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MODULES SATISFYING THE DIVISIBILITY CONDITION ON DESCENDING CHAINS

HWANKOO KIM*, SALAH EDDINE MAHDOU, AND OUSSAMA AYMANE ES SAFI

Abstract. This paper deals with modules satisfying the divisibility condition on descending chains, denoted by DCC_d. This notion was introduced by R. Dastanpour and A. Ghorbani as a generalization of Artinian modules. The goal of this paper is to further explore this class of modules. For instance, we study the behavior of finitely generated and principal ideals within descending chains. Among the main results, we present a theorem concerning modules over a DCC_d ring. In the final part of the paper, we examine some properties of DCC_d modules. For example, we show that there are no free DCC_d (respectively, Artinian) modules over a domain that is not principal. Finally, we conclude this work by investigating the transfer of the DCC_d property to trivial ring extensions.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity, and all modules are unital. If A is a ring, we denote by $Nil(A)$ the set (ideal) of all nilpotent elements of A, and by $tq(A) = A_{A\setminus Z(A)}$ the total ring of quotients of A. An ideal is said to be *regular* if it contains a regular element (i.e., a non-zero-divisor).

In [4], R. Dastanpour and A. Ghorbani introduced a generalization of modules satisfying the descending chain condition (DCC). A module M is said to satisfy the *epi-DCC* on submodules if, in every descending chain of submodules of M, all but finitely many terms are homomorphic images of their predecessors.

In this paper, we define a subclass of such modules as follows: an A-module M is said to satisfy the *descending chain condition on divisibility* (denoted DCC_d) if, in every descending chain of submodules

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$$

of M, there exists an integer k such that for all $i \geq k$, there exists an element $a_i \in A$ with $a_i N_i = N_{i+1}$. If each a_i is a unit in A, then M is Artinian.

Let A be a ring and E an A-module. The *trivial ring extension* of A by E is the ring whose underlying additive group is $A \oplus E$, and whose multiplication is defined

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* : Corresponding author.

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by

$$(a, e)(b, f) := (ab, af + be)$$

for all $a, b \in A$ and $e, f \in E$. The basic properties of trivial ring extensions are detailed in [6] and [7], and this construction has been widely used to produce new examples of rings with specific properties (see [1, 2, 5, 8]).

For simplicity, we refer to modules satisfying the descending chain condition on divisibility by the abbreviation DCC_d.

The remainder of this paper consists of two sections. In the first section, we study the distribution of finitely generated submodules in descending chains of DCC_d submodules. Specifically, we show that in such a chain, either the number of finitely generated submodules or the number of non-finitely generated submodules is finite.

We also investigate localization, direct products, and homomorphic images of DCC_d modules. Before closing the section with results concerning the transfer of the DCC_d condition to trivial ring extensions, we establish several important results. For example, we show that if a torsion A-module satisfies DCC_d, then A itself is a DCC_d ring. As a consequence, over a domain that is not principal, there are no free DCC_d modules. In particular, such domains admit no free Artinian modules.

It is well known that if M is an Artinian A-module, then every monomorphism in $\mathrm{Hom}_A(M, M)$ is an epimorphism. By providing a counterexample, we show that this property does not generally hold for DCC_d modules. Furthermore, we give sufficient conditions under which a DCC_d module does satisfy this property.

2. Results

In this section, we investigate modules satisfying the divisibility condition on descending chains of submodules.

Definition 2.1. An A-module M is said to satisfy the *descending chain condition on divisibility* (DCC_d) if for every descending chain

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots \supseteq N_n \supseteq \cdots$$

of submodules of M, there exists an integer n such that for all $i \geq n$, there exists an element $a_i \in A$ with $N_i = a_i N_{i+1}$.

Example 2.2. Let A be a DCC_d ring. Then every ideal of A is a DCC_d A-module. This holds since every submodule of a given ideal is itself an ideal of A.

Example 2.3. The ring $\mathbb{Z}[X]$ does not satisfy DCC_d as a $\mathbb{Z}$-module. Consider the descending chain

$$X\mathbb{Z}[X] \supset X^2\mathbb{Z}[X] \supset \cdots \supset X^n\mathbb{Z}[X] \supset \cdots .$$

For every $m \in \mathbb{N}$, we have $mX^n\mathbb{Z}[X] \neq X^{n+1}\mathbb{Z}[X]$. Hence, the divisibility condition fails.

Remark 2.4. Let A be a ring. It is well known that the submodules of A (viewed as a module over itself) are precisely its ideals. Therefore, A is a DCC_d ring if and only if it is a DCC_d A-module.

The following result concerns the distribution of finitely generated submodules in a descending chain of a DCC_d module.

Proposition 2.5. *Let A be a ring and M an A-module satisfying the DCC_d condition. Then the following assertions hold:*

(1) If the number of finitely generated submodules in a descending chain $M_1 \supset M_2 \supset \cdots$ is infinite, then there exists an integer K such that for all $n \geq K$, M_n is finitely generated.
(2) If the number of cyclic submodules in a descending chain $M_1 \supset M_2 \supset \cdots$ is infinite, then there exists an integer K such that for all $n \geq K$, M_n is cyclic.

Proof. Let $M_1 \supset M_2 \supset \cdots$ be a descending chain of submodules of an A-module M satisfying the DCC_d condition.

For part (1), assume that the number of finitely generated submodules in the chain is infinite. Since M satisfies the DCC_d condition, there exists an integer k such that for all $n \geq k$, we have $M_{n+1} = a_n M_n$ for some $a_n \in A$. Let $p \geq k$ be an integer. Since the number of finitely generated submodules is infinite, there exists $m \geq p$ such that M_m is finitely generated. But then $M_p = a_{pm} M_m$ for some $a_{pm} \in A$, and hence M_p is also finitely generated. Repeating this argument shows that all M_n with $n \geq m$ are finitely generated.

Part (2) follows by the same argument, replacing "finitely generated" with "cyclic" and observing that the image of a cyclic module under multiplication remains cyclic. □

In the following proposition, we investigate the transfer of the DCC_d property to direct sums of modules.

Proposition 2.6. *Let A be a ring and I an index set. Then:*

(1) If $A^{(I)}$ is a DCC_d A-module, then $A^{(I)}$ is a DCC_d ring.
(2) If I is an infinite set, then $A^{(I)}$ does not satisfy DCC_d as an A-module.

Proof. Let A be a ring and I an index set.

(1) Assume that $A^{(I)}$ is a DCC_d A-module. We show that $A^{(I)}$ is a DCC_d ring. Let

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

be a descending chain of ideals of $A^{(I)}$. Since each ideal of $A^{(I)}$ is an A-submodule and $A^{(I)}$ satisfies the DCC_d condition, there exists an element $a \in A$ and an integer K such that for all $n \geq K$, we have $I_{n+1} = aI_n$. Since scalar multiplication in $A^{(I)}$ is componentwise, this implies that for all $x = (x_i) \in I_n$, we have $ax = (ax_i) \in I_{n+1}$. Hence, the chain of ideals satisfies the divisibility condition, and so $A^{(I)}$ is a DCC_d ring.

(2) Let I be an infinite index set. Suppose for contradiction that $A^{(I)}$ satisfies DCC_d on submodules. Then, by part (1), $A^{(I)}$ would also satisfy DCC_d as a ring. However, according to [3], this is not the case. Therefore, $A^{(I)}$ cannot satisfy DCC_d on submodules. □

Proposition 2.7. *Let A be a DCC_d ring. If A^2 is a DCC_d A-module, then A is Artinian.*

Proof. Let A be a DCC_d ring such that A^2 is a DCC_d A-module. We show that A is Artinian. Let

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

be a descending chain of ideals of A. Then

$$I_1 \oplus A \supset I_2 \oplus A \supset \cdots \supset I_n \oplus A \supset \cdots$$

is a descending chain of submodules of the A-module $A \oplus A = A^2$. Since A^2 satisfies the DCC_d condition, there exists an element $a \in A$ and an integer K such that for all $n \geq K$,

$$I_{n+1} \oplus A = a(I_n \oplus A).$$

This implies that $I_{n+1} = aI_n$ and $A = aA$. Therefore, a is a unit in A and so $I_{n+1} = I_n$ for all $n \geq K$. Hence, the chain stabilizes, and A is Artinian. □

Corollary 2.8. *Let A be a ring and I an index set. Then $A^{(I)}$ need not satisfy the DCC_d condition as an A-module.*

Proposition 2.9. *Let M and N be A-modules, and let $f : M \to N$ be an epimorphism. If M is a DCC_d A-module, then N is also a DCC_d A-module.*

Proof. Let $N_1 \supset N_2 \supset \cdots$ be a descending chain of submodules of N. Consider the corresponding chain of preimages under f:

$$f^{-1}(N_1) \supset f^{-1}(N_2) \supset \cdots,$$

which is a descending chain of submodules of M. Since M satisfies the DCC_d condition, there exists an integer K and elements $a_n \in A$ such that for all $n \geq K$, we have

$$f^{-1}(N_{n+1}) = a_n f^{-1}(N_n).$$

Applying f and using that f is an epimorphism, we get

$$N_{n+1} = f(f^{-1}(N_{n+1})) = f(a_n f^{-1}(N_n)) = a_n f(f^{-1}(N_n)) = a_n N_n,$$

for all $n \geq K$. Thus, N satisfies the DCC_d condition. □

Corollary 2.10. *Let M be a DCC_d A-module and N a submodule of M. Then:*
- *(1) N is a DCC_d A-module.*
- *(2) M/N is a DCC_d A-module.*

Proof. (1) Let N be a submodule of a DCC_d A-module M. Any descending chain of submodules in N is also a descending chain in M, hence it satisfies the divisibility condition. Therefore, N is DCC_d.

(2) Since the canonical projection $M \twoheadrightarrow M/N$ is an epimorphism, the result follows from Proposition 2.9. □

Corollary 2.11. *Let A be a DCC_d ring. Then every finitely generated A-module satisfies the DCC_d condition.*

Proof. Let M be a finitely generated A-module. Then there exists an integer $n \in \mathbb{N}$ and a surjective homomorphism $f : A^n \to M$. Since A is a DCC_d ring, the free module A^n is a DCC_d A-module. Hence, by Proposition 2.9, M is also a DCC_d A-module. □

Definition 2.12. Let A be a DCC_d ring. An A-module M is said to be A-*finite* if there exists a nonzero element $a \in A$ and a finitely generated submodule N of M such that $aM \subseteq N$. If the element a is regular (i.e., not a zero-divisor), then M is called A_g-*finite*.

Corollary 2.13. *Let A be a DCC_d ring. Then every A_g-finite A-module satisfies the DCC_d condition.*

Proof. Since M is A_g-finite, there exists a regular (i.e., non-zero-divisor) element $a \in A$ and a finitely generated submodule N of M such that $aM \subseteq N$. The regularity of a implies that $M \cong aM \subseteq N$, and thus M is a submodule of a finitely generated module. Since A is a DCC_d ring, N is DCC_d by earlier results, and hence so is M. □

Definition 2.14. Let A be a ring, M an A-module, and T a family of submodules of M. An element $N \in T$ is said to be *weakly A-minimal in T* if for every $L \in T$ with $N \subset L$, there exists a nonzero element $a \in A$ such that $aL \subseteq N$. A submodule N of M is called *weakly A-minimal* if it is weakly A-minimal in the set of all nonzero submodules of M.

Proposition 2.15. *Let A be a ring and M an A-module that is weakly A-minimal. If $M = M_1 \oplus M_2$, then M_1 and M_2 are torsion modules.*

Proof. Assume $M = M_1 \oplus M_2$ with $M_1 \neq 0$ and $M_2 \neq 0$. Since $M_1 \subseteq M$, and M is weakly A-minimal, there exists a nonzero element $a_1 \in A$ such that $a_1(M_1 \oplus M_2) \subseteq M_1$. In particular, $a_1M_2 \subseteq M_1 \cap M_2 = 0$. Hence, $a_1M_2 = 0$, showing that M_2 is a torsion module. A symmetric argument shows that M_1 is also a torsion module. □

Let M be an A-module and $a \in A$. We say that a is a *non-zero-divisor on M* if for every $m \in M$, the relation $am = 0$ implies $m = 0$.

Proposition 2.16. *Let A be a ring and M an A-module. Then the following statements are equivalent:*

(1) M is a DCC_d A-module.
(2) Every nonempty set of submodules of M has a weakly A-minimal element.

Proof. (1) ⇒ (2): Let T be a nonempty set of submodules of M that contains no weakly A-minimal element. Choose $N_1 \in T$. Since N_1 is not weakly A-minimal, there exists $N_2 \in T$ with $N_2 \subsetneq N_1$ such that for all nonzero $a \in A$, $aN_2 \not\subseteq N_1$. Proceeding inductively, we obtain a strictly descending chain

$$N_1 \supsetneq N_2 \supsetneq N_3 \supsetneq \cdots,$$

such that no N_i satisfies the divisibility condition with respect to the next. This contradicts the DCC_d property of M. Hence, every nonempty set of submodules must have a weakly A-minimal element.

(2) ⇒ (1): Let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be a descending chain of submodules of M. Let $T = \{N_i \mid i \geq 1\}$. By hypothesis, T has a weakly A-minimal element, say N_k. Then for each $n \geq k$, there exists a nonzero element $a_n \in A$ such that $a_nN_n \subseteq N_k \subseteq N_{n+1} \subseteq N_n$, so $a_nN_n = N_{n+1}$. This shows that the chain satisfies the divisibility condition, and thus M is a DCC_d A-module. □

Remark 2.17. Let A be a ring and I an ideal of A. Then:

(1) A/I is a DCC_d ring if and only if it satisfies the DCC_d condition as an A-module.
(2) If A is a DCC_d ring, then A/I satisfies the DCC_d condition as an A-module for every ideal I of A.

Proposition 2.18. *Let A be a ring, and let $I \subset J$ be ideals of A. If A/J satisfies the DCC_d condition as an A-module, then A/I also satisfies the DCC_d condition as an A-module.*

Proof. Assume that A/J satisfies the DCC_d condition as an A-module. Let

$$I_1/I \supseteq I_2/I \supseteq \cdots \supseteq I_n/I \supseteq \cdots$$

be a descending chain of submodules of A/I. Since $I \subset J$, this induces a descending chain

$$I_1/J \supseteq I_2/J \supseteq \cdots \supseteq I_n/J \supseteq \cdots$$

in A/J. By assumption, there exists an integer K such that for all $n \geq K$, there exists $a_n \in A$ with $I_{n+1}/J = a_n(I_n/J)$. It follows that $I_{n+1} = a_n I_n$, and thus $I_{n+1}/I = a_n(I_n/I)$. Therefore, A/I satisfies the DCC_d condition as an A-module. □

Proposition 2.19. *Let M be a DCC_d A-module and S a multiplicative subset of A. Then $S^{-1}M$ is a DCC_d $S^{-1}A$-module.*

Proof. Let

$$S^{-1}N_1 \supseteq S^{-1}N_2 \supseteq \cdots \supseteq S^{-1}N_i \supseteq \cdots$$

be a descending chain of submodules of $S^{-1}M$, where each N_i is a submodule of M. Since M is a DCC_d A-module, there exists an integer K and elements $a_n \in A$ such that $N_{n+1} = a_n N_n$ for all $n \geq K$. Then,

$$S^{-1}N_{n+1} = (a_n/1) \cdot S^{-1}N_n,$$

for all $n \geq K$. Therefore, $S^{-1}M$ satisfies the DCC_d condition as an $S^{-1}A$-module. □

Proposition 2.20. *Let A be a ring and M an A-module satisfying the DCC_d condition, such that $Z(M) \neq 0$. Then A satisfies the DCC_d condition.*

Proof. Let $x \in M$ be such that $\mathrm{Ann}(x) = 0$. Then the map $\varphi : A \to M$ defined by $\varphi(a) = ax$ is injective and identifies A with the submodule $Ax \subseteq M$. Since M is a DCC_d A-module and Ax is a submodule, Ax is also DCC_d, and hence $A \cong Ax$ satisfies the DCC_d condition. Therefore, A is a DCC_d ring. □

Corollary 2.21. *Let A be a ring and M an A-module satisfying the DCC_d condition on submodules. If A is a domain and M is a torsion-free A-module, then A is a principal ideal domain.*

Proof. Let A be a domain and M a torsion-free A-module satisfying the DCC_d condition. By Proposition 2.20, A satisfies the DCC_d condition on submodules, and hence on ideals. According to [3], a domain satisfying DCC_d on ideals must be a principal ideal domain. □

Remark 2.22. Over a domain that is not principal, there exists no free module satisfying the DCC_d condition. In particular, there are no free Artinian modules over such a ring.

Corollary 2.23. *Let A be a ring. Then the following statements are equivalent:*

(1) *A is a DCC_d ring.*
(2) *A is a DCC_d A-module.*
(3) *Every regular ideal of A is a DCC_d A-module.*
(4) *There exists a regular element $x \in A$ such that the principal ideal Ax is a DCC_d A-module.*

Proof. (1) $\iff$ (2): This equivalence follows directly from Remark 2.4, which states that the DCC_d property of a ring coincides with that of the ring viewed as an A-module.

(2) $\Rightarrow$ (3): Every regular ideal is a submodule of A, and thus inherits the DCC_d property by Corollary 2.10.

(3) $\Rightarrow$ (4): This is a special case, taking any regular element $x \in A$ and considering the principal ideal Ax.

(4) $\Rightarrow$ (2): If x is a regular element of A, then the A-module Ax is isomorphic to A, so the DCC_d property of Ax implies that A is a DCC_d A-module. □

Example 2.24. (1) Let A be a domain which is not a field. Then $A[X]$ does not satisfy the DCC_d condition on submodules. Hence, $A[X]$ is not Artinian as an A-module.
(2) Let A be a domain which is not a field. Then $A[X, Y]$ is not a DCC_d $A[X]$-module. Therefore, $A[X, Y]$ is not Artinian as an $A[X]$-module.
(3) $\mathbb{Z}[X, Y]$ is not a DCC_d $\mathbb{Z}[X]$-module. Therefore, $\mathbb{Z}[X, Y]$ is not Artinian as a $\mathbb{Z}[X]$-module.

Proposition 2.25. *Let A be a ring and M an A-module.*
(1) Suppose that A satisfies the DCC_d condition, and let $N_1 \subset N_2 \subset \cdots$ be an ascending chain of submodules of M. Then the chain $\mathrm{Ann}\,(N_1) \supseteq \mathrm{Ann}\,(N_2) \supseteq \cdots$ is stationary.
(2) Suppose that M is a torsion-free A-module satisfying DCC_d, and let $N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of submodules of M. Then the chain $\mathrm{Ann}\,(N_1) \supseteq \mathrm{Ann}\,(N_2) \supseteq \cdots$ is stationary.

Proof. (1) Let A be a DCC_d ring and $N_1 \subset N_2 \subset \cdots$ an ascending chain of submodules of M. Then the corresponding chain of annihilators

$$\mathrm{Ann}\,(N_1) \supseteq \mathrm{Ann}\,(N_2) \supseteq \cdots$$

is descending in A, which satisfies DCC_d. Therefore, there exists an integer n such that for all $k \geq n$, $\mathrm{Ann}\,(N_{k+1}) = a_k \,\mathrm{Ann}\,(N_k)$ for some $a_k \in A$.

To show that $\mathrm{Ann}\,(N_{k+1}) = \mathrm{Ann}\,(N_k)$, let $x \in \mathrm{Ann}\,(N_k)$ and $y \in N_{k+1}$. Since $\mathrm{Ann}\,(N_{k+1}) = a_k \,\mathrm{Ann}\,(N_k)$, we may write $y = a_k z$ for some $z \in N_k$. Then,

$$xy = x(a_k z) = a_k(xz) = 0,$$

which implies $x \in \mathrm{Ann}\,(N_{k+1})$. Hence, $\mathrm{Ann}\,(N_k) = \mathrm{Ann}\,(N_{k+1})$, and the chain is stationary.

(2) Suppose M is a torsion-free A-module satisfying DCC_d, and let $N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of submodules. Then there exists K such that for all $n \geq K$, we have $N_{n+1} = a_n N_n$ for some $a_n \in A$.

Let $x \in \mathrm{Ann}\,(N_{n+1})$ and $y \in N_n$. Then $a_n y \in N_{n+1}$, so

$$a_n xy = x a_n y = 0.$$

Since M is torsion-free, it follows that $xy = 0$, i.e., $x \in \mathrm{Ann}\,(N_n)$. Thus,

$$\mathrm{Ann}\,(N_{n+1}) \subseteq \mathrm{Ann}\,(N_n),$$

and the chain of annihilators is descending and eventually constant. □

Corollary 2.26. *Let A be a ring and M a free torsion-free A-module satisfying the DCC_d condition. Then the following assertions hold:*

(1) If $N_1 \subset N_2 \subset \cdots$ is an ascending chain of submodules of M, then the chain $\mathrm{Ann}\,(N_1) \supseteq \mathrm{Ann}\,(N_2) \supseteq \cdots$ is stationary.
(2) If $N_1 \supseteq N_2 \supseteq \cdots$ is a descending chain of submodules of M, then the chain $\mathrm{Ann}\,(N_1) \subseteq \mathrm{Ann}\,(N_2) \subseteq \cdots$ is stationary.

Proposition 2.27. *Let A be a ring such that every element of A is either a regular element or a unit, and let M be a DCC_d A-module. Suppose $f : M \to M$ is a monomorphism. Then one of the following holds:*

(1) f is surjective;
(2) There exists a nonzero element $a \in A$ such that $aM = 0$.

To prove this, we need the following lemma:

Lemma 2.28. *Let A be a ring, M an A-module, and $f : M \to M$ a monomorphism. If there exists an integer n such that $\mathrm{Im}(f^n) = \mathrm{Im}(f^{n+1})$, then f is surjective.*

Proof. Let $y \in M$. Then $f^n(y) \in \mathrm{Im}(f^n) = \mathrm{Im}(f^{n+1})$. So there exists $x \in M$ such that $f^n(y) = f^{n+1}(x) = f^n(f(x))$. Since f is injective, it follows that $y = f(x)$, hence f is surjective. □

Proof of Proposition 2.27. Let A be a ring in which every element is either a zero-divisor or a unit, M a DCC_d A-module, and $f : M \to M$ a monomorphism. The chain

$$M \supseteq \mathrm{Im}(f) \supseteq \mathrm{Im}(f^2) \supseteq \cdots$$

is a descending chain of submodules of M, so by the DCC_d condition, there exists an integer K and elements $a_n \in A$ such that for all $n \geq K$,

$$\mathrm{Im}(f^{n+1}) = a_n \,\mathrm{Im}(f^n).$$

If some a_n is a unit, then $\mathrm{Im}(f^{n+1}) = \mathrm{Im}(f^n)$, and by Lemma 2.28, f is surjective.

Otherwise, a_n is a zero-divisor. Then there exists $b_n \in A$ such that $b_n a_n = 0$. We claim that $b_n M = 0$.

Indeed, let $z \in M$. Then $f^{n+1}(z) \in \mathrm{Im}(f^n)$. Since $\mathrm{Im}(f^{n+1}) = a_n \,\mathrm{Im}(f^n)$, there exists $y \in M$ such that $f^{n+1}(z) = a_n f^n(y)$, and so

$$f^n(f(z)) = f^n(a_n y) = a_n f^n(y).$$

Applying b_n and using $b_n a_n = 0$, we get

$$f^n(f(b_n z)) = b_n f^{n+1}(z) = b_n a_n f^n(y) = 0.$$

Since f is injective, this implies $f^n(b_n z) = 0$, and thus by injectivity again, $b_n z = 0$. Therefore, $b_n M = 0$. □

Corollary 2.29. *Let n be a positive integer and let M be a DCC_d $\mathbb{Z}/n\mathbb{Z}$-module such that $\mathrm{Ann}(M) = 0$. Then every monomorphism of M is surjective.*

Proof. The result follows from the fact that every element of $\mathbb{Z}/n\mathbb{Z}$ is either a zero-divisor or a unit. □

Corollary 2.30. *(1) Let A be a ring, R its total ring of quotients, and M a DCC_d R-module such that $\mathrm{Ann}(M) = 0$. Then every monomorphism of M is surjective.*
(2) Let A be a DCC_d ring and R its total ring of quotients. Then every monomorphism of R is surjective.

Proof. (1) This follows from the fact that in a total ring of quotients, every element is either a unit or a zero-divisor.

(2) Since R is the total ring of a DCC_d ring, R is also a DCC_d ring. Apply part (1) with $M = R$. □

Corollary 2.31. *Let A be an Artinian ring and M a DCC_d A-module such that* $\text{Ann}(M) = 0$*. Then every monomorphism of M is surjective.*

Proof. This holds because in an Artinian ring, every element is either a zero-divisor or a unit. □

We now present a main result concerning the transfer of the DCC_d property to trivial ring extensions.

Theorem 2.32. *Let A be a ring and E an A-module, and let $R := A \propto E$ be the trivial ring extension of A by E. If R satisfies the DCC_d condition as an A-module, then the following statements hold:*

(1) A is a DCC_d ring and E is a DCC_d A-module.
(2) If for every $a \in A$ with $a \neq 1$, we have $E \neq aE$, then A is Artinian.

Proof. (1) Suppose R satisfies the DCC_d condition as an A-module. Let

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

be a descending chain of ideals of A, and

$$N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$$

a descending chain of submodules of E. Then the chain

$$I_1 \propto N_1 \supseteq I_2 \propto N_2 \supseteq \cdots \supseteq I_n \propto N_n \supseteq \cdots$$

is a descending chain of submodules of R. Since R is DCC_d, there exists an integer K and elements $a_n \in A$ such that for all $n \geq K$,

$$I_{n+1} \propto N_{n+1} = a_n(I_n \propto N_n).$$

This implies $I_{n+1} = a_n I_n$ and $N_{n+1} = a_n N_n$. Therefore, A is a DCC_d ring and E is a DCC_d A-module.

(2) Again, suppose R satisfies DCC_d, and consider the descending chain

$$I_1 \propto E \supseteq I_2 \propto E \supseteq \cdots \supseteq I_n \propto E \supseteq \cdots .$$

As before, there exists an integer K and elements $a_n \in A$ such that for all $n \geq K$,

$$I_{n+1} \propto E = a_n(I_n \propto E),$$

which gives $I_{n+1} = a_n I_n$ and $E = a_n E$. By hypothesis, $a_n = 1$ for all $n \geq K$. Hence, $I_{n+1} = I_n$ and the chain stabilizes, showing that A is Artinian. □

Corollary 2.33. *Let E be a $\mathbb{Z}$-module such that for all $a \in \mathbb{Z}$ with $a \neq 1$, we have $E \neq aE$. Then $\mathbb{Z} \propto E$ is not a DCC_d $\mathbb{Z}$-module.*

Example 2.34. Let A be a ring. Then $\mathbb{Z} \propto A[X]$ is not a DCC_d $\mathbb{Z}$-module.

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Hwankoo Kim, Division of Computer Engineering, Hoseo University, Republic of Korea

Email address: hkkim@hoseo.edu

Salah Eddine Mahdou, Laboratory of Modelling and Mathematical Structures, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.

Email address: salahmahdoulmtiri@gmail.com

Oussama Aymane Es Safi, Laboratory of Modelling and Mathematical Structures, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.

Email address: essafi.oussamaaymane@gmail.com