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ON THE ZERO ATTRACTOR OF A SEQUENCE OF TWO VARIABLES POLYNOMIALS

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Abstract. The paper investigates the zeros of a family of polynomials in two variables arising from a linear recurrence sequence associated with binomial coefficient triangle. Using an analytical method based on conformal mappings, we analyse the attractor of these zeros. This study provides information on the distribution and behaviour in polynomial families tied to combinatorial structures, offering insights into their underlying patterns and properties.

1. Introduction

Through many centuries the zeros of polynomials and their asymptotic distribution were regarded as significant issues in mathematics. Despite several studies dealing with different situations and various approaches, the author's concentration has always been on one-variable polynomials. For instance, in [10] He *et al* considered a sequence of Fibonacci polynomials, which can be regarded as a sequence of orthogonal polynomials with constant coefficients. Indeed, Fibonacci polynomials are particular class of the classical Jacobi polynomials studied by Dilcher [7], therefore many informations on their zeros are available in literature. Among different generalizations of the classical Fibonacci polynomials, He and Ricci in [9] provide a weighted generalization of Faber polynomials in complex domain, exploring their connection to the classical Fibonacci polynomials, and determine the asymptotic distribution of their zeros. Other more challenging cases of polynomials satisfying either three term recurrence relations with polynomials coefficients or higher order recurrence relations are thrown in literature. We quote for instance the case of polynomials satisfying a four-term recurrence relation [4, 8, 11], see also the references therein.

Since then, Raab ideas are concerned with the directions $(1, q)$ in binomial coefficients triangle for all positive integer q.

In [3], by assimilating the binomial coefficients $\binom{n}{k}$ to the lattice $\mathbb{Z}\times\mathbb{Z}$ via the map $(n, k) \longmapsto \binom{n}{k}$ with the convention $\binom{n}{k} = 0$ for $k > n$ or $k < 0$, the authors remarked that the grid $(n, 0)$ and the direction (r, q), with $r \geq 2$, $r+q > 0$, define the diagonal ray of binomial coefficients triangle containing the elements

$$U(n,k) = \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk}, \quad k = 0, ..., \left\lfloor \frac{n-p}{r+q} \right\rfloor ,$$

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where x and y are nonzero real or complex variables. The set of these rays, when n runs across the set of natural numbers, forms a vector field of direction (r,q) of binomial coefficients triangle.

The main purpose of this paper the investigate the zeros' properties of the following two variables polynomial

$$U_{n+1}^{(r,q,p)}(x,y) = \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} U(n,k) = \sum_{k\geq 0} \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk}.$$

It is worthwhile to mention that the situation of two variables polynomials is much more complicated, at least for a one reason that the zero attractor, if it exists, it's not a point in general but a plane curve. See for instance [5, 6, 16, 17].

The paper is organized as follows. Section 2, we characterize the zeros of a bivariate polynomials defined by the terms of a sequence $\{U_n(x,y)\}_{n\geq 0}$ that satisfies a four term recurrence relation linked to binomial coefficients triangle with rational generating function with the denominator

$$D(t) = g_x(t) - y^2 t^3,$$

where $g_x(t)$ is a polynomial of degree two, having only one zero of multiplicity 2 at $t = 1/x$. At the end of the section we plot some special cases in order to reinforce our conjecture about the zeros of $U_n(x,y)$ and their distribution along the algebraic curve $\mathcal{I}m\left(y^2 + x^3\right) = 0$. Section 3, is devoted to study the zeros of a particular class of the polynomials defined by the terms of the sequence satisfying (5) via Cauchy integral representation formula with a specialization on the curve $y^2 = x^3$. Section 4 is devoted to describe how we obtain the zeros by the conformal mappings. While in the last section we prove that the set of attractor is characterized by the zeros with the same modulus.

2. Statement of the problem

As mentioned above, the connection between Fibonacci numbers and binomial coefficients triangle as well as its generalization has been the subject of several studies. One of them was the work of Belbachir-Komatsu-Szalay [3] who characterized the linear recurrent sequence associated to rays in binomial coefficients triangle by assimilating the binomial coefficients $\binom{n}{k}$ to the lattice $\mathbb{Z}\times\mathbb{Z}$ via the following map $(n,k) \longmapsto \binom{n}{k}$ with the convention $\binom{n}{k} = 0$ for $k > n$ or $k < 0$.

Therein, for $n \in \mathbb{N}\cup\{0\}$, $r \in \mathbb{N}$ and $q \in \mathbb{Z}$ with $q + r > 0$, the diagonal ray of binomial coefficient triangle was defined by the grid point $(n,0)$ and the direction (r,q). The diagonal ray of the weighted binomial coefficient triangle contains the elements

$$U^{(r,q)}(n,k) = \binom{n-qk}{rk} x^{n-(q+r)k} y^{rk}, \quad k = 0, ..., \left\lfloor \frac{n}{r+q} \right\rfloor,$$

where x and y are real or complex variables.

For $r \geq 2$, the intermediate rays of order p, $p = 1, 2, ..., r-1$, was defined by

$$U^{(r,q,p)}(n,k) = \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk}, \quad k = 0, ..., \left\lfloor \frac{n-p}{r+q} \right\rfloor.$$

Also, for a fixed direction (r, q), a fixed value of p and the grid point starts at (n, j), $0 \leq j \leq p-1$, we consider a sequence with general term given by the sum of elements laying on the corresponding ray, i.e.

$$(1) \quad U_{n+1}^{(r,q,p)} := \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} U^{(r,q,p)}(n,k) = \sum_{k \geq 0} \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk},$$

with the convention $U_0 = 0$.

Notice that because the sum over empty set is zero, we also have $U_1 = \cdots = U_p = 0$ and

$$(2) \qquad U_j = \binom{j-1}{p} x^{j-p-1} y^p, \qquad p+1 \leq j \leq r+q+p-1.$$

In that paper, the authors focused on linear recurrence relations associated to the above sequence. Among other results, they obtained the following.

Theorem 2.1. [2] *The sequence defined in (1) satisfy the linear recurrence relation*

$$(3) \qquad \sum_{k=0}^{r} (-x)^k \binom{r}{k} U_{n-k} = y^r U_{n-r-q},$$

and its generating function is given by [3]

$$(4) \qquad G(t) := \sum_{n \geq 0} U_{n+1}^{(r,q,p)} t^n = \frac{y^p t^{p+1} (1-xt)^{r-p-1}}{(1-xt)^r - y^r t^{q+r}}.$$

Our main goal in the present paper is to investigate the bivariate polynomial's zeros defined by (1) as well as their limiting behaviour. For instance, we shall focus our work on the special case $r = 2$, $q = 1$ and $p = 1$ known in the literature as the direction $(2, 1)$ in binomial coefficients triangle. Its worthy to notice that the case $p = 0$ represents the principal direction, i.e. the summation starts from the first column.

To the best of our knowledge, all the previous studies are concerned with the simplest case $r = 1$. In this contribution, we shall focus on the case $r > 1$. Besides, the latter cases generate new situations according to the principal array and its intermediate arrays, it is given by the recurrence relation

$$(5) \qquad U_n = 2xU_{n-1} - x^2 U_{n-2} + y^2 U_{n-3}, \quad n \geq 3,$$

with $U_0 = U_1 = 0$, $U_2 = y$ and generating function [3]

$$G(t) := \sum_{n \geq 0} U_{n+1}^{(2,1,1)} t^n = \frac{yt^2}{(1-xt)^2 - y^2 t^3}.$$

In this case, the relation (1) becomes

$$U_{n+1}^{(2,1,1)}(x,y)=\sum_{k=0}^{\lfloor n-1/3\rfloor}U^{(2,1,1)}(n,k)=\sum_{k\geq 0}\binom{n-k}{1+2k}x^{n-1-3k}y^{1+2k}.$$

The Figure 1 shows the direction $(r,q,p)=(2,1,1)$. Notice that the red colors corresponding to the standard case $p=0$. In the triplet (r,q,p), the first component p represents the pitch between two successive elements, second the q indicates the direction (up or down), while the p specifies the starting point, i.e. the corresponding column.

The following table gives the first few terms of the sequence defined by (5)

The first few terms of the sequence	The sum of the coefficients
$U_0=U_1=0$	0
$U_2=y$	1
$U_3=2xy$	2
$U_4=3x^2y$	3
$U_5=y\left(4x^3+y^2\right)$	5
$U_6=xy\left(5x^3+4y^2\right)$	9
$U_7=x^2y\left(6x^3+10y^2\right)$	16
$U_8=y\left(20x^3y^2+7x^6+y^4\right)$	28
$U_9=xy\left(35x^3y^2+8x^6+6y^4\right)$	49
$U_{10}=x^2y\left(56x^3y^2+9x^6+21y^4\right)$	86

The sum of the coefficients is cited in OEIS $A005314$ [15].

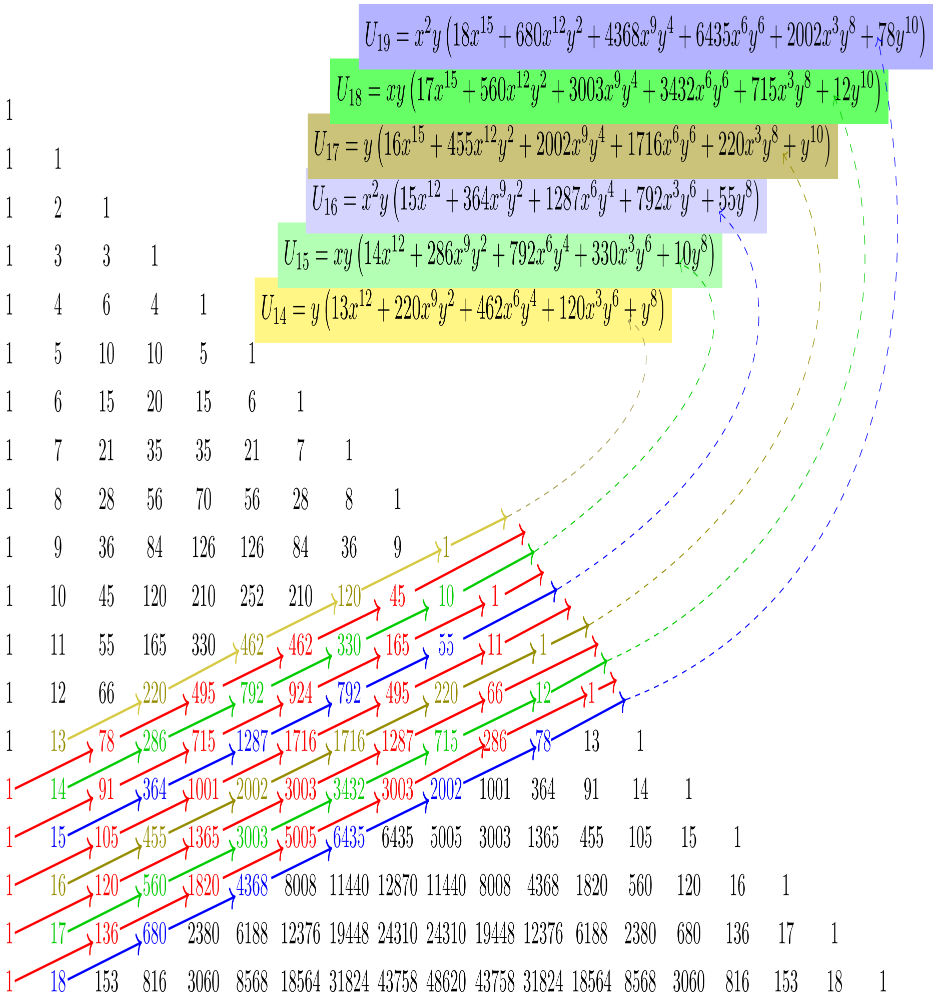


FIGURE 1. Illustration of vector field through the direction $(r,q,p) = (2,1,1)$

From another hand, it is well known that the zeros of polynomials depend on their coefficients, thus as a bonus from the above table, we merely deduce the following

Theorem 2.2. *The coefficients of the polynomials defined by the terms of sequence* $\left\{U_{n+1}^{(2,1,1)}(x,y)\right\}_{n\geq 0}$ *given by (1) satisfy the following four term recurrence relation*

$$U_{n+3} = 2U_{n+2} - U_{n+1} + U_n, \quad U_0 = U_1 = 0, \quad U_2 = 1 \tag{6}$$

with generating function

$$f(t) = \frac{t}{1 - 2t + t^2 - t^3}.$$

Furthermore, the sequence $\{U_n\}_{n\geq 0}$ *defined by the relation (6) is not log-concave.*

Indeed, the third coefficient squared is less than the second times the fourth coefficients, i.e. $3 \times 3 < 2 \times 5$ provides a counterexample.

Now, let us recall the following

Definition 2.3. *([18]) We say that a recurrence sequence is simply periodic if it is periodic and that it returns to its first term.*

The determinant of the companion matrix $A = \begin{pmatrix} 2x & -x^2 & y^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ of (5) equal y^2, and then it's a nonzero determinant, which asserts that this sequence is simply periodic. Therefore, by using the periodicity we merely deduce that

$$U_n = yP_n\left(x^3, y^2\right)x^{(n+1)} \mod [3] \tag{7}$$

where P_n are a bivariate polynomials in the variables x^3 and y^2.

Let us denote by $Z(U_j)$ the set of zeros of the bivariate polynomials defined in terms of $\{U_n\}_{n\in\mathbb{N}}$. Motivated by some experimental investigations, we conjecture the following

Conjecture 2.4. (1) *The zero attractor of* $Z(U_j)$ *is characterized by* $\mathcal{I}m\left(y^2 + x^3\right) = 0$.

(2) *The zeros are distributed along the curve* $\mathcal{I}m\left(y^2 + x^3\right) = 0$.

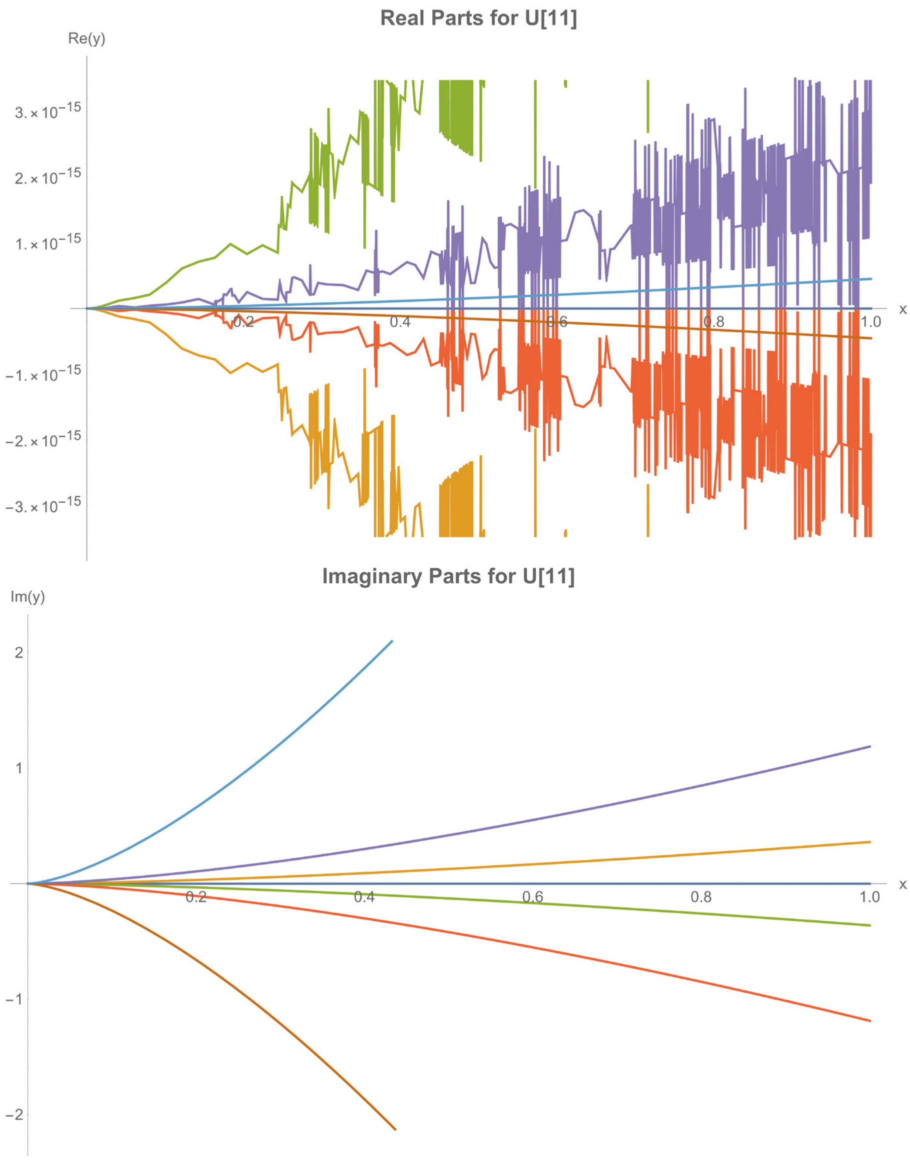

Real Parts for U[11]
Re(y)
3. × 10^−15
2. × 10^−15
1. × 10^−15
−1. × 10^−15
−2. × 10^−15
−3. × 10^−15
0.2
0.4
0.6
0.8
1.0
x
Imaginary Parts for U[11]
Im(y)
2
1
−1
−2
0.2
0.4
0.6
0.8
1.0
x

Figure 2. U[11]

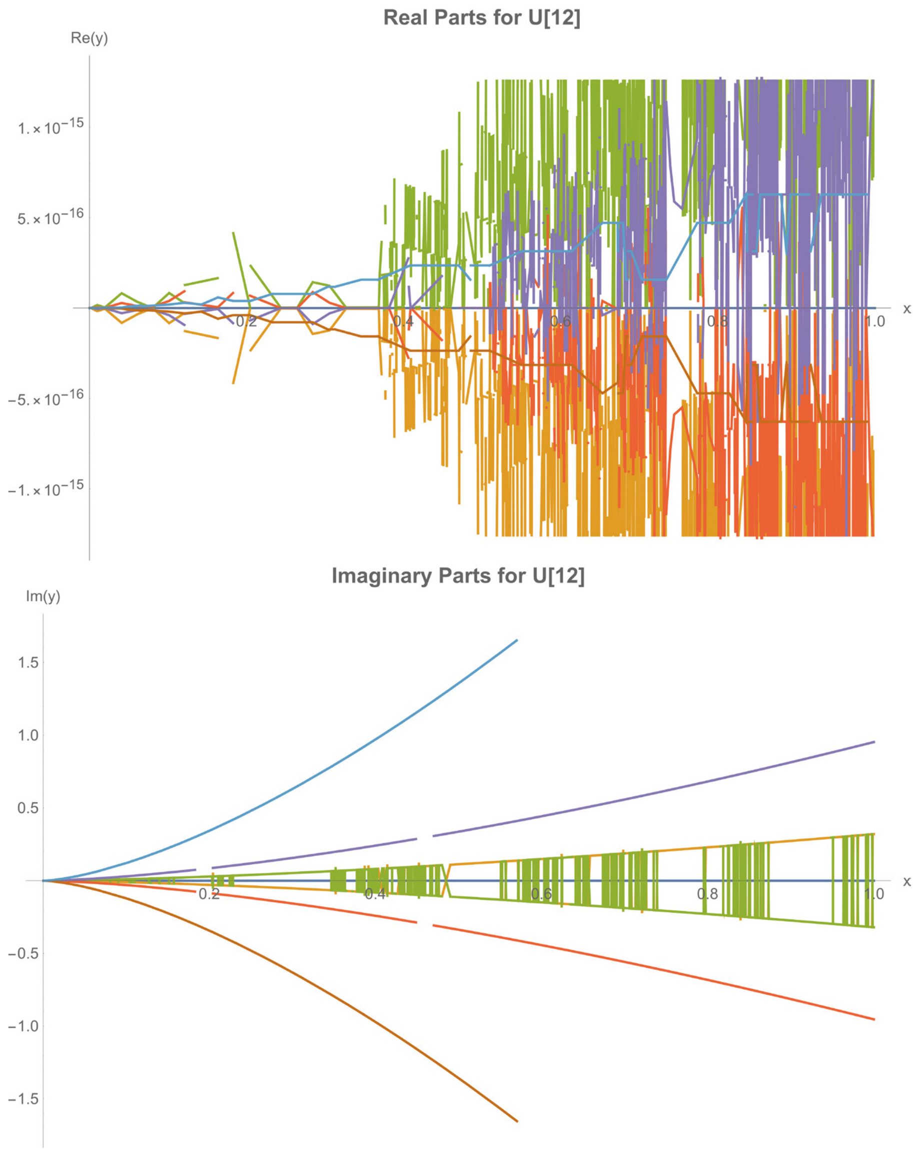

Real Parts for U[12]
Re(y)
1. × 10^−15
5. × 10^−16
−5. × 10^−16
−1. × 10^−15
0.2
0.4
0.6
0.8
1.0
x
Imaginary Parts for U[12]
Im(y)
1.5
1.0
0.5
−0.5
−1.0
−1.5
0.2
0.4
0.6
0.8
1.0
x

Figure 3. U[12]

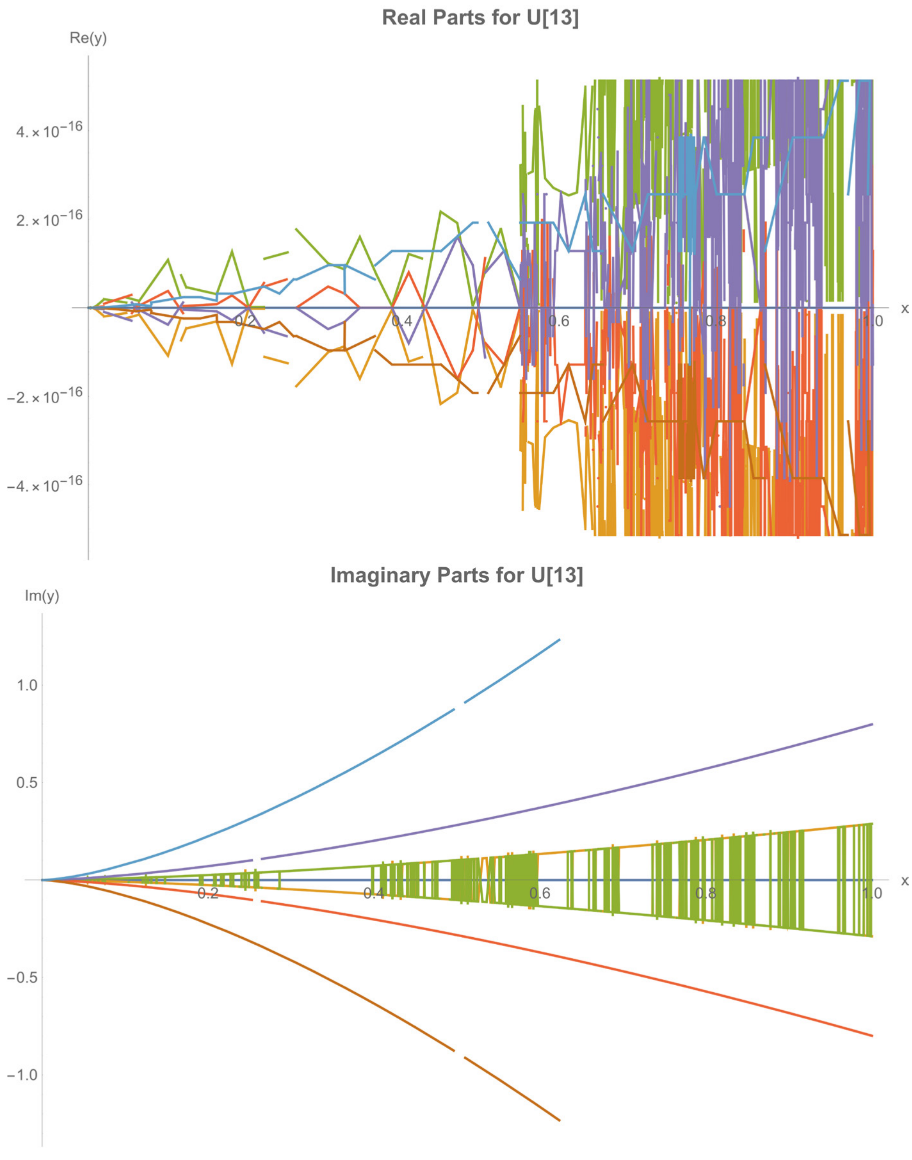


FIGURE 4. U[13]

3. Integral Representation

In this section, we shall mimic the idea of Ricci [8] in order to analyze the limiting behavior of the zeros of specific class of polynomials associated with the $(2, 1, 1)$ direction in binomial coefficients triangle, restricting to the case of polynomials in one variable, using Cauchy theory and conformal mappings. For this end, we shall specify the case $y^2 = x^3$. Now using

Cauchy integral formula, the polynomial sequence $U_n(x,y)$ defined by (5) has the following representation.

Lemma 3.1. *For any x, y nonzero real or complex parameters there exists a non negative real number $\rho_{x,y} > 0$ such that*

$$U_n(x,y) = \frac{1}{2\pi i} \oint\limits_{|t|=\rho_{x,y}} \frac{y}{(1-xt)^2 - y^2t^3} \frac{dt}{t^n}. \tag{8}$$

Proof. Since $\frac{y}{(1-xt)^2-y^2t^3} \longrightarrow y$ for $t \longrightarrow 0$ then we can find $\rho_{x,y} > 0$ such that $\left|(1-xt)^2 - y^2t^3\right| \geq 1$ for $|t| = \rho_{x,y}$. Hence the integral (8) is well-defined.

Let us consider the integral

$$\widetilde{U_n}(x,y) = \frac{1}{2\pi i} \oint\limits_{|t|=\rho_{x,y}} \frac{y}{(1-xt)^2 - y^2t^3} \frac{dt}{t^n}.$$

We can directly verify that $\widetilde{U_n}(x,y)$ satisfies (5) for $n \geq 0$. Next, since the Taylor expansion of $\frac{y}{(1-xt)^2-y^2t^3} = y + 2xyt + O\left(t^2\right)$, by residue Theorem we have $\widetilde{U_0}(x,y) = 0$, $\widetilde{U_1}(x,y) = 0$ and $\widetilde{U_2}(x,y) = y$. Hence the initial values in (2) are satisfied, thus the integral representation $\widetilde{U_n}(x,y)$ is a solution to the recurrence (3) and since the solution of the linear recurrence sequence is unique, then $\widetilde{U_n}(x,y) = U_n(x,y)$. The uniqueness of the solution completes the proof of Lemma 3.1. □

Under the specialization $y^2 = x^3$, the Equation (8) gives

$$\begin{aligned} U_n(x,y) &= \frac{1}{2\pi i} \oint\limits_{|t|=\rho_{x,y}} \frac{y}{(1-xt)^2 - y^2t^3} \frac{dt}{t^n} \\ &= \frac{(x\sqrt{x})}{2\pi i} \oint\limits_{|t|=\rho_x} \frac{1}{(1-xt)^2 - x^3t^3} \frac{dt}{t^n}. \end{aligned} \tag{9}$$

The latter equation becomes, after replacing t by t/x in the Equation (9),

$$U_n(x,y) = \frac{-x^{n+\frac{1}{2}}}{2\pi i} \oint\limits_{|t|=\rho_x} \frac{1}{t^3 - (1-t)^2} \frac{dt}{t^n} = \frac{-x^{n+\frac{1}{2}}}{2\pi i} \oint\limits_{|t|=\rho_x} \frac{1}{P(t)} \frac{dt}{t^n},$$

where

$$P(t) = t^3 - (t-1)^2. \tag{10}$$

Next, we shall denote by

$$\omega_n(x) := \frac{-1}{2\pi i} \oint\limits_{|t|=\rho_x} \frac{1}{P(t)} \frac{dt}{t^n}, \tag{11}$$

also from Lemma 3.1

$$\omega_n(x) = x^{-\left(n+\frac{1}{2}\right)} U_n(x,y) \tag{12}$$

accordingly, the zeros of $U_n(x,y)$ could be obtained from those of $\omega_n(x)$ after multiplying by $x^{n+\frac{1}{2}}$.

Lemma 3.2. (a) *The polynomial $P(t)$ defined in (10) has no zero of order 3.*

(b) *$P(t)$ has zero of order 2 if and only if $t_1 = \frac{1-i\sqrt{5}}{3}$ or $t_2 = \frac{1+i\sqrt{5}}{3}$.*

Proof. (1) The derivative polynomial $P'(t)$ has no zeros of order 2, consequently $P(t)$ has no zero of order 3.

(2) $P'(t)$ has two complex zeros of order 1, $t_1 = \frac{1-i\sqrt{5}}{3}$ and $t_2 = \frac{1+i\sqrt{5}}{3}$, and in this case the zeros of $P(t)$ are: $\frac{1}{3} + i\frac{\sqrt{5}}{3}$, $\frac{1}{3} + i\frac{\sqrt{5}}{3}$, $-1 - i\frac{\sqrt{5}}{2}$ when $t = t_2$, and when $t = t_1$ the zeros of $P(t)$ are: $\frac{1}{3} - i\frac{\sqrt{5}}{3}$, $\frac{1}{3} - i\frac{\sqrt{5}}{3}$, $\frac{-1}{2} - i\frac{\sqrt{5}}{5}$. □

Let t_1, t_2, t_3 be the zeros of $P(t)$ arranged via their magnitudes, i.e.

$$|t_1| \leq |t_2| \leq |t_3|, \tag{13}$$

$P(t)$ has distinct zeros if $t \neq t_1$ and $t \neq t_2$. Therefore, after developing the partial fraction decomposition for $\frac{1}{P(t)}$, we obtain

$$\frac{1}{P(t)} = \frac{1}{P'(t_1)}\frac{1}{t-t_1} + \frac{1}{P'(t_2)}\frac{1}{t-t_2} + \frac{1}{P'(t_3)}\frac{1}{t-t_3}.$$

Using Equation (11), the integration term by term and using the residue Theorem, we get

$$\omega_n(x) = \left[\frac{1}{P'(t_1)\,t_1^{n-1}} + \frac{1}{P'(t_2)\,t_2^{n-1}} + \frac{1}{P'(t_3)\,t_3^{n-1}}\right]. \tag{14}$$

According to (14), it's clear that the asymptotic behavior of $\omega_n(x)$ depends on the magnitude of the zeros of $P(t) = 0$.

4. Conformal mappings

In the present section, we describe how the zeros of $P(t)$ can be obtained by a sequence of conformal mappings.

Set $t = z + \frac{1}{3}$, then $P(t) = 0$ returns to its canonical form:

$$z^3 + \frac{5}{3}z - \frac{11}{27} = 0. \tag{15}$$

Moreover, for $z = \lambda q$ with $\lambda = \frac{i\sqrt{5}}{3}$, we have

$$q^3 - 3q + \beta = 0, \tag{16}$$

with

$$\beta = \frac{-11\sqrt{5}i}{25}. \tag{17}$$

It is clear that β is not conformal. On the other hand, for $q = p + \frac{1}{p}$, Equation (16) in p is

$$p^6 + \beta p^3 + 1 = 0. \tag{18}$$

Equivalently for $p^3 = s$, we get

$$s^2 + \beta s + 1 = 0, \tag{19}$$

which implies

$$\beta = -\left(s + \frac{1}{s}\right). \tag{20}$$

Now, it is clear how to obtain the zeros of $P(t)$ by going through a sequence of conformal mappings starting from the β-plane and subsequently ending in the t-plane .

The map : $J(\zeta) = \zeta + \frac{1}{\zeta}$ is called Joukowski map [13], it is conformal in the regions $|\zeta| < 1$ and $|\zeta| > 1$.

The β-plane is mapped into the exterior region to the unit circle in the s-plane under $J^{-1}(-\beta)$, the region is mapped into the exterior region to the unit circle in the $p-$plane under $p = s^{1/3}$.

Remark 4.1. *The map $p = s^{1/3}$ is multiple-valued.*

Then, according to $q = J(p)$, $z = \frac{i\sqrt{5}}{3}q$ and $t = z + \frac{1}{3}$, we recover the zeros of $P(t)$ in the p-plane and finally t to t/x in the x-plane. We can express this situation symbolically as

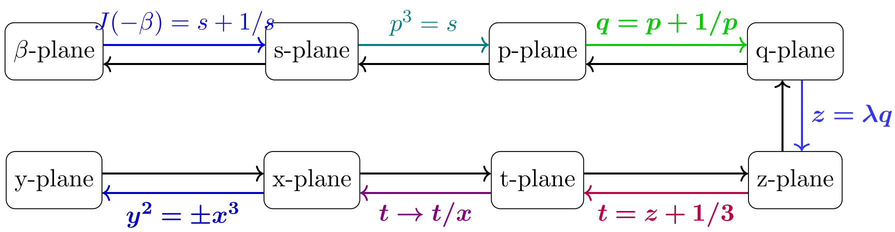


FIGURE 5. Description Diagram

As Joukowski map: $J(\zeta) = \zeta + \frac{1}{\zeta}$ is conformal on $|\zeta| < 1$ and $|\zeta| > 1$. Let us focus on $|\zeta| > 1$. We have the following result

Theorem 4.2. *The Joukowski conformal mapping $J(\zeta)$ maps circles onto ellipses and the straight lines onto hyperbolas.*

Proof. Considering the complex ζ plane (for $|\zeta| > 1$), put $\zeta = R\exp\{i\theta\}$, so that for $R = const$, the circle $\zeta = R\exp\{i\theta\}$ comes out.

$$\begin{aligned} J(\zeta) &= R\exp\{i\theta\} + R^{-1}\exp\{-i\theta\} \\ &= (R + R^{-1})\cos\theta + i(R - R^{-1})\sin\theta. \end{aligned}$$

The ellipses come by setting $u = (R + R^{-1})\cos\theta$ and $v = (R - R^{-1})\sin\theta$. Therefore, we obtain the following equation of the ellipse

$$u^2/(R + R^{-1})^2 + v^2/(R - R^{-1})^2 = 1,$$

whose semi-axis are $a = (R+R^{-1})$ and $b = |R-R^{-1}|$. Since $a^2-b^2 = c^2 = 4$, the foci are in points $c = 2$ and $c = -2$.

For $\theta = -T = const$ (straight lines), put $\zeta = R\exp\{-iT\}$.

$$\begin{aligned} J(\zeta) &= R\exp\{-iT\} + R^{-1}\exp\{iT\} \\ &= (R + R^{-1})\cos T - i(R - R^{-1})\sin T. \end{aligned}$$

The hyperbolas (transformation of straight lines) come by setting $u = (R + R^{-1}\}) \cos T$, $v = (R - R^{-1}) \sin T$, i.e.

$$u^2/(R+R^{-1})^2 - v^2/(R-R^{-1})^2 = 1.$$

□

4.1. **Zero analysis and structure in the p-plane.** Recall that t_1, t_2, t_3 are the zeros of $P(t) = 0$ such that they satisfy (10). In this subsection, we will study the set Λ defined by:

$$\Lambda = \{|t_1(x)| = |t_2(x)|\}. \tag{21}$$

To represent the set Λ in the $x-$plane, it is better to represent the set of points in the $p-$plane that leads to $|t_1(x)| = |t_2(x)|$ as shown in Figure 5.

$$\begin{aligned} q = J(p) &= R\exp(i\theta) + R^{-1}\exp(-i\theta) \\ &= \left(R + R^{-1}\right)\cos\theta + i\left(R - R^{-1}\right)\sin\theta, \\ z = \lambda q &= \frac{i\sqrt{5}}{3}\left(R + R^{-1}\right)\cos\theta + \frac{\sqrt{5}}{3}\left(-R + R^{-1}\right)\sin\theta \end{aligned}$$

and

$$t = z + \frac{1}{3} = \frac{i\sqrt{5}}{3}\left(R + R^{-1}\right)\cos\theta + \frac{\sqrt{5}}{3}\left(-R + R^{-1}\right)\sin\theta + \frac{1}{3}.$$

Then

$$\begin{aligned} \left|t^2\right| &= \left(\frac{\sqrt{5}}{3}\left(-R + R^{-1}\right)\sin\theta + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\left(R + R^{-1}\right)\cos\theta\right)^2 \\ &= \frac{5}{9}(R^2 + R^{-2}) + \frac{10}{9}\cos 2\theta + \frac{1}{9} + \frac{2\sqrt{5}}{9}\left(R^{-1} - R\right)\sin\theta. \end{aligned}$$

Now let $R\exp(i\theta_0)$, $R\exp(i\left(\theta_0 + \frac{2\pi}{3}\right))$ and $R\exp(i\left(\theta_0 + \frac{4\pi}{3}\right))$ be the images of a point x in the $P-$plane.

For example, we may assume that $R\exp(i\theta_0)$ leads to t_1 in the $t-$plane and $R\exp(i\left(\theta_0 + \frac{4\pi}{3}\right))$ leads to t_2 .

Thus $\left|t_1^2\right| = \left|t_2^2\right|$ implies

$$\begin{aligned} &\frac{5}{9}(R^2 + R^{-2}) + \frac{10}{9}\cos 2\theta_0 + \frac{1}{9} + \frac{2\sqrt{5}}{9}\left(R^{-1} - R\right)\sin\theta_0 \\ &= \frac{5}{9}(R^2 + R^{-2}) + \frac{10}{9}\cos 2\left(\theta_0 + \frac{4\pi}{3}\right) + \frac{1}{9} \\ &+ \frac{2\sqrt{5}}{9}\left(R^{-1} - R\right)\sin\left(\theta_0 + \frac{4\pi}{3}\right). \end{aligned}$$

Then we get after simplification:

$$\begin{aligned} &\frac{10}{9}\left[\cos 2\theta_0 - \cos 2\left(\theta_0 + \frac{4\pi}{3}\right)\right] \\ &= \frac{2\sqrt{5}}{9}\left(R^{-1} - R\right)\left[-\sin\theta_0 + \sin\left(\theta_0 + \frac{4\pi}{3}\right)\right]. \end{aligned}$$

Using the trigonometric identity

$$\cos 2a - \cos 2b = -2(\sin a - \sin b)(\sin a + \sin b)$$

we get

$$\frac{2\sqrt{5}}{9}\left(R^{-1}-R\right)\left[2\sin\frac{2\pi}{3}\cos\left(\theta_0+\frac{2\pi}{3}\right)\right] \tag{22}$$
$$=-\frac{20}{9}\left[2\sin\frac{2\pi}{3}\cos\left(\theta_0+\frac{2\pi}{3}\right)\right]\left[2\cos\frac{2\pi}{3}\sin\left(\theta_0+\frac{2\pi}{3}\right)\right].$$

Two cases to discuss

The first one is: if $\cos\left(\theta_0+\frac{2\pi}{3}\right)=0$ this implies that $\theta_0=\frac{-\pi}{6}$ or $\theta_0=\frac{-7\pi}{6}$, then any $R\geq 1$ satisfies (22).

The second one is: if $\cos\left(\theta_0+\frac{2\pi}{3}\right)\neq 0$, then we get, after cancelling the common factor, $\cos\left(\theta_0+\frac{2\pi}{3}\right)$ leads to

$$\frac{\sqrt{5}}{9}\left(R^{-1}-R\right)=\frac{10}{9}\sin\left(\theta_0+\frac{2\pi}{3}\right),$$

which is equivalent to the rectangular equation

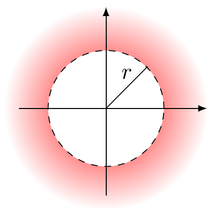
$$\left(x+\frac{\sqrt{15}}{2}\right)^2+\left(y-\frac{\sqrt{5}}{2}\right)^2=6.$$

We summarize the previous discussions in the following Lemma.

Lemma 4.3. *Let $t_1=R\exp\{i\theta_0\}$ and $t_2=R\exp\{i\theta_0+\frac{4\pi}{3}\}$ be two zeros of the polynomial $P(t)$ (10). In the p-plane, the above complex numbers have the same magnitude if one of the following conditions is satisfied*

(a) $p_1=R\exp\{-i\frac{\pi}{6}\}$ *or* $p_1=R\exp\{-i\frac{7\pi}{6}\}$ *for all* $R\geq 1$.

(b) p_1 *lies in the circular arc of the circle in the region* $R\geq 1$.



$\left(x+\frac{\sqrt{15}}{2}\right)^2+\left(y-\frac{\sqrt{5}}{2}\right)^2=6$

Notice that p_2 is the image of p_1 by the rotation of angle $\frac{4\pi}{3}$.

The condition $|t_1|<|t_3|$ is needed to make sure that $|t_1|\leq|t_2|\leq|t_3|$. Recall that t_3 corresponds to $R\exp\left(i\left(\theta_0+\frac{2\pi}{3}\right)\right)$ in the $p-$plane, since $|t_1|<|t_3|$ this implies, $|t_1|^2<|t_3|^2$ then we have

$$\frac{2\sqrt{5}}{9}\left(R^{-1}-R\right)\left[2\sin\left(\frac{-\pi}{3}\right)\cos\left(\theta_0+\frac{\pi}{3}\right)\right]$$
$$>\frac{-20}{9}\left[2\sin\left(\frac{-\pi}{3}\right)\cos\left(\theta_0+\frac{\pi}{3}\right)\right]\left[2\sin\left(\theta_0+\frac{\pi}{3}\right)\cos\left(\frac{-\pi}{3}\right)\right].$$

To solve this inequality, we have two cases to discuss:

1- If $\cos\left(\theta_0+\frac{\pi}{3}\right)>0$, which is equivalent to

$$\frac{-5\pi}{6}<\theta_0<\frac{\pi}{6}. \tag{23}$$

Then after cancelling $\cos\left(\theta_0+\frac{\pi}{3}\right)$, we get

$$\frac{2\sqrt{5}}{9}\left(R^{-1}-R\right)<\frac{-20}{9}\sin\left(\theta_0+\frac{\pi}{3}\right),$$

or, in rectangular form,

$$\left(x+\frac{\sqrt{15}}{2}\right)^2+\left(y-\frac{\sqrt{5}}{2}\right)^2>6. \tag{24}$$

The corresponding region in this case follows, after combining (23) and (24), namely

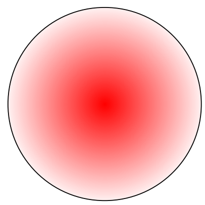
$$\left(x+\frac{\sqrt{15}}{2}\right)^2+\left(y-\frac{\sqrt{5}}{2}\right)^2>6,\ R\geq 1,\ \text{and}\ \frac{-5\pi}{6}<\theta_0<\frac{\pi}{6}. \tag{25}$$

2- If $\cos\left(\theta_0+\frac{2\pi}{3}\right)<0$ this is equivalent to:

$$\frac{\pi}{6}<\theta_0<\frac{7\pi}{6},$$

and we have to solve

$$\frac{2\sqrt{5}}{9}\left(R^{-1}-R\right)>\frac{-10}{9}\sin\left(\theta_0+\frac{\pi}{3}\right).$$



$\left(x+\frac{\sqrt{15}}{2}\right)^2+\left(y-\frac{\sqrt{5}}{2}\right)^2<6.$

In a similar way, we get the region for the case 2:

$$R\geq 1\quad\text{and}\ \frac{\pi}{6}<\theta_0<\frac{7\pi}{6}. \tag{26}$$

Let the region Γ_1 and the region Γ_2 corresponding to (25) and (26), respectively. Then

$$\Gamma_1\cup\Gamma_2=\{\text{points in } p-\text{ plane that correspond to } |t_1|<|t_3|\}.$$

Now, imposing the condition stated in Lemma 4.3, we get the point set C_1 of the point p_1 that corresponds to root t_1.

Explicitly, C_1 is the point set defined as

$$C_1=\{\Gamma_1\cup\Gamma_2\}\cap\left\{\left\{\text{ ray}:\theta=\frac{-\pi}{6}\right\}\cup\left\{\left(x+\frac{\sqrt{15}}{2}\right)^2+\left(y-\frac{\sqrt{5}}{2}\right)^2=6\right\}\right\}.$$

Similarly, we defined C_2 as the rotation of C_1 through an angle of $\frac{4\pi}{3}$ and C_3 through a rotation of angle $\frac{2\pi}{3}$.

It is clear that C_2 consists of points in the $p-$plane that corresponds to t_2 and C_3 corresponds to t_3. Thus we have proved the following

Theorem 4.4. *The condition for the points in the $p-$plane for which $|t_1|=|t_2|<|t_3|$ is $p_i\in C_i$, for $i=1,2,3$.*

5. Zero attractor

The concept of attractors was introduced in 1964 by Auslander et al. [1], it plays an important role mainly in dynamical systems [12].

Definition 5.1. *Let* $\{q_n(x)\}_{n\geq 0}$ *be a sequence of polynomials, where the degree of* $q_n(x)$ *increases to infinity as* $n \longrightarrow \infty$.

A set A *in the* $x-$*plane is called the asymptotic zero attractor of zeros of* $\{q_n(x)\}_{n\geq 0}$ *if the following two conditions holds:*

(1) Let $A_\varepsilon = \cup_{x\in A} B(x,\varepsilon)$ *where* $B(x,\varepsilon)$ *is the open disc centered at* x *with radius* ε, A_ε *is a neighborhood of* A, *there exists* $n_0(\varepsilon)$ *such that for all* $n \geq n_0$ *all the zeros of* $q_n(x)$ *are in* A_ε.

(2) For all $x \in A$, *for all* $\varepsilon > 0$, *there exists* $n_1(x,\varepsilon) \in \mathbb{N}$ *and a zero* r *of* $\{q_{n_1}(x)\}_{n\geq 0}$ *such that* $r \in B(x,\varepsilon)$.

Let $Z(\omega_n)$ denoted the zero attractor of $\omega_n(x)$.

To see the zero attractor of $U_n(x,y)$, by using (12) we have $\omega_n(x) = x^{-\left(\frac{1}{2}+n\right)} U_n(x,y)$.

Assume $x_0 \in Z(\omega_n)$ again by Equation (12) $\omega_n(x_0) = x_0^{-\left(\frac{1}{2}+n\right)} U_n(x_0,y_0)$ where $y_0 = x_0^{\frac{3}{2}}$. Since $\omega_n(x_0) = 0$, we get $U_n(x_0,y_0) = 0$. Hence $x_0^{-\left(\frac{1}{2}+n\right)} \in Z(U_n)$.

Let C be the curve defined by $\{$The x values for which $|t_1| = |t_2|.\}$.

For all $\varepsilon > 0$, let C_ε be the $\varepsilon-$ neighborhood of C in the $x-$plane.

Now we will give a justification that, for all large n, all the zeros of $\omega_n(x)$ are contained in C_ε.

According to 13 the asymptotics of $\omega_n(x)$ depends on the magnitudes of the zeros of $P(t) = 0$.

Let $B = \{x \in x-\text{plane} : |t_1(x)| < |t_2(x)|\}$. Obviously, B is an open region in the $x-$plane. Then we have

Lemma 5.2. *There exists a non negative real number* ρ *such that for all large* n, *the zeros of* $\omega_n(x)$ *are contained in the disc* $D_\rho = \{x : |x| \leq \rho\}$.

Proof. According to Lemma 4, the zeros of $P(t) = 0$ excluding $t \neq \frac{1\pm i\sqrt{5}}{3}$ are simple and consist of one real zero ζ_1 and two complex conjugate zeros, ζ_2 and ζ_3, with $|\zeta_1| < |\zeta_2|$.

Since ζ_2 and ζ_3 have the same magnitude their choice is arbitrary.

Let $P'(t) = 3t^2 - 2t + 2$ be the derivative polynomial of $P(t)$. Define $\zeta_2 = \mu \exp i\alpha$ and $\zeta_3 = \overline{\zeta_2}$ where the bar denotes the complex conjugate.

We then compute

$$P'(\zeta_1) = 3\left(\zeta_1 - \frac{1}{3}\right)^2 + \frac{5}{3},$$

$$P'(\zeta_2) = 3\left(\mu\exp(i\alpha) - \frac{1}{3}\right)^2 + \frac{5}{3},$$

$$P'(\zeta_3) = 3\left(\mu\exp(-i\alpha) - \frac{1}{3}\right)^2 + \frac{5}{3}.$$

Using Equation (14) and we take the modulus, we drive the following estimate

$$|\omega_n(x)| = \left|\frac{1}{P'(\zeta_1)\,\zeta_1^{n-1}}\right| \left|1 + \frac{P'(\zeta_1)}{P'(\zeta_2)}\left(\frac{\zeta_1}{\zeta_2}\right)^{n-1} + \frac{P'(\zeta_1)}{P'(\zeta_3)}\left(\frac{\zeta_1}{\zeta_3}\right)^{n-1}\right| > 0,$$

for all sufficiently large n.

This establishes the result and completes the proof of the lemma. □

The following lemma can be used to determine the location of the zeros of $P(t) = 0$.

Lemma 5.3. *Let L be a compact set of B, then $P'(\zeta_1)\,\zeta_1^{n-1}\omega_n(x) \longrightarrow 1$ uniformly for all $x \in L$ as $n \longrightarrow \infty$.*

Proof. Using Lemma 5.2, if $x \in L$, then $P(t)$ has only simple zeros with $P^{'}(\zeta_i)$, for $i = 1, 2, 3$.

Since L is a compact set, this implies $0 < m \leq \left|P^{'}(\zeta_i)\right| \leq M$ uniformly for $x \in L$ and $i = 1, 2, 3$.

Using again Equation 14, we get

$$P'(\zeta_1)\,\zeta_1^{n-1}\omega_n(x) = \left|1 + o\left(\left|\frac{\zeta_1}{\zeta_3}\right|^{n-1}\right)\right|,$$

where the small o terms approach zero uniformly. This confirms our result and completes the proof of the lemma. □

From the results of the previous Lemma, we conclude the following

Corollary 5.4. *The set B has no zeros of $\omega_n(x)$, all the zeros of $\omega_n(x)$ are belong to C_ε.*

Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

Data Sets

There is no data sets available.

References

[1] J. Auslander, N.P. Bhatia, P. Seibert, Attractors in dynamical systems. Bol. Soc. Mat. Mex. 9 (1964), 55-66.

[2] H. Belbachir, T. Komatsu, L. Szalay, Characterization of linear recurrences associated to rays in Pascal's triangle. In: Diophantine analysis and related fields 2010. AIP Conf. Proc., 1264, Amer. Inst. Phys., Melville, NY, 2010, pp. 90–99.

[3] H. Belbachir, T. Komatsu, L. Szalay, Linear recurrences associated to rays in Pascal's triangle and combinatorial identities, Math. Slovaca 64 (2014), 287–300.

[4] V. Botta, E. F. Filho, On polynomials generated by a four-term recurrence relation and linear coefficients, Calcolo (2023), 60:33.

[5] R. Boyer, W.M.Y. Goh, Polynomials associated with partitions: asymptotics and zeros, Special functions and orthogonal polynomials, 33–45, Contemp. Math., 471, Amer. Math. Soc., Providence, RI, 2008.

[6] R. Boyer, W.M.Y. Goh, Appell polynomials and their zero attractors, Gems in experimental mathematics, 69–96, Contemp. Math., 517, Amer. Math. Soc., Providence, RI, 2010.

[7] K. Dilcher, A Generalization of Fibonacci Polynomials and a Representation of Gegenbauer Polynomials of Integer Order, Fibonacci Quart. 25:4 (1987), 300-3003.

[8] W. Goh, M.X. He, P.E. Ricci, On the universal zero attractor of the Tribonacci-related polynomials. Calcolo 46 (2009), 95-129.

[9] M. He, P.E. Ricci, Asymptotic distribution of the zeros of weighted Fibonacci polynomials, Complex Var. 28 (1996), 375-384.
[10] M.X. He, P.E. Ricci, D. Simon, Numerical results on the zeros of generalized Fibonacci polynomials. Calcolo 34 (1997), 25-40.
[11] J. Luong, K. Tran, Zeros of a table of polynomials satisfying a four-term contiguous relation, Z. Anal. Anwend. 41 (2022), no. 1/2, 81-91.
[12] J. Milnor, On the Concept of Attractor, Commun. Math. Phys. 99,177-195 (1985).
[13] Z. Nehari, Conformal mappings (McGraw-Hill, New York, 1952).
[14] J.A. Raab, A generalization of the connection between the Fibonacci sequence and Pascal's Triangle, The Fibonacci Quart. 1(3) (1963), 21-31.
[15] N.J.A. Sloane, The on-line encyclopedia of integer sequences, The On-Line Encyclopedia of Integer Sequences
[16] K. Tran, The root distribution of polynomials with a three-term recurrence, J. Math. Anal. Appl. 421 (2015), 878–892.
[17] K. Tran, Connections between discriminants and the root distribution of polynomials with rational generating function, J. Math. Anal. Appl. 410 (2014), 330–340.
[18] A. Vince, Period of a linear recurrence, Acta Arith. 39 (1981), 303-311.

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