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TIME-CHANGED CONVOLUTED POISSON PROCESS

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Abstract. In this article, we study the time-changed convoluted Poisson process (TCPP) which is obtained by time-changing the convoluted Poisson process (CPP) by Lévy subordinator. We derive distributional properties of the TCPP, such as, the probability mass function, probability generating function, moments and covariance. We study the asymptotic behaviour of the correlation function which proves the long-memory in the considered process. We also present a compound Poisson representation of the TCPP. We define the convoluted negative binomial process (CNBP) by time-changing the CPP with an independent gamma subordinators. We derive the governing differential equation of the CNBP and studied its dependence behavior. The Lévy measure density of the CNBP is also derived. Finally, we explore some other important examples of the TCPP and their connection with the fractional differential equations.

1. Introduction

Poisson processes with fluctuating intensities find widespread application in diverse fields. For instance, they are employed in telecommunications (see [17]) to depict varying call arrivals throughout the day, in biological systems (see [19]) to represent changing neuron firing rates, and in numerous dynamic systems across various domains. It would be interesting to study the probabilistic models for the Poisson process with varying intensity. In this direction, [9] and [24], defined the convoluted fractional Poisson process and the convoluted fractional Poisson process of order k, respectively. The need for generalized models in convoluted Poisson processes arises from the diverse range of real-world phenomena they describe, requiring flexible frameworks that can accommodate varying intensities, time dependencies, and complex event patterns.

The method of subordination (see [6]) has garnered notable interest in methods of creating new stochastic processes (see also [23] for subordinated stochastic processes). It involves substituting the time component of the original process with an independent stochastic process, preferably characterized by non-decreasing sample paths. In this paper, we study the convoluted Poisson process (CPP) time-changed by Lévy subordinator and its inverse as time-changing the CPP will give flexibility to the count models studied in the literature and it will also help generalize the existing theoretical results. In this article, we introduce time-changed convoluted Poisson process (TCPP) and study its several distributional properties, such as, the probability mass function, probability generating function, mean and second order properties. Using the asymptotic behavior of the correlation function we show that the TCPP has long range dependence (LRD) property. We also present its compound Poisson representation and work out several special cases of the TCPP which includes

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convoluted negative binomial process, convoluted Poisson process with tempered-stable inverse Gaussian, Mittag-Leffler and multistable subordinator time changes. Additionally, some non Lévy process such as inverse tempered stable and inverse of inverse Gaussian are used to time-change the CPP. The governing differential equations and Lévy measures are also derived in several special cases.

The paper is organized as follows. Section 2 contains some preliminary results which are required for the rest of the paper. In Section 3, we introduce and present results related to the TCPP. We define the CNBP and discuss its various important characteristics in Section 4. Finally, we explore some special time-changed variants of the TCPP in Section 5.

2. Preliminaries

In this section, we establish the notations and definitions that will be referenced in the subsequent sections. The set of natural numbers is denoted by$\mathbb{N}$, while the set of non-negative integers is represented as $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{C}$ to represent the set of complex numbers.

2.1. **Definitions.**

(i) Let f and g be two real-valued functions with support from the set of integers. The discrete convolution of the function f and g is defined as (see [8])

$$(f * g)(n) = \sum_{i=-\infty}^{\infty} f(i)g(n-i),$$

satisfying $\sum_{i=\infty}^{\infty} |f(i)| < \infty$ and $\sum_{i=\infty}^{\infty} |g(i)| < \infty$.

(ii) The shift operator denoted by $e^{-k\partial_t}$ for $K \in \mathbb{R}$ is given by (see [2])

$$e^{-k\partial_t} f(t) = \sum_{n=0}^{\infty} f(t) \frac{(-k\partial_t)^n}{n!} = f(t-k).$$

(iii) Three parameters Mittag-Leffler function $L^{\alpha}_{\beta,\gamma}(z)$ is defined as (see [20], [21])

$$L^{\alpha}_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\gamma+\beta k)} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad \beta, \gamma, \alpha, z \in \mathbb{C} \text{ and } \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\alpha) > 0.$$

(iv) The generalized Wright function is defined by ([11])

$${}_p\psi_q \left[z \left| \begin{matrix} (\alpha_i, \beta_i)_{1,p} \\ (a_j, b_j)_{1,q} \end{matrix} \right. \right] = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + \beta_i k)}{\prod_{j=1}^{q} \Gamma(a_j + b_j k)}, \quad z, \alpha_i, a_i \in \mathbb{C} \text{ and } \beta_i, b_i \in \mathbb{R}.$$

(v) For $0 < \tau < 1$, the Caputo fractional derivative of order τ is defined as (see [20])

$$\partial_t^{\tau} f(t) = \frac{1}{\Gamma(1-\tau)} \int_0^t (t-u)^{-\tau} f'(u) du.$$

2.2. **Lévy Subordinators.**

Let $\{S_f(t)\}_{t\geq 0}$ with $S_f(0) = 0$ be a Lévy subordinator with Bernŝtein function f whose sample path are non-decreasing. Then, the Laplace trandsform of $S_f(t)$ is given by

$$\mathbb{E}\left(e^{-uS_f(t)}\right) = e^{-tf(u)}, \quad u > 0, \tag{1}$$

where

$$f(u) = bu + \int_0^\infty \left(1 - e^{-ux}\right)\mu(dx), \quad b > 0.$$

Here b denotes the drift and μ is a non-negative Lévy measure satisfying $\mu([0,\infty]) = \infty$ and $\int_0^\infty \min(x,1)\mu(dx) < \infty$.

2.2.1. *Gamma Process.* Let $\mu > 0$, $\rho > 0$, and $\{\Gamma(t)\}_{t\geq 0}$ be a gamma process, where $\Gamma(t) \sim G(\mu, \rho t)$, which denotes the gamma distribution with scale parameter μ^{-1} and shape parameter ρt. Its probability density function (pdf) is given by

$$f_G(x,t) = \frac{\mu^{\rho t}}{\Gamma(\rho t)} x^{\rho t - 1} e^{-\mu x}, \quad x > 0. \tag{2}$$

The fractional order moment of the gamma process is given by

$$\mathbb{E}\left[\Gamma^l(t)\right] = \frac{\Gamma(\rho t + l)}{\mu^l\, \Gamma(\rho t)}, \quad l > 0. \tag{3}$$

For $x, t \geq 0$, the pmf in (2) satisfies the following cauchy system (see [2])

$$e^{-\frac{1}{\rho}\partial_t} f_G(x,t) = \frac{1}{\mu}\frac{d}{dx} f_G(x,t) + f_G(x,t), \tag{4}$$

with $f_G(x,0) = \delta(x)$ and $\lim_{|x|\to\infty} f_G(x,t) = 0$ and $\delta(x)$ is the Dirac delta function.

2.2.2. *Convoluted Poisson Process.* Kataria [10] introduced the convoluted Poisson process (CPP) denoted by $\{N^c(t)\}_{t\geq 0}$ by verying intensity as a function of states with the help of the discrete convolution. Let $\{\lambda_i, i \in \mathbb{Z}\}$ is a sequence of intensity parameters such that $\lambda_i = 0$ for all $i < 0$ and $\lambda_i > \lambda_{i+1} > 0$ for all $i \geq 0$ with $\sum_{i=0}^\infty (\lambda_{i-1} - \lambda_i) = 0$. The probability mass function (pmf) of the CPP is given by

$$p_1^c(n,t) = \begin{cases} e^{-\lambda_0 t}, & \text{if } n = 0, \\ \displaystyle\sum_{k=1}^n \sum_{\Lambda_n^k} k! \prod_{i=1}^n \frac{(\lambda_{i-1} - \lambda_i)^{k_i}}{k_i!} t^k e^{-\lambda_0 t}, & \text{if } n \geq 1, \end{cases} \tag{5}$$

where $\Lambda_n^k = \{(k_1, k_2, \dots, k_n) : \sum_{i=1}^n k_j = k, \sum_{i=1}^n jk_j = k, \;\; k_j \in \mathbb{N}_0\}$.

The pmf in (5) satisfies the following fractional stochastic differential equation of the form (see [10])

$$\begin{aligned} \frac{d}{dt} p_1^c(n,t) &= -\lambda_n * p_1^c(n,t) + \lambda_{n-1} * p_1^c(n-1,t) \\ &= -\lambda_0 p_1^c(n,t) + \sum_{i=1}^n (\lambda_{i-1} - \lambda_i) p_1^c(n-i,t). \end{aligned} \tag{6}$$

2.3. **Convoluted Fractional Poisson Process.**

Kataria [10] introduced the convoluted fractional Poisson process (CFPP) denoted by $\{N^c_\beta(t)\}_{t\geq 0}$ by varying intensity as a function of states with the help of the discrete convolution. Let $\{\lambda_i, i \in \mathbb{Z}\}$ is a sequence of intensity parameters such that $\lambda_i = 0$ for all $i < 0$ and $\lambda_i > \lambda_{i+1} > 0$ for all $i \geq 0$ with $\sum_{i=0}^{\infty}(\lambda_{i-1} - \lambda_i) = 0$. The probability mass function (pmf) of the CFPP is given by

$$p^c_\beta(n,t) = \begin{cases} L^1_{\beta,1}(-\lambda_0 t^\beta), & \text{if } n = 0, \\ \sum_{k=1}^{n} \sum_{\Lambda^k_n} k! \prod_{i=1}^{n} \frac{(\lambda_{i-1} - \lambda_i)^{k_i}}{k_i!} t^{k\beta} L^{k+1}_{\beta,k\beta+1}(-\lambda_0 t^\beta), & \text{if } n \geq 1, \end{cases} \tag{7}$$

where $\Lambda^k_n = \{(k_1, k_2, \ldots, k_n) : \sum_{i=1}^n k_j = k, \sum_{i=1}^n jk_j = k, \;\; k_j \in \mathbb{N}_0\}$. The pmf in (7) satisfies the following fractional stochastic differential equation (see [10])

$$\partial^\beta_t p^c_\beta(n,t) = -\lambda_n * p^c_\beta(n,t) + \lambda_{n-1} * p^c_\beta(n-1,t), \;\; n \geq 0, \tag{8}$$

with initial conditions

$$p^c_\beta(n,0) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1, \end{cases}. \tag{9}$$

For $\beta = 1$, (8) reduces the differential equation for the convoluted Poisson process $\{N^c_1(t)\}_{t\geq 0}$ of the form (see [10])

$$\begin{aligned} \frac{d}{dt} p^c_1(n,t) &= -\lambda_n * p^c_1(n,t) + \lambda_{n-1} * p^c_1(n-1,t) \\ &= -\lambda_0 p^c_1(n,t) + \sum_{i=1}^{n} (\lambda_{i-1} - \lambda_i) p^c_1(n-i,t). \end{aligned} \tag{10}$$

3. Time-Changed Convoluted Poisson Process

We introduce the time-changed convoluted Poisson process (TCPP) by time changing in convoluted Poisson process (CFPP) with an independent Lévy subordinator as

$$N^c_f(t) = N^c(S_f(t)), t \geq 0. \tag{11}$$

Let $h_f(x,t)$ be the pdf for the Lévy subordinator. Then, with the help of (5), the pmf for the TCPP is derived as

$$\begin{aligned} q^c_f(n,t) = P\{N^c_f(t) = n\} &= \int_0^\infty p^c(n,y) h_f(y,t) dy \\ &= \begin{cases} \int_0^\infty \left(\sum_{i=0}^{\infty} \frac{(-\lambda_0 y)^i}{\Gamma(i+1)} \right) h_f(y,t) dy, & \text{if } n = 0, \\ \int_0^\infty \sum_{k=1}^{n} \sum_{\Lambda^k_n} k! \prod_{i=1}^{n} \frac{(\lambda_{i-1} - \lambda_i)^{k_i}}{k_i!} \frac{y^k}{\Gamma(k+1)} \left(\sum_{i=0}^{\infty} \frac{(-\lambda_0 y)^i}{i!} \right) h_f(y,t) dy, & \text{if } n \geq 1, \end{cases} \end{aligned}$$

$$= \begin{cases} \sum_{i=0}^{\infty} \frac{(-\lambda_0)^i}{\Gamma(i+1)} \int_0^{\infty} y^i h_f(y,t)dy, & \text{if } n=0, \\ \\ \sum_{k=1}^{n} \sum_{\Lambda_n^k} k! \prod_{i=1}^{n} \frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!\Gamma(k+1)} \sum_{i=0}^{\infty} \frac{(-\lambda_0)^i}{i!} \int_0^{\infty} y^{(k+i)} h_f(y,t)dy, & \text{if } n \geq 1, \end{cases}$$

$$= \begin{cases} \sum_{i=0}^{\infty} \frac{(-\lambda_0)^i}{\Gamma(i\beta+1)} \mathbb{E}\left[S_f^i(t)\right], & \text{if } n=0, \\ \\ \sum_{k=1}^{n} \sum_{\Lambda_n^k} k! \prod_{i=1}^{n} \frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!\Gamma(k+1)} \sum_{i=0}^{\infty} \frac{(-\lambda_0)^i}{i!} \mathbb{E}\left[S_f^{k+i}(t)\right], & \text{if } n \geq 1, \end{cases}$$

$$= \begin{cases} \mathbb{E}e^{-\lambda_0 S_f(t)}, & \text{if } n=0, \\ \\ \sum_{k=1}^{n} \sum_{\Lambda_n^k} k! \prod_{i=1}^{n} \frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!} \mathbb{E}\left[S_f^k(t) e^{-\lambda_0 S_f(t)}\right], & \text{if } n \geq 1, \end{cases} \tag{12}$$

Considering (12) and using the results for exponential functions as a special case of the Mittag-Leffler function and the definition of the Bell polynomials encountered in Beghin [4] and Comtet [7, pp. 133-137], respectively, we may note that

$$\begin{aligned} \sum_{n=0}^{\infty} q_f^c(n,t) &= q_f^c(0,t) + \sum_{n=1}^{\infty} q_f^c(n,t) \\ &= \mathbb{E}e^{-\lambda_0 S_f(t)} + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{\Lambda_n^k} k! \prod_{i=1}^{n} \frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!} \mathbb{E}\left[S_f^k(t) e^{-\lambda_0 S_f(t)}\right] \\ &= \mathbb{E}e^{-\lambda_0 S_f(t)} + \sum_{k=1}^{\infty} \mathbb{E}\left[S_f^k(t) e^{-\lambda_0 S_f(t)}\right] \sum_{n=k}^{\infty} \sum_{\Lambda_n^k} k! \prod_{i=1}^{n} \frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!} \\ &= \sum_{k=0}^{\infty} (\lambda_0)^k \mathbb{E}\left[S_f^k(t) e^{-\lambda_0 S_f(t)}\right] = \mathbb{E}[1] = 1. \end{aligned}$$

Remark 3.1. *Let T_f be first passage time distribution of the TCPP. Then, we have*

$$P\{T_f > t\} = P\{N_f^c(t) = 0\} = \mathbb{E}e^{-\lambda_0 S_f(t)}, \ t > 0, \tag{13}$$

which coincides with the Laplace transform of the subordinator with Bernŝtein function f.

Next, we obtain the probability generating function (pgf) of the TCPP.

Theorem 3.1. *The pgf of the TCPP is given by*

$$\mathcal{H}_f^c(u,t) = \exp\left(-tf\left(\sum_{i=0}^{\infty}((\lambda_{i-1}-\lambda_i))u^i\right)\right) \tag{14}$$

Proof. It is known that the pgf of CPP have the following form ([10])

$$\mathcal{G}^c(u,t) = \exp\left(\sum_{i=0}^{\infty} u^i(\lambda_{i-1}-\lambda_i)t\right). \tag{15}$$

We consider the formula

$$\mathcal{H}_f^c(u,t) = \int_0^\infty \mathcal{G}^c(u,y)h_f(y,t)dy \tag{16}$$

$$= \int_0^\infty \left(\exp\left(\sum_{i=0}^{\infty} u^i(\lambda_{i-1}-\lambda_i)y\right)\right) h_f(y,t)dy \tag{17}$$

$$= \mathbb{E}\left[\exp\left(S_f(t)\sum_{i=0}^{\infty} u^i(\lambda_{i-1}-\lambda_i)\right)\right]. \tag{18}$$

Since $S_f(t)$ is a subordinator, therefore following the definition (1), we get the result. □

Remark 3.2. *It may be noted that the moment generating function (mgf) for the TCPP can be obtained through (*14*) and is given by*

$$M_f^c(u,t) = \exp\left(-tf\left(\sum_{i=0}^{\infty}((\lambda_{i-1}-\lambda_i))e^{ui}\right)\right). \tag{19}$$

In the following theorem, we derive the formula for the mean, variance and the autocovariance of the TCPP.

Theorem 3.2. *For $0 < s \le t < \infty$, the mean, variance and autocovariance of the TCPP are given by*
(i) $\mathbb{E}[N_f^c(t)] = R\mathbb{E}\left[S_f(t)\right].$
(ii) $Var[N_f^c(t)] = R^2\, Var\left[S_f(t)\right] + (R+2S)\mathbb{E}\left[S_f(t)\right].$
(iii) $Cov[N_f^c(s), N_f^c(t)] = R^2\, Var\left[S_f(s)\right] + (R+2S)\mathbb{E}\left[S_f(s)\right],$
where $R = \sum_{i=0}^{\infty}\lambda_i$ *and* $S = \sum_{i=1}^{\infty} i\lambda_i$.

Proof. The mean and variance of the CPP are given by (see [10])

$$\mathbb{E}\left[N^c(t)\right] = Rt, \tag{20}$$

$$\text{Var}\left[N^c(t)\right] = (R+2S)t, \tag{21}$$

respectively. Using (20), we get

$$\mathbb{E}[N_f^c(t)] = \mathbb{E}\left[\mathbb{E}[N^c(S_f(t))|S_f(t)]\right] = \mathbb{E}\left[RS_f(t)\right] = R\mathbb{E}\left[S_f(t)\right]. \tag{22}$$

Now, we compute the variance function for the TCPP as

$$\text{Var}[N_f^c(t)] = \text{Var}\left[\mathbb{E}[N^c(S_f(t))|S_f(t)]\right] + \mathbb{E}\left[\text{Var}[N^c(S_f(t))|S_f(t)]\right] \tag{23}$$

$$= \text{Var}\left[RS_f(t)\right] + \mathbb{E}\left[(R+2S)S_f(t)\right] \tag{24}$$

$$= R^2\text{Var}\left[S_f(t)\right] + (R+2S)\mathbb{E}\left[S_f(t)\right]. \tag{25}$$

Using the independent increment property of the Lévy subordinator, we get

$$\text{Cov}\left[S_f(s), S_f(t)\right] = \text{Cov}\left[S_f(s), (S_f(t)-S_f(s)) + S_f(s)\right]$$

$$= \text{Cov}\left[S_f(s), S_f(t) - S_f(s)\right] + \text{Cov}\left[S_f(s), S_f(s)\right]$$
$$= \text{Var}\left[S_f(s)\right].$$

Hence, the proof of the part (iii) follows from the Leonenko [15] covariance formula and the above relation as

$$\text{Cov}\left[N_f^c(s), N_f^c(t)\right] = \text{Var}\left[N^c(1)\right]\mathbb{E}(S_f(s)) + \left(\mathbb{E}\left[N^c(1)\right]\right)^2 \text{Var}\left[S_f(s)\right] \tag{26}$$

$$= R^2 \text{Var}\left[S_f(s)\right] + (R + 2S)\mathbb{E}\left[S_f(s)\right]. \tag{27}$$

□

Now, we present an intresting distributional equality of the TCPP.

Theorem 3.3. *Let $\{N(t)\}_{t\geq 0}$ br the Poisson process with parameter λ_0 and let $X_i, i = 1, 2, \ldots$ be the sequence of the independent and identically distributed (i.i.d.) random variables with pmf*

$$P\{X_1 = k\} = \frac{\lambda_{k-1} - \lambda_k}{\lambda_0}, \quad k \geq 1. \tag{28}$$

Then, we have

$$N_f^c(t) \stackrel{d}{=} \sum_{i=1}^{N(S_f(t))} X_i, \quad t \geq 0,$$

where $N(S_f(t))$ is a Poisson process subordinated by the Lévy subordinator introduced in Orsingher and Toaldo [18] *and $X_i's$ are independent of $N(S_f(t))$.*

Proof. It is known that the process $\{N(S_f(t))\}_{t\geq 0}$ has the pgf of the form

$$G_f(u, t) = e^{-tf(\lambda_0(1-u))}. \tag{29}$$

Also, the pgf of the $X_i's$ can be derived which have the following form

$$\mathbb{E}\left[u^{X_i}\right] = \frac{1}{\lambda_0}\sum_{i=1}^{\infty}(\lambda_{i-1} - \lambda_i)u^i. \tag{30}$$

We consider

$$\mathbb{E}\left[u^{N_f^c(t)}\right] = \mathbb{E}\left[u^{\sum_{i=1}^{N(S_f(t))} X_i}\right] = \mathbb{E}\left[\mathbb{E}\left[u^{\sum_{i=1}^{N(S_f(t))} X_i} \,\middle|\, N(S_f(t))\right]\right] = \mathbb{E}\left[\left(\mathbb{E}\left[u^{X_1}\right]\right)^{N(S_f(t))}\right]$$
$$= \mathbb{E}\left[\frac{1}{\lambda_0}\sum_{i=1}^{\infty}(\lambda_{i-1} - \lambda_i)u^i\right]^{N(S_f(t))} = \exp\left(-tf\left(\lambda_0\left(1 - \frac{1}{\lambda_0}\sum_{i=1}^{\infty}(\lambda_{i-1} - \lambda_i)u^i\right)\right)\right).$$

Using simple algebra, we get the pgf similar to (14). Hence, this completes the proof. □

4. Convoluted Negative Binomial Process

We define the convoluted negative binomial process (CNBP) by time changing in convoluted Poisson process (CPP) with an independent gamma subordinator as

$$\mathcal{Q}_1^c(t) = N^c(\Gamma(t)). \tag{31}$$

For the gamma subordinator, (12) reduces to

$$q_1^c(n,t) = \begin{cases} \dfrac{1}{\Gamma(\rho t)} \displaystyle\sum_{k=0}^{\infty} \left(\frac{-\lambda_0}{\mu}\right)^k \frac{\Gamma(\rho t + k)}{k!}, & \text{if } n = 0, \\ \\ \dfrac{1}{\Gamma(\rho t)} \displaystyle\sum_{k=1}^{n} \frac{1}{\mu^k} \sum_{\Lambda_n^k} \prod_{i=1}^{n} \frac{(\lambda_{i-1} - \lambda_i)^{k_i}}{k_i!} \sum_{j=0}^{\infty} \left(\frac{-\lambda_0}{\mu}\right)^j \frac{\Gamma(\rho t + j + k)}{j!}, & \text{if } n \geq 1, \end{cases} \tag{32}$$

First, we recall the following lemma from [25] which is useful in the subsequent result.

Lemma 4.1. *The governing PDE for the gamma subordinator* $\{\Gamma(t)\}_{t\geq 0}$ *is given by (see* [25]*)*

$$\frac{\partial}{\partial t} f_1(x,t) = \rho \left[\log \alpha + \log y - \psi(\rho t)\right] f_1(x,t), \quad y > 0 \text{ and } f_1(x,0) = 0,$$

where $\psi(x)$ *is the digamma function.*

Theorem 4.1. *The pmf of the CNBP satisfies the following differential equation*

$$\frac{\partial}{\partial t} q_1^c(n,t) = \rho \left(\log \mu - \psi(\rho t)\right) q_1^c(n,t) + \rho \int_0^{\infty} p_1^c(n,y)(\log y) f_G(y,t)dy. \tag{33}$$

Proof. We consider

$$\begin{aligned} \frac{\partial}{\partial t} q_1^c(n,t) &= \frac{\partial}{\partial t} \int_0^{\infty} p_1^c(n,y) f_G(y,t)dy \\ &= \int_0^{\infty} p_1^c(n,y) \frac{\partial}{\partial t} f_G(y,t)dy \\ &= \int_0^{\infty} p_1^c(n,y) \left[\log \mu + \log y - \psi(\rho t)\right] f_G(y,t)dy \\ &= \rho \int_0^{\infty} p_1^c(n,y) \left(\log \mu - \psi(\rho t)\right) f_G(y,t)dy + \rho \int_0^{\infty} p_1^c(n,y)(\log y) f_G(y,t)dy. \end{aligned}$$

Arranging the terms suitably, we get the result. □

Theorem 4.2. *The pmf in (7) satisfies the following differential equation*

$$\mu \left(1 - e^{-\frac{1}{\rho}\partial_t}\right) = -\lambda_n * q_1^c(n,t) + \lambda_{n-1} * q_1^c(n-1,t). \tag{34}$$

Proof. We apply the pmf definition for $q_1^c(n,t)$ together with (4) and (10) as

$$\begin{aligned} e^{-\frac{1}{\rho}\partial_t} q_1^c(n,t) &= \int_0^{\infty} p_1^c(n,y) e^{-\frac{1}{\rho}\partial_t} f_G(y,t)dy \\ &= \int_0^{\infty} p_1^c(n,y) \left(\frac{1}{\mu}\frac{d}{dy} f_G(x,t) + f_G(y,t)\right) dy \\ &= \int_0^{\infty} p_1^c(n,y) f_G(y,t)dy + \frac{1}{\mu} \int_0^{\infty} p_1^c(n,y) \frac{d}{dy} f_G(y,t)dy \end{aligned}$$

$$
\begin{aligned}
&= q_1^c(n,t) - \frac{1}{\mu}\int_0^\infty \frac{d}{dy} p_1^c(n,y) f_G(x,t)dy \\
&= q_1^c(n,t) - \frac{1}{\mu}\int_0^\infty \left(-\lambda_0 p_1^c(n,y) + \sum_{i=1}^{n}(\lambda_{i-1}-\lambda_i)p_1^c(n-i,y)\right) f_G(y,t)dy \\
&= q_1^c(n,t) + \frac{1}{\mu}\lambda_0 q_1^c(n,t) - \frac{1}{\mu}\sum_{i=1}^{n}(\lambda_{i-1}-\lambda_i)q_1^c(n-i,t) \\
&= q_1^c(n,t) - \frac{1}{\mu}\left(-\lambda_n * q_1^c(n,t) + \lambda_{n-1} * q_1^c(n-1,t)\right).
\end{aligned}
$$

On rearranging the terms, we get the theorem. □

Next, we explore the dependence behavior and the Lévy measure of the CNBP.

Theorem 4.3. *The mean, variance and autocovariance of the CNBP are given by*
(i) $\mathbb{E}[\mathcal{Q}_1^c(t)] = \frac{\Gamma(\rho t)}{\mu} R.$
(ii) $Var[\mathcal{Q}_1^c(t)] = \frac{\rho t}{\mu}\left(\frac{R^2}{\mu} + R + 2S\right).$
(iii) $Cov[\mathcal{Q}_1^c(s), \mathcal{Q}_1^c(t)] = \frac{\rho s}{\mu}\left(\frac{R^2}{\mu} + R + 2S\right).$

where $R = \sum_{i=0}^{\infty}\lambda_i$ *and* $S = \sum_{i=1}^{\infty} i\lambda_i$.

Proof. The proof follows on a similar line as in the proof of Theorem 3.2. □

Proposition 4.1. *The Lévy measure of the CNBP is given by*

$$
\mathcal{V}_1(\cdot) = \rho \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} k! \sum_{\Lambda_n^k}\prod_{i=1}^{n}\frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!}\frac{\Gamma(k)}{(\mu+\lambda_0)^k}. \tag{35}
$$

Proof. It is known that the Lévy measure $pi(\cdot)$ of the gamma subordinator is $\pi_G(\cdot) = \rho y^{-1}e^{-\mu y}$. Using the formula given in [23] and with the help of (5), we get

$$
\mathcal{V}_1(\cdot) = \int_0^\infty \sum_{n=1}^{\infty} p^c(n,y)\delta_n(\cdot)\pi_G dy \tag{36}
$$

$$
= \int_0^\infty \sum_{n=1}^{\infty} p^c(n,y)\delta_n(\cdot)\rho y^{-1}e^{-\mu y}dy \tag{37}
$$

$$
= \rho \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{\Lambda_n^k} k! \prod_{i=1}^{n}\frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!}\int_0^\infty y^{k-1}e^{-(\lambda_0+\mu)y}dy. \tag{38}
$$

Using the known identity $\int_0^\infty x^n e^{-an} = n! a^{-n-1}$, we get

$$
\mathcal{V}_1(\cdot) = \rho \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} k! \sum_{\Lambda_n^k}\prod_{i=1}^{n}\frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!}\frac{\Gamma(k)}{(\mu+\lambda_0)^k}. \tag{39}
$$

Hence, we complete the proof. □

4.1. **LRD Property.**

Definition 4.1. *For $0 < s < t$, let the correlation function $Corr[X(s), X(t)]$ for a stochastic process $\{X(t)\}_{t\geq 0}$ satisfies the following relation (see* [16]*)*

$$c_1(s)t^{-d} \leq \ Corr[X(s), X(t)] \leq c_2(s)t^{-d}$$

for large t, $d > 0$, $c_1(s) > 0$ and $c_2(s) > 0$. Expressly

$$\lim_{t\to\infty} \frac{Corr[X(s), X(t)]}{t^{-d}} = c(s),$$

for some $c(s) > 0$ and $d > 0$. The process $\{X(t)\}_{t\geq 0}$ is said to have the LRD property if $d \in (0, 1)$.

Theorem 4.4. *The CFNBP exhibits the LRD property.*

Proof. The correlation function for the CFNBP is given by

$$\text{Corr}\left[\mathcal{Q}_1^c(s), \mathcal{Q}_1^c(t)\right] = \frac{\text{Cov}[\mathcal{Q}^c(s), \mathcal{Q}^c(t)]}{\sqrt{\text{Var}\left[\mathcal{Q}_1^c(s)\right]}\sqrt{\text{Var}\left[\mathcal{Q}_1^c(t)\right]}} \tag{40}$$

Using Part (ii) and Part(iii) of the Theorem(4.3), we get

$$\text{Corr}\left[\mathcal{Q}_1^c(s), \mathcal{Q}_1^c(t)\right] = \frac{\frac{\rho s}{\mu}\left(\frac{R^2}{\mu} + R + 2S\right)}{\sqrt{\frac{\rho s}{\mu}\left(\frac{R^2}{\mu} + R + 2S\right)}\sqrt{\frac{\rho t}{\mu}\left(\frac{R^2}{\mu} + R + 2S\right)}} \tag{41}$$

$$= t^{-\frac{1}{2}} s^{\frac{1}{2}} \frac{\frac{\rho}{\mu}\left(\frac{R^2}{\mu} + R + 2S\right)}{\sqrt{\frac{\rho}{\mu}\left(\frac{R^2}{\mu} + R + 2S\right)}\sqrt{\frac{\rho}{\mu}\left(\frac{R^2}{\mu} + R + 2S\right)}}. \tag{42}$$

This implies that

$$\lim_{t\to\infty} \frac{\text{Corr}\left[\mathcal{Q}_1^c(s), \mathcal{Q}_1^c(t)\right]}{t^{-\frac{1}{2}}} = s^{\frac{1}{2}}. \tag{43}$$

Therefore, using the Definition 4.1 for $d = \frac{1}{2}$, we observe that CFNBP has the LRD property. □

5. Some Other Time-Changed Versions

5.1. **CPP Time-Changed by Tempered Stable Subordinator.** For $0 < \alpha < 1$, the tempered α-stable subordinator $D_{\alpha,\theta}(t)$ is defined by the Laplace transform given by (see [22])

$$\mathbb{E}[e^{-uD_{\alpha,\theta}(t)}] = e^{-t((\alpha + u)^{\theta} - \alpha^{\theta})}, \ \ \theta > 0.$$

We define the CPP time-changed by the tempered α-stable subordinator (TSS) as

$$\mathcal{Q}_2^c(t) = N^c\left(D_{\alpha,\theta}(t)\right), \ \ t \geq 0. \tag{44}$$

Theorem 5.1. *The pmf $q_2^c(n,t)$ of the process $\mathcal{Q}_{\in}(t)$ satisfies the following differential equation*

$$\left(\alpha^{\theta} - \frac{\partial}{\partial t}\right)^{1/\theta} q_2^c(n,t) = (\alpha + \lambda_0)q_2^c(n,t) - \sum_{i=1}^{n}(\lambda_{i-1} - \lambda_i)q_2^c(n-i,t). \tag{45}$$

Proof. Since, it is known that the pdf $f_1(x,t)$ of the TSS satisfies (see [3])

$$\frac{\partial}{\partial x}f_2(x,t) = -\alpha f_2(x,t) + \left(\alpha^\theta - \frac{\partial}{\partial t}\right)^{1/\theta} f_2(x,t), \tag{46}$$

with $f_2(x,0) = \delta_0(x)$ and $f_2(0,t) = 0$.

Consider

$$\left(\alpha^\theta - \frac{\partial}{\partial t}\right)^{1/\theta} q_2^c(n,t) = \int_0^\infty p_1^c(n,y)\left(\alpha^\theta - \frac{\partial}{\partial t}\right)^{1/\theta} f_2(y,t)dy \tag{47}$$

$$= \int_0^\infty p_1^c(n,y)\left(\frac{\partial}{\partial y}f_2(y,t) + \alpha f_2(y,t)\right)dy \tag{48}$$

$$= \alpha q_2^c(n,t) - \int_0^\infty f_2(y,t)\frac{d}{dy}p_1^c(n,y)dy. \tag{49}$$

With the help of (6), we get

$$\begin{aligned}\left(\alpha^\theta - \frac{\partial}{\partial t}\right)^{1/\theta} q_2^c(n,t) &= \alpha q_2^c(n,t) - \int_0^\infty f_1(y,t)\left(-\lambda_0 p_1^c(n,y) + \sum_{i=1}^n(\lambda_{i-1}-\lambda_i)p_1^c(n-i,y)\right)dy\\ &= (\alpha+\lambda_0)q_2^c(n,t) - \sum_{i=1}^n(\lambda_{i-1}-\lambda_i)q_2^c(n-i,t).\end{aligned}$$

Hence, this proved the required differential equation. □

5.2. **CPP Time-Changed by the Inverse Gaussian Subordinator.** For $\eta,\mu>0$, the inverse Gaussian subordinator $G_{\eta,\nu}(t)$ is given by (see [1, Equation (1.27)])

$$\mathbb{E}[e^{-uG_{\eta,\nu}(t)}] = e^{-t\eta\left(\sqrt{2u+\mu^2}-\mu\right)},\ u>0. \tag{50}$$

We define the CPP time-changed by the inverse Gaussion subordinator (IGS) as

$$\mathcal{Q}_3^c(t) = N^c\left(G_{\eta,\nu}(t)\right),\ \ t\geq 0. \tag{51}$$

Theorem 5.2. *The pmf $q_3^c(n,t)$ of the process $\mathcal{Q}_3^c(t)$ satisfies*

$$\left(\frac{d^2}{dt^2} - 2\eta\mu\frac{d}{dt}\right)q_3^c(n,t) = 2\lambda_0\eta^2 q_3^c(n,t) - 2\eta^2\sum_{i=1}^n(\lambda_{i-1}-\lambda_i)q_3^c(n-i,t). \tag{52}$$

Proof. Let $f_2(x,t)$ be the pdf of the IG subordinator. It satisfies the following differential equation (see [13])

$$\frac{\partial^2}{\partial t^2}f_2(x,t) = 2\eta^2\frac{\partial}{\partial x}f_2(x,t) + 2\eta\mu\frac{\partial}{\partial t}f_2(x,t). \tag{53}$$

Therefore, we have

$$\begin{aligned}\left(\frac{d^2}{dt^2} - 2\eta\mu\frac{d}{dt}\right)q_3^c(n,t) &= \int_0^\infty p_1^c(n,y)\left(\frac{d^2}{dt^2} - 2\eta\mu\frac{d}{dt}\right)f_3(y,t)dy\\ &= 2\eta^2\int_0^\infty p_1^c(n,y)\frac{\partial}{\partial y}f_3(y,t)dy\\ &= -2\eta^2\int_0^\infty f_3(y,t)\frac{d}{dy}p_1^c(n,y)dy\\ &= -2\eta^2\int_0^\infty f_3(y,t)\left(-\lambda_0 p_1^c(n,y) + \sum_{i=1}^n(\lambda_{i-1}-\lambda_i)p_1^c(n-i,y)\right)dy\end{aligned}$$

$$= 2\lambda_0\eta^2 q_3^c(n,t) - 2\eta^2 \sum_{i=1}^{n} (\lambda_{i-1} - \lambda_i) q_3^c(n-i,t).$$

Hence, we get the required differential equation. □

5.3. **CPP Time-Changed by the Mittag-Leffler Lévy Subordinator.** The Mittag-Leffler (ML) Lévy process $M_{\alpha,\rho,\mu}(t)$ is defined by the Laplace transform given by (see [14])

$$\mathbb{E}[e^{-uM_{\alpha,\rho,\mu}(t)}] = \left(\frac{\mu}{\mu + u^\alpha}\right)^{\rho t}, \; u > 0. \tag{54}$$

For $\tau \geq 1$, its pdf $f_3(t)$ solves the following fractional differential equation (see [14])

$$\frac{\partial^\tau}{\partial t^\tau} f_{M_{\alpha,\rho,\mu}(t)}(x) = \rho \frac{\partial^{\tau-1}}{\partial t^{\tau-1}} \left[(\log\mu - \psi(\rho t)) f_{M_{\alpha,\rho,\mu}(t)}(x) + \int_0^\infty g_\alpha(x,y)(\log y) f_G(y,t) dy \right].$$

We derive the differential equation solving the pmf $q_4(n,t)$ of the CPP time-changed by the ML Lévy subordinator as

$$\begin{aligned}
\frac{1}{\rho}\frac{d^\tau}{dt^\tau} q_4^c(n,t) &= \int_0^\infty p_1^c(n,s) \frac{\partial^\tau}{\partial t^\tau} f_{M_{\alpha,\rho,\mu}(t)}(s) ds \\
&= \int_0^\infty p_1^c(n,s) \frac{\partial^{\tau-1}}{\partial t^{\tau-1}} \left[(\log\mu - \psi(\rho t)) f_{M_{\alpha,\rho,\mu}(t)}(s) + \int_0^\infty g_\alpha(s,y)(\log y) f_G(y,t) dy \right] ds \\
&= \frac{\partial^{\tau-1}}{\partial t^{\tau-1}} \left((\log\mu - \psi(\rho t)) q_4^c(n,t) + \int_0^\infty \int_0^\infty g_\alpha(s,y)(\log y) f_G(y,t) dy ds \right).
\end{aligned}$$

Hence, we get

$$\frac{1}{\rho}\frac{d^\tau}{dt^\tau} q_4^c(n,t) = \frac{\partial^{\tau-1}}{\partial t^{\tau-1}} \left((\log\mu - \psi(\rho t)) q_4^c(n,t) + \int_0^\infty \int_0^\infty g_\alpha(s,y)(\log y) f_G(y,t) dy ds \right). \tag{55}$$

5.4. **CPP Time-Changed by the Inverse Tempered Stable Subordinator.** For $0 < \alpha < 1, \theta > 0$, let $\{\mathcal{E}_{\theta,\alpha}(t)\}_{t\geq 0}$ be the inverse TSS defined as (see [13])

$$\mathcal{E}_{\theta,\alpha}(t) = \inf\{s \geq 0 : D_{\alpha,\theta}(s) > t\}, \tag{56}$$

where $D_{\alpha,\theta}(t)$ is tempered stable subordinator.

We define the CPP time-changed by the inverse TSS as

$$\mathcal{Q}_5^c(t) = N^c\left(\mathcal{E}_{\theta,\alpha}(t)\right), \;\; t \geq 0. \tag{57}$$

Let $f_5(x,t)$ be the pdf of the inverse TSS solves the following differential equation (see [13])

$$\frac{\partial}{\partial t} f_5(x,t) + \delta_0(t)\delta_0(x) = \sum_{i=1}^{m} \binom{m}{i} \theta^{(1-i/m)} (-1)^i \frac{\partial^i}{\partial x^i} f_5(x,t), \tag{58}$$

where $\delta_0(x) = f_5(x,0)$ and $\lim_{x\to\infty} f_5(x,t) = 0$.

Let $q_5^c(n,t)$ be the pmf of the CPP time-changed by the inverse tempered stable subordinator. Then, for $m = 1/\nu \geq 2$ with the help of (58), we have

$$\begin{aligned}
\frac{d}{dt} q_5^c(n,t) &= \int_0^\infty p_1^c(n,s) \frac{\partial}{\partial t} f_5(s,t) ds \\
&= \int_1^\infty p_1^c(n,s) \left(\sum_{i=0}^{m} \binom{m}{i} \theta^{(1-i/m)} (-1)^i \frac{\partial^i}{\partial s^i} f_3(s,t) - \delta_0(t)\delta_0(s) \right) ds
\end{aligned}$$

$$= \sum_{i=0}^{m} \binom{m}{i} \theta^{(1-i/m)}(-1)^i \int_0^\infty p_1^c(n,s) \frac{\partial^i}{\partial s^i} f_3(s,t) ds - \delta_0(t) p_1^c(n,0)$$
$$= \sum_{i=1}^{m} \binom{m}{i} \theta^{(1-i/m)}(-1)^i \sum_{j=1}^{i} (-1)^j \frac{d^{j-1}}{ds^{j-1}} p_1^c(n,s) \frac{\partial^{i-k}}{\partial s^{i-k}} f_5(s,t)\Big|_{s=0}$$
$$+ \sum_{i=1}^{m} \binom{m}{i} \theta^{(1-i/m)} \int_0^\infty f_5(s,t) \frac{d^i}{ds^i} p_1^c(n,s) ds - \delta_0(t) p_1^c(n,0).$$

Proposition 5.1. *The pmf $q_5^c(n,t)$ solve the following differential equation*

$$\left(\theta + \frac{d}{dt}\right)^\alpha q_5^c(n,t) = (\theta^\alpha - \lambda_0) q_5^c(n,t) + \sum_{i=1}^{n} (\lambda_{i-1} - \lambda_i) q_5^c(n-i,t)$$
$$+ p_1^c(n,s) f_5(s,t)|_{s=0} - t^{-\alpha} E_{1,1-\alpha}^{1-\alpha}(-\theta t) p_1^c(n,0).$$

Proof. It is known that (see [12])

$$\frac{\partial}{\partial x} f_5(x,t) + t^{-\alpha} E_{1,1-\alpha}^{1-\alpha}(-\theta t)\delta_0(x) + \left(\theta + \frac{\partial}{\partial t}\right)^\alpha f_5(x,t) = \theta^\alpha f_5(x,t). \tag{59}$$

Hence, we derive the differential equation as

$$\left(\theta + \frac{d}{dt}\right)^\alpha q_5^c(n,t) = \int_0^\infty p_1^c(n,s) \left(\theta^\alpha f_5(s,t) - t^{-\alpha} E_{1,1-\alpha}^{1-\alpha}(-\theta t)\delta_0(s) - \frac{\partial}{\partial x} f_5(s,t)\right) ds$$
$$= \theta^\alpha q_5^c(n,t) + p_1^c(n,s) f_5(s,t)|_{s=0} - t^{-\alpha} E_{1,1-\alpha}^{1-\alpha}(-\theta t) p_1^c(n,0)$$
$$+ \int_0^\infty \left(-\lambda_0 p_1^c(n,s) + \sum_{i=1}^{n} (\lambda_{i-1} - \lambda_i) p_1^c(n-i,s)\right) f_5(s,t) ds$$
$$= (\theta^\alpha - \lambda_0) q_5^c(n,t) + \sum_{i=1}^{n} (\lambda_{i-1} - \lambda_i) q_5^c(n-i,t)$$
$$+ p_1^c(n,s) f_5(s,t)|_{s=0} - t^{-\alpha} E_{1,1-\alpha}^{1-\alpha}(-\theta t) p_1^c(n,0).$$

Hence, we get the result. □

5.5. **CPP Time-Changed by the First Hitting Time of the Inverse Gaussian Subordinator.** Let $\xi(t)$ be the first hitting-time of the IG subordinator $G_{\eta,\nu}(t)$ defined as

$$\xi(t) = \inf\{s \geq 0 : G_{\eta,\nu}(s) > t\}, \quad t \geq 0.$$

We define the CPP time changed by the first hitting-time of IG subordinator as

$$\mathcal{Q}_6^c(t) = N^c\left(\xi(t)\right), \ \ t \geq 0. \tag{60}$$

Proposition 5.2. *The pmf $q_6^c(n,t) = P\{\mathcal{Q}_6^c(t) = n\}$ solves the following differential equation*

$$\eta \left(\nu^2 + 2\frac{d}{dt}\right)^{1/2} q_6^c(n,t) = p_1^c(n,0) f_6(0,t) - (\eta\nu + \lambda_0) q_6^c(n,t) - \frac{\eta\sqrt{2}e^{-\eta^2 t/2}}{\sqrt{\pi t}} p_1^c(n,0)$$
$$- \sum_{i=1}^{n} (\lambda_{i-1} - \lambda_i) q_6^c(n-i,t).$$

Proof. The density $f_6(x,t)$ of the $\xi(t)$ satisfies the following result (see [26]

$$\frac{\partial}{\partial x}f_6(x,t)+\eta\left(\nu^2+2\frac{\partial}{\partial t}\right)^{1/2}+\frac{\eta\sqrt{2}e^{-\eta^2 t/2}}{\sqrt{\pi t}}\delta_0(x)=\eta\nu f_6(x,t),$$

with $f_6(x,0)=\delta_0(x)$. Hence, we get

$$\begin{aligned}\eta\left(\nu^2+2\frac{d}{dt}\right)^{1/2}q_6^c(n,t) &= \int_0^\infty p_1^c(n,s)\left(\eta\nu f_6(s,t)-\frac{\partial}{\partial s}f_6(s,t)-\frac{\eta\sqrt{2}e^{-\eta^2 t/2}}{\sqrt{\pi t}}\right)ds\\ &= p_1^c(n,0)f_6(0,t)-\eta\nu q_6(n,t)-\frac{\eta\sqrt{2}e^{-\eta^2 t/2}}{\sqrt{\pi t}}p_1^c(n,0)\\ &- \int_0^\infty\left(-\lambda_0 p_1^c(n,s)+\sum_{i=1}^n(\lambda_{i-1}-\lambda_i)p_1^c(n-i,s)\right)f_6(s,t)ds\\ &= p_1^c(n,0)f_6(0,t)-(\eta\nu+\lambda_0)q_6^c(n,t)-\frac{\eta\sqrt{2}e^{-\eta^2 t/2}}{\sqrt{\pi t}}p_1^c(n,0)\\ &- \sum_{i=1}^n(\lambda_{i-1}-\lambda_i)q_6^c(n-i,t).\end{aligned}$$

Hence, we completes the proof. □

5.6. **CPP Time-Changed by the Multistable Subordinator.** Let $M(t)$ be a multistable subordinator with stability index $\alpha(t)$ and the Lévy measure given by (see [5])

$$\pi_t(dx)=\frac{\alpha(t)}{\Gamma(1-\alpha(t))}x^{-\alpha(t)-1}dx,\quad x>0. \tag{61}$$

We consider the CPP time changed by the multistable subordinator as

$$Y^c(t)=N^c\left(M(t)\right),\quad t\geq 0.$$

The Lévy measure of $Y^c(t)$ is calculated as

$$\begin{aligned}\pi_t(dx) &= \int_0^\infty\sum_{n=1}^\infty p_1^c(n,s)\delta_n(dx)\pi_t(ds)\\ &= \int_0^\infty\sum_{n=1}^\infty\sum_{k=1}^n\sum_{\Lambda_n^k}k!\prod_{i=1}^n\frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!}s^k e^{-\lambda_0 s}\delta_n(dx)\frac{\alpha(t)}{\Gamma(1-\alpha(t))}s^{-\alpha(t)-1}ds\\ &= \sum_{n=1}^\infty\sum_{k=1}^n\sum_{\Lambda_n^k}k!\prod_{i=1}^n\frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!}\delta_n(dx)\frac{\alpha(t)}{\Gamma(1-\alpha(t))}\int_0^\infty s^{k-\alpha(t)-1}e^{-\lambda_0 s}ds\\ &= \sum_{n=1}^\infty\sum_{k=1}^n\sum_{\Lambda_n^k}k!\prod_{i=1}^n\frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!}\delta_n(dx)\frac{\alpha(t)}{\Gamma(1-\alpha(t))}\frac{\Gamma(k-\alpha(t))}{\lambda_0^{k-\alpha(t)}}\\ &= \sum_{n=1}^\infty\sum_{k=1}^n\sum_{\Lambda_n^k}k!\prod_{i=1}^n\binom{\alpha(t)}{k}\frac{(\lambda_{i-1}-\lambda_i)^{k_i}}{k_i!}\delta_n(dx)\frac{k!(-1)^{k+1}}{\lambda_0^{k-\alpha(t)}},\end{aligned}$$

where δ_n is Dirac delta function.

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