chain is stationary. □

Proof of Theorem 2.15 Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in A. Since A satisfies ACC_d, there exists N such that $I_k = x_k I_{k+1}$ for all $k \geq N$. Because the number of zero-divisors in A is finite, there exists $K \geq N$ such that I_k contains no zero-divisor for all $k \geq K$. By hypothesis, the associated descending chain DCP(I) is stationary. Hence, by the lemma, the chain (I) is stationary. Therefore, A is Noetherian. □

Corollary 2.17. *Let A be a domain satisfying ACC_d, and let*

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be an ascending chain of ideals. If the associated descending chain of principal ideals DCP(I) *is stationary, then the chain (I) is also stationary.*

Proof. This follows from the fact that a domain has no zero-divisors, so the lemma applies immediately.

Corollary 2.18. *Let A be a ring satisfying ACC_d, and let*

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be an ascending chain of ideals that contains a prime ideal. If the associated descending chain of principal ideals DCP(I) *is stationary, then the chain (I) is also stationary.*

Proof. Let $P = I_1$ be a prime ideal, and consider the chain modulo P:

$$I_1/P \subseteq I_2/P \subseteq \cdots,$$

and the associated descending chain

$$a_k A/P \supseteq a_{k+1} A/P \supseteq \cdots.$$

Since A/P is a domain and satisfies ACC_d, and DCP$(I)/P$ is stationary, Corollary 2.17 implies that the chain I_n/P is stationary. Hence, the original chain is also stationary. □

Proposition 2.19. *Let R be a ring. Then the following statements hold:*

(1) If R satisfies the ACC_d condition, then so does $S^{-1}R$ for any multiplicative subset S of R.
(2) If R is a semi-local ring and $S^{-1}R$ satisfies ACC_d for each $S = R \backslash M$, where M is a maximal ideal of R, then R satisfies ACC_d.

Proof. (1) Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending chain of ideals in R. Then the localized chain

$$S^{-1}I_1 \subseteq S^{-1}I_2 \subseteq S^{-1}I_3 \subseteq \cdots$$

is an ascending chain of ideals in $S^{-1}R$. Since R satisfies ACC_d, there exists an integer $k \in \mathbb{N}$ such that for all $i \geq k$, there exists $a_i \in R$ with $I_i = a_i I_{i+1}$. It follows that

$$S^{-1}I_i = S^{-1}(a_i I_{i+1}) = \left(\frac{a_i}{1}\right) S^{-1}I_{i+1},$$

so $S^{-1}R$ satisfies ACC_d.

(2) Let R be a semi-local ring with maximal ideals $M_1, M_2, \ldots, M_t$. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in R. For each $i = 1, 2, \ldots, t$, let $S_i = R \setminus M_i$. Then the localized chain

$$S_i^{-1}I_1 \subseteq S_i^{-1}I_2 \subseteq \cdots$$

is an ascending chain of ideals in $S_i^{-1}R$, which satisfies ACC_d by assumption. Thus, for each i, there exists $k_i \in \mathbb{N}$ such that for all $j \geq k_i$, there exists $\frac{a_j}{s_j} \in S_i^{-1}R$ with

$$S_i^{-1}I_j = \left(\frac{a_j}{s_j}\right) S_i^{-1}I_{j+1} = S_i^{-1}(a_j I_{j+1}).$$

Let $k = \max\{k_1, k_2, \ldots, k_t\}$. Then for all $j \geq k$, the equality $S_i^{-1}(a_j I_j) = S_i^{-1}I_{j+1}$ holds for all i. By [9, Corollary, p. 164], it follows that $a_j I_j = I_{j+1}$ in R for all $j \geq k$. Therefore, R satisfies ACC_d. □

Corollary 2.20. *Let R be a semi-local ring. Then the following statements are equivalent:*

(1) R satisfies ACC_d.
(2) $S^{-1}R$ satisfies ACC_d for each $S = R \setminus M$, where M is a maximal ideal of R.

Proof. This follows directly from Proposition 2.19. □

Let us recall two related constructions. Let A and B be rings, J an ideal of B, and $f : A \to B$ a ring homomorphism. The *amalgamation of A and B along J with respect to f* is the subring of $A \times B$ defined by

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A,\ j \in J\}.$$

This construction generalizes the *amalgamated duplication* of a ring along an ideal, introduced and studied by D'Anna, Finocchiaro, and Fontana in [1, 2, 3, 4, 5]. See [8] for a survey on this topic.

Theorem 2.21. *Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. Then:*

(1) If $A \bowtie^f J$ satisfies the ACC_d condition, then so does A.
(2) If $A \bowtie^f J$ satisfies the ACC_d condition and $J \subseteq \mathrm{Ann}(\mathrm{Im}\, f \setminus \{1\})$, then $J^2 = J$.

Proof. Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B.

(1) Assume that $A \bowtie^f J$ satisfies the ACC_d condition. Then, by [6, Example 4.2(3)], the quotient

$$(A \bowtie^f J)/(0 \bowtie^f J) \cong A$$

also satisfies ACC_d. Hence, A inherits the ACC_d property.

(2) Let

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

be a descending chain of ideals of A. Then the corresponding chain

$$I_1 \bowtie^f J \supseteq I_2 \bowtie^f J \supseteq \cdots \supseteq I_n \bowtie^f J \supseteq \cdots$$

is a descending chain of nonzero ideals in $A \bowtie^f J$. Since $A \bowtie^f J$ satisfies ACC_d, there exists an integer k such that for all $i \geq k$,

$$I_{i+1} \bowtie^f J = (b_i, f(b_i) + \ell_i)(I_i \bowtie^f J)$$

for some element $(b_i, \ell_i) \in A \bowtie^f J$.

Let $a_{i+1} \in I_{i+1}$ and $j_{i+1} \in J$. Then there exist $a_i \in I_i$ and $j_i \in J$ such that:

$$(a_{i+1}, f(a_{i+1}) + j_{i+1}) = (b_i, f(b_i) + \ell_i)(a_i, f(a_i) + j_i).$$

Multiplying in $A \bowtie^f J$, we obtain:

$$a_{i+1} = b_i a_i,$$

$$f(a_{i+1}) + j_{i+1} = b_i f(a_i) + b_i j_i + a_i \ell_i + \ell_i f(a_i) + \ell_i j_i.$$

Using the assumption $J \subseteq \mathrm{Ann}(\mathrm{Im}\, f \setminus \{1\})$, we have:

$$b_i f(a_i) = 0, \quad b_i j_i = 0, \quad a_i \ell_i = 0, \quad \ell_i f(a_i) = 0.$$

Hence,

$$f(a_{i+1}) + j_{i+1} = \ell_i j_i,$$

so $j_{i+1} = \ell_i j_i$. This is true for all $j_{i+1} \in J$. Therefore, $J^2 = J$, as claimed. □

3. Modules with Ascending Divisibility on Submodules

In this section, we present results concerning modules that satisfy ascending divisibility on submodules. This notion was introduced by R. Dastanpour and A. Ghorbani in [6] as follows:

Let M be an R-module. Then M is said to satisfy the *ascending chain condition on divisibility* (ACC_d) if for every ascending chain of submodules

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq N_n \subseteq \cdots,$$

there exists, for each $i \geq 0$, an endomorphism $\varphi_i \in \mathrm{End}_R(M)$ such that $N_i = \varphi_i(N_{i+1})$.

In this paper, we consider a special case where each φ_i is multiplication by an element $a_i \in A$, i.e., $\varphi_i(x) = a_i x$ for all $x \in M$. This leads us to the following definition:

Definition 3.1. An A-module M is said to satisfy the *ascending chain condition on divisibility* (ACC_d) if for every ascending chain of submodules

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots,$$

there exists an integer k such that for all $i \geq k$, there exists $a_i \in A$ with

$$N_i = a_i N_{i+1}.$$

Proposition 3.2. *Let M be an A-module satisfying the ACC_d condition. Then every free submodule of M is finitely generated.*

Proof. Let M be an A-module satisfying ACC_d, and let $N \subseteq M$ be a free submodule. Let $(x_i)_{i \in I}$ be a basis of N. Suppose for contradiction that I is infinite. Define a chain of submodules by

$$N_i := Ax_1 + Ax_2 + \cdots + Ax_i \quad \text{for each } i \in \mathbb{N}.$$

Then $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain in M. Since M satisfies ACC_d, there exists an integer k such that $N_i = a_i N_{i+1}$ for some $a_i \in A$ for all $i \geq k$.

In particular, $x_{i+1} \in N_{i+1}$ implies that $a_i x_{i+1} \in N_i$. So there exist $b_1, b_2, \ldots, b_i \in A$ such that

$$a_i x_{i+1} = b_1 x_1 + b_2 x_2 + \cdots + b_i x_i.$$

This contradicts the assumption that $(x_i)_{i \in I}$ is a free set, since x_{i+1} would be A-linearly dependent on $\{x_1, \ldots, x_i\}$. Therefore, I must be finite, and N is finitely generated. □

Corollary 3.3. *Let A be a ring satisfying ACC_d, and suppose that the module $A^{(I)}$ satisfies ACC_d. Then the index set I must be finite.*

Proof. The module $A^{(I)}$ is a free A-module. By the proposition above, any free submodule of an ACC_d module is finitely generated. Hence, I must be finite. □

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