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NEIGHBORS STRESS SUM ENERGY OF GRAPHS

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Abstract. In this article, we introduced the concept of the Neighbors Stress Sum of a vertex and a new matrix for a connected graph G. In this matrix, the sum of the i-th row and the i-th column are both equal to the Neighbors Stress Sum of the i-th vertex. Additionally, we define a new graph energy variant called the Neighbors Stress Sum Energy, denoted as $E_{NSS}(G)$, for a graph G. Further, we established bounds, and characterized the largest eigenvalue of $NSS(G)$.

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1. Introduction

In this article, we will be focusing on finite, unweighted, simple, and undirected graphs. Let $G = (V, E)$ denote a graph. The degree of a vertex v in G is denoted by $d(v)$. The distance between two vertices u and v in G, denoted $d(u, v)$, is the number of edges in the shortest path (or geodesic) connecting them. A geodesic path P is said to pass through a vertex v if v is an internal vertex of P, meaning v lies on P but is not an endpoint of P. Neighborhood of a vertex v is defined as

$$N_G(v) = \{u \in V(G) \mid uv \in E(G)\}.$$

Gutman [6] introduced the concept of graph energy, denoted by $E(G)$, as the sum of the absolute values of the eigenvalues of its adjacency matrix, $A(G)$. Eigenvalues are fundamental to the study of graphs, as they are intricately linked to many important graph invariants and extreme properties. As a result, graph energy, while being a specific form of matrix norm, has garnered significant attention from both pure and applied mathematicians. Spectral graph theory, which focuses on matrices associated with graphs, their eigenvalues, and graph energies, plays a crucial role in analyzing graph structures through the application of matrix theory and linear algebra. Various types of graph energies related to topological indices have been introduced and extensively examined in the literature. Numerous matrices can be related to a graph, and their spectrums provide certain helpful information about the graph [1, 3, 5, 7, 10–12, 18, 20–22, 33, 40, 41].

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In 1953, Alfonso Shimbel [36] introduced the notion of vertex stress for graphs as a centrality measure. Stress of a vertex v in a graph G is the number of shortest paths (geodesics) passing through v. This concept has many applications including the study of biological and social networks. Many stress related concepts in graphs and topological indices have been defined and studied by several authors [2,4,9,13–17,19,23–32,34,35,37–39]. A graph G is k-stress regular [4] if $str(v) = k$ for all $v \in V(G)$. The following indices have been defined using stress of vertices in a graph to explain some properties of chemical compounds.

The square stress sum index $SSS(G)$ [31] of a simple graph $G(V, E)$ is defined as

$$SSS(G) = \sum_{uv \in E(G)} \left[str(u)^2 + str(v)^2\right].$$

The stress-sum index $SS(G)$ [27] of a simple graph $G(V, E)$ is defined as

$$SS(G) = \sum_{uv \in E(G)} [str(u) + str(v)] = \sum_{v \in V(G)} d(v)str(v).$$

The second stress index $S_2(G)$ [28] of a simple graph $G(V, E)$ is defined as

$$S_2(G) = \sum_{uv \in E(G)} str(u)str(v).$$

The first stress index $S_1(G)$ [28] of a simple graph $G(V, E)$ is defined as

$$S_1(G) = \sum_{v \in E(G)} str(v)^2.$$

In this paper, we introduce the neighbors stress sum matrix of a graph G and define the neighbors stress sum energy $E_{NSS}(G)$ based on its eigenvalues. This new approach extends the concept of graph energy to incorporate stress-related measures, offering a fresh perspective on graph invariants. We also establish bounds for $E_{NSS}(G)$ in relation to other graph invariants.

2. Neighbors stress sum of a vertex, Neighbors stress sum matrix and the Neighbors stress sum energy

In this section, neighbors stress sum of a vertex, Neighbors stress sum matrix and the Neighbors stress sum energy of a graph are discussed.

Neighbors stress sum of a vertex v, denoted by $N_s(v)$, is defined as sum of stress of adjacent vertices of v, that is:

$$N_s(v) = \sum_{u \in N_G(v)} str(u).$$

Inspired by the works on topological indices and related matrices, eigenvalues and bounds, we introduce the following matrix and energy associated with stress sum index for a graph G and find the bounds of the same for a connected graph.

Neighbors stress sum matrix for a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is defined as $NSS(G) = [a_{ij}]_{n\times n}$, where

$$a_{ij} = \begin{cases} -d(v_i)str(v_i), & i = j; \\ str(v_i) + str(v_j), & i \neq j \text{ and } v_i \sim v_j; \\ 0, & \text{otherwise,} \end{cases}$$

where $str(v_i)$ denotes the stress of the vertex v_i and $d(v_i)$ denotes the degree of the vertex v_i.

The neighbors stress sum polynomial of a graph G is defined as

$$P_{NSS(G)}(\lambda) = |N_\lambda I - NSS(G)|,$$

where I is an $n \times n$ unit matrix.

All the roots of the equation $P_{NSS(G)}(\lambda) = 0$ are real because the matrix $NSS(G)$ is real and symmetric. Therefore, these roots can be ordered as $N_{\lambda_1} \geq N_{\lambda_2} \geq \cdots \geq N_{\lambda_n}$, with N_{λ_1} being the largest and N_{λ_n} being the smallest eigenvalue. The neighbors stress sum energy of a graph is represented by

$$E_{NSS}(G) = \sum_{i=1}^{n} |N_{\lambda_i}|.$$

Example: Consider the graph G given in Figure 1. The stresses of the

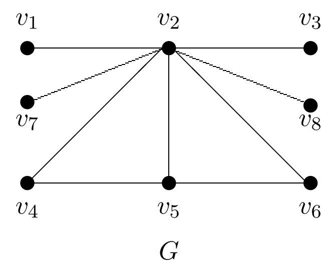

FIGURE 1. A graph G

vertex of G are as follows:
str(v_1) = str(v_3) = str(v_7) = str(v_8) = str(v_4) = str(v_6) = 0, str(v_2) = 19, str(v_5) = 1.

The neighbors stress of the vertex of G are as follows:
$N_s(v_1) = N_s(v_3) = N_s(v_5) = N_s(v_7) = N_s(v_8) = 19$,$N_s(v_2) = 1$ and $N_s(v_4) = N_s(v_6) = 20$,
Then, the neighbors stress sum matrix of G is given by

$$NSS(G) = \begin{bmatrix} 0 & 19 & 0 & 0 & 0 & 0 & 0 & 0 \\ 19 & -133 & 19 & 19 & 20 & 19 & 19 & 19 \\ 0 & 19 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 19 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 20 & 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 19 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 19 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 19 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Preliminary Results

In this section, we will document the necessary results to support our main findings in section 4.

Theorem 3.1. *Let c_i and $d_i, for\ 1 \leq i \leq n$, be non-negative real numbers. Then*

$$\sum_{i=1}^{n} c_i^2 \sum_{i=1}^{n} d_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} c_i d_i\right)^2,$$

where $M_1 = \max_{1\leq i\leq n}\{c_i\}; M_2 = \max_{1\leq i\leq n}\{d_i\}; m_1 = \min_{1\leq i\leq n}\{c_i\}$ and $m_2 = \min_{1\leq i\leq n}\{d_i\}$.

Theorem 3.2. *Let c_i and $d_i, for\ 1 \leq i \leq n$ be positive real numbers. Then*

$$\sum_{i=1}^{n} c_i^2 \sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} c_i d_i\right)^2 \leq \frac{n^2}{4}\left(M_1M_2 - m_1m_2\right)^2,$$

where $M_1 = \max_{1\leq i\leq n}\{c_i\}; M_2 = \max_{1\leq i\leq n}\{d_i\}; m_1 = \min_{1\leq i\leq n}\{c_i\}$ and $m_2 = \min_{1\leq i\leq n}\{d_i\}$.

Theorem 3.3. *(BPR Inequality) Let c_i and $d_i, for 1 \leq i \leq n$ be non-negative real numbers. Then*

$$\left|n\sum_{i=1}^{n} c_i d_i - \sum_{i=1}^{n} c_i \sum_{i=1}^{n} d_i\right| \leq \alpha(n)(A-a)(B-b),$$

where a, b, A and B are real constants, that for each $i, 1 \leq i \leq n, a \leq c_i \leq A$ and $b \leq d_i \leq B$. Further, $\alpha(n) = n\left\lceil\frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$.

Theorem 3.4. *(Diaz–Metcalf Inequality) If c_i and $d_i, 1 \leq i \leq n$, are non-negative real numbers. Then*

$$\sum_{i=1}^{n} d_i^2 + rR\sum_{i=1}^{n} c_i^2 \leq (r+R)\left(\sum_{i=1}^{n} c_i d_i\right),$$

where r and R are real constants, so that for each $i, 1 \leq i \leq n$, holds $rc_i \leq d_i \leq Rc_i$.

Theorem 3.5. *(The Cauchy-Schwarz inequality) If* $c = (c_1, c_2, \ldots, c_n)$ *and* $d = (d_1, d_2, \ldots, d_n)$ *are real* n*-vectors, then*

$$\left(\sum_{i=1}^{n} c_i d_i\right)^2 \leq \left(\sum_{i=1}^{n} c_i^2\right)\left(\sum_{i=1}^{n} d_i^2\right).$$

4. Bounds for the eigenvalues and energy

In this section, we discuss about bounds for the eigenvalues and energy.

Lemma 4.1. *For any graph* G *we have*
(i) $\sum_{v\in V(G)} N_s(v) = SS(G)$
(ii) $\sum_{v\in V(G)} str(v)N_s(v) = 2S_2(G)$.

Theorem 4.2. *Let* G *be a graph with the vertex set* $V(G)$ *and edge set* $E(G)$ *with the neighbors stress sum matrix* $NSS(G)$*. If* $P_{NSS(G)}(N_\lambda) = C_0 N_\lambda^n + C_1 N_\lambda^{n-1} + C_2 N_\lambda^{n-2} + \cdots + C_n$ *be the characteristic polynomial of* $N(G)$*, then*
(i) $C_0 = 1$
(ii) $C_1 = SS(G)$
(iii) $C_2 = \frac{1}{2}\left[(SS(G))^2 - \sum_{v\in V(G)} d(v)str(v)^2\right] - SSS(G) - 2S_2(G)$.

Proof.
(i) From the definition of $P_{NSS(G)}(\lambda)$, it follows that $C_0 = 1$.
(ii) From the definition of $P_{NSS(G)}(\lambda)$, we have

$$\begin{aligned}(-1)C_1 &= \text{ sum of all } 1\times 1 \text{ principal minors of } N(G)\\ &= \sum_{i=1}^{n} a_{ii}\\ &= -\sum_{v\in V(G)} d(v)str(v)\\ &= -\sum_{uv\in E(G)} [str(u) + str(v)]\\ &= -SS(G).\end{aligned}$$

(iii) From the definition of $P_{NSS(G)}(\lambda)$, we have

$$\begin{aligned}(-1)^2 C_2 &= \text{ sum of all } 2\times 2 \text{ principal minors of } NSS(G)\\ &= \sum_{1\leq i<j\leq n} \begin{vmatrix} a_{ii} & a_{ij}\\ a_{ji} & a_{jj}\end{vmatrix}\\ &= \sum_{1\leq i<j\leq n} a_{ii}a_{jj} - \sum_{1\leq i<j\leq n} a_{ij}^2\\ &= \frac{1}{2}\left[\left(\sum_{i=1}^{n} a_{ii}\right)^2 - \sum_{i=1}^{n} a_{ii}^2\right] - SSS(G) - 2S_2(G)\end{aligned}$$

$$\implies C_2 = \frac{1}{2}\left[(SS(G))^2 - \sum_{v\in V(G)} d(v)str(v)^2\right] - SSS(G) - 2S_2(G).$$

□

Theorem 4.3. *Let $NSS(G) = [a_{ij}]_{n\times n}$ be the neighbors stress sum matrix of a graph G and $N_{\lambda_1} \geq N_{\lambda_2} \geq \cdots \geq N_{\lambda_n}$ be its eigenvalues. Then*
(i) $\sum_{i=1}^{n} N_{\lambda_i} = -SS(G)$.
(ii) $\sum_{i=1}^{n} N_{\lambda_i^2} = S + 2SSS(G) + 4S_2(G)$, where $S = \sum_{v\in V(G)}(d(v)str(v))^2$.

Proof.
(i) It follows by the trace of matrix.

$$\sum_{i=1}^{n} N_{\lambda_i} = \mathrm{Tr}(NSS(G)) = -SS(G).$$

(ii) We have,

$$\begin{aligned}
\sum_{i=1}^{n} N_{\lambda_i^2} &= \mathrm{Tr}(NSS(G))^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 \\
&= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j} a_{ij}^2 \\
&= \sum_{v\in V(G)} (d(v)str(v))^2 + 2\sum_{uv\in E(G)} (str(u)+str(v))^2
\end{aligned}$$

$\Longrightarrow \sum_{i=1}^{n} N_{\lambda_i^2} = S + 2SSS(G) + 4S_2(G)$, where $S = \sum_{v\in V(G)}(d(v)str(v))^2$. □

Proposition 4.4. *The characteristic polynomial of the complete graph K_n on n vertices is N_λ^n and hence energy is zero.*

Theorem 4.5. *For a friendship graph F_n, $n \geq 2$ on $2n+1$ vertices, we have $P_{NSS}(F_n) = N_\lambda^{2n}\left(8n^4 - 12n^3 + 4n^2 - N_\lambda\right)$*

Proof. In F_n graph, the stress of central vertex is $2n(n-1)$ and remaining $2n$ vertices have stress 0. By the definition of neighbors stress sum, we have

$$P_{NSS}(F_n) = \begin{vmatrix} 4n^2(n-1) - N_\lambda & 2n(n-1) & 2n(n-1) & \cdots & 2n(n-1) \\ 2n(n-1) & -N_\lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 2n(n-1) & 0 & 0 & \cdots & -N_\lambda \end{vmatrix}$$

$= N_\lambda^{2n}\left(8n^4 - 12n^3 + 4n^2 - N_\lambda\right)$ □

Theorem 4.6. *Let G be any graph with n-vertices. Then*

$$E_{NSS}(G) \leq \sqrt{(S + 2SSS(G) + 4S_2(G))n}.$$

Proof. Setting $c_i = 1, d_i = N_{\lambda_i}$, for $i = 2, 3, \ldots, n$ in theorem 3.5, we have

$$\begin{aligned} \left(\sum_{i=1}^{n} |N_{\lambda_i}|\right)^2 &\leq n \sum_{i=1}^{n} N_{\lambda_i}^2 \\ \Longrightarrow \ (E_{NSS}(G))^2 &\leq n((S + 2SSS(G) + 4S_2(G))) \\ \Longrightarrow \ E_{NSS}(G) &\leq \sqrt{n(S + 2SSS(G) + 4S_2(G))}. \end{aligned}$$

□

Theorem 4.7. *If G is a graph with n vertices and $E_{NSS}(G)$ be the neighbors stress sum energy of G, then*

$$\sqrt{S + 2SSS(G) + 4S_2(G)} \leq E_{NSS}(G).$$

Proof. By the definition of $E_{NSS}(G)$, we have

$$[E_{NSS}(G)]^2 = \left(\sum_{i=1}^{n} |N_{\lambda_i}|\right)^2 \geq \sum_{i=1}^{n} |N_{\lambda_i}|^2 = S + 2SSS(G) + 4S_2(G)$$

which gives

$$\sqrt{S + 2SSS(G) + 4S_2(G)} \leq E_{NSS}(G).$$

□

Theorem 4.8. *For any graph G, $SS(G) \leq E_{NSS}(G)$*

Proof. Let $N_{\lambda_i}; i = 1, 2, \ldots, n$ be eigenvalues of G. By triangle inequality of real numbers, we have

$$\begin{aligned} &|N_{\lambda_1} + N_{\lambda_2} + \cdots + N_{\lambda_n}| \leq |N_{\lambda_1}| + |N_{\lambda_2}| + \cdots + |N_{\lambda_n}| \\ \Longrightarrow \ &|-SS(G)| \leq E_{NSS}(G) \\ \Longrightarrow \ &SS(G) \leq E_{NSS}(G). \end{aligned}$$

Equality is attained if and only if either all $\lambda_i; i = 1, 2, \ldots, n$ are of same sign or except one all eigen values are zero. □

Theorem 4.9. *Let G be any graph with n-vertices and $\triangle$ be the absolute value of the determinant of the neighbors stress sum matrix $NSS(G)$. Then*

$$\sqrt{(S + 2SSS(G) + 4S_2(G)) + n(n-1)\triangle^{2/n}} \leq E_{NSS}(G)$$

Proof. By the definition of Neighbors stress sum energy, we find that

$$\begin{aligned} (E_{NSS}(G))^2 &= \left(\sum_{i=1}^{n} |N_{\lambda_i}|\right)^2 \\ &= \sum_{i=1}^{n} |N_{\lambda_i}|^2 + 2\sum_{i<j} |N_{\lambda_i}| \left|N_{\lambda_j}\right| \\ &= (S + 2SSS(G) + 4S_2(G)) + \sum_{i \neq j} |N_{\lambda_i}| \left|N_{\lambda_j}\right|. \end{aligned}$$

Since for nonnegative number the Arithmetic mean is greater than Geometric mean,

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}|N_{\lambda_i}||N_{\lambda_j}| &\geq \left(\prod_{i\neq j}|N_{\lambda_i}||N_{\lambda_j}|\right)^{\frac{1}{n(n-1)}}\\ &= \left(\prod_{i=1}^{n}|N_{\lambda_i}|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}\\ &= \prod_{i=1}^{n}|N_{\lambda_i}|^{2/n}\\ &= \triangle^{2/n}.\end{aligned}$$

Therefore,

$$\sum_{i\neq j}|N_{\lambda_i}|\left|N_{\lambda_j}\right| \geq n(n-1)\Delta^{\frac{2}{n}}$$

$$\implies [E_{NSS}(G)]^2 \geq S + 2SSS(G) + 4S_2(G) + n(n-1)\Delta^{\frac{2}{n}}$$

$$\implies E_{NSS}(G) \geq \sqrt{S + 2SSS(G) + 4S_2(G) + n(n-1)\Delta^{\frac{2}{n}}}.$$

Equality in AM-GM inequality is attained if and only if all $N_{\lambda_i}; i = 1, 2, \ldots, n$ are equal. □

Lemma 4.10. *Let $c_1, c_2, \ldots, c_n$ be non-negative numbers. Then*

$$\begin{aligned}n\left[\frac{1}{n}\sum_{i=1}^{n}c_i - \left(\prod_{i=1}^{n}c_i\right)^{1/n}\right] &\leq n\sum_{i=1}^{n}c_i - \left(\sum_{i=1}^{n}\sqrt{c_i}\right)^2\\ &\leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n}c_i - \left(\prod_{i=1}^{n}c_i\right)^{1/n}\right].\end{aligned}$$

Theorem 4.11. *Let G be a connected graph with n vertices. Then*

$$\begin{aligned}&\sqrt{(S + 2SSS(G) + 4S_2(G)) + n(n-1)\phi^{2/n}}\\ &\qquad\leq E_{NSS}(G)\\ &\qquad\leq \sqrt{(S + 2SSS(G) + 4S_2(G))(n-1) + n\phi^{2/n}}.\end{aligned}$$

Proof. Let $c_i = |N_{\lambda_i}|^2, i = 1, 2, \ldots, n$ and

$$\begin{aligned}V &= n\left[\frac{1}{n}\sum_{i=1}^{n}|N_{\lambda_i}|^2 - \left(\prod_{i=1}^{n}|N_{\lambda_i}|^2\right)^{1/n}\right]\\ &= n\left[\frac{(S + 2SSS(G) + 4S_2(G))}{n} - \left(\prod_{i=1}^{n}|N_{\lambda_i}|\right)^{2/n}\right]\\ &= n\left[\frac{(S + 2SSS(G) + 4S_2(G))}{n} - \phi^{2/n}\right]\\ &= (S + 2SSS(G) + 4S_2(G)) - n\phi^{2/n}.\end{aligned}$$

From lemma 4.10, we obtain

$$V \leq n\sum_{i=1}^{n}|N_{\lambda_i}|^2 - \left(\sum_{i=1}^{n}|N_{\lambda_i}|\right)^2 \leq (n-1)V$$

that is

$$\begin{aligned}(S+2SSS(G)+4S_2(G)) - n\phi^{2/n} &\leq n(S+2SSS(G)+4S_2(G)) - (E_{NSS}(G))^2 \\ &\leq (n-1)\left((S+2SSS(G)+4S_2(G)) - n\phi^{2/n}\right).\end{aligned}$$

Upon Simplification of above equation, we find that

$$\begin{aligned}&\sqrt{(S+2SSS(G)+4S_2(G)) + n(n-1)\phi^{2/n}} \\ &\qquad \leq E_{NSS}(G) \\ &\qquad \leq \sqrt{(S+2SSS(G)+4S_2(G))(n-1) + n\phi^{2/n}}.\end{aligned}$$

□

Theorem 4.12. *If G is any graph with n vertices with $NSS(G) = [a_{ij}]_{n\times n}$ being its neighbors stress sum matrix and $N_{\lambda_1} \geq N_{\lambda_2} \geq \cdots \geq N_{\lambda_n}$ are its eigenvalues, then*

$$N_{\lambda_1} \leq \sqrt{\frac{(n-1)}{n}\left(S+2SSS(G)+4S_2(G) - \frac{(SS(G))^2}{n}\right)} - \frac{SS(G)}{n}$$

Proof. Taking $c_i = N_{\lambda_i}$ and $d_i = 1$ for $i = 2, 3, \ldots, n$ in Theorem 3.5, we have

$$\left(\sum_{i=2}^{p} N_{\lambda_i}\right)^2 \leq (n-1)\left(\sum_{i=2}^{p} N_{\lambda_i}\right).$$

From Theorem 4.3, we have

$$\sum_{i=2}^{n} N_{\lambda_i} = -SS(G) - N_{\lambda_1}, \quad \text{and} \quad \sum_{i=2}^{n} N_{\lambda_i}^2 = S+2SSS(G)+4S_2(G) - N_{\lambda_1}^2.$$

Then

$$\begin{aligned}&(-SS(G) - N_{\lambda_1})^2 \leq (n-1)\left(S+2SSS(G)+4S_2(G) - N_{\lambda_1}^2\right) \\ \Longrightarrow\quad & SS(G)^2 + N_{\lambda_1}^2 + 2N_{\lambda_1}SS(G) \leq (n-1)(S+2SSS(G)+4S_2(G)) - nN_{\lambda_1}^2 + N_{\lambda_1}^2 \\ \Longrightarrow\quad & SS(G)^2 + 2N_{\lambda_1}SS(G) + nN_{\lambda_1}^2 \leq (n-1)(S+2SSS(G)+4S_2(G)) - (SS(G))^2 \\ \Longrightarrow\quad & \left(\sqrt{n}N_{\lambda_1} + \frac{SS(G)}{\sqrt{n}}\right)^2 \leq (n-1)(S+2SSS(G)+4S_2(G)) - \frac{(n-1)}{n}SS(G)^2 \\ \Longrightarrow\quad & \sqrt{n}N_{\lambda_1} + \frac{SS(G)}{\sqrt{n}} \leq \sqrt{(n-1)\left(S+2SSS(G)+4S_2(G) - \frac{SS(G)^2}{n}\right)} \\ \Longrightarrow\quad & N_{\lambda_1} \leq \sqrt{\frac{(n-1)}{n}\left(S+2SSS(G)+4S_2(G) - \frac{SS(G)^2}{n}\right)} - \frac{SS(G)}{n}.\end{aligned}$$

□

Theorem 4.13. *Let G be a graph of order n. Then*

$$E_{NSS}(G) \geq \sqrt{(S + 2SSS(G) + 4S_2(G))n - \frac{n^2}{4}\left(N_{\lambda_1} - N_{\lambda\min}\right)^2},$$

where $N_{\lambda_1} = N_{\lambda\max} = \max_{1\leq i\leq n}|N_{\lambda_i}|$ and $N_{\lambda\min} = \min_{1\leq i\leq n}|N_{\lambda_i}|$.

Proof. Suppose $N_{\lambda_1}, N_{\lambda_2}, \ldots, N_{\lambda_n}$ are the eigenvalues of $NSS(G)$. Setting $c_i = 1$ and $d_i = |N_{\lambda_i}|$ in theorem 3.2 , we have

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |N_{\lambda_i}|^2 - \left(\sum_{i=1}^{n} |N_{\lambda_i}|\right)^2 \leq \frac{n^2}{4}\left(N_{\lambda_1} - N_{\lambda\min}\right)^2$$

$$\implies (S + 2SSS(G) + 4S_2(G))n - (E_{NSS}(G))^2 \leq \frac{n^2}{4}\left(N_{\lambda_1} - N_{\lambda\min}\right)^2$$

$$\implies E_{NSS}(G) \geq \sqrt{(S + 2SSS(G) + 4S_2(G))n - \frac{n^2}{4}\left(N_{\lambda_1} - N_{\lambda\min}\right)^2}.$$

□

Theorem 4.14. *Suppose zero is not an eigenvalue of $NSS(G)$, then*

$$E_{NSS}(G) \geq \frac{2\sqrt{N_{\lambda_1}N_{\lambda\min}}\sqrt{(S + 2SSS(G) + 4S_2(G))n}}{N_{\lambda_1} + N_{\lambda\min}}.$$

where $N_{\lambda_1} = N_{\lambda\max} = \max_{1\leq i\leq n}|N_{\lambda_i}|$ and $N_{\lambda\min} = \min_{1\leq i\leq n}|N_{\lambda_i}|$.

Proof. Suppose $N_{\lambda_1}, N_{\lambda_2}, \ldots, N_{\lambda_n}$ are the eigenvalues of $NSS(G)$. Setting $c_i = |N_{\lambda_i}|$ and $d_i = 1$ in theorem 3.1 , we have

$$\sum_{i=1}^{n} |N_{\lambda_i}|^2 \sum_{i=1}^{n} 1^2 \leq \frac{1}{4}\left(\sqrt{\frac{N_{\lambda_1}}{N_{\lambda\min}}} + \sqrt{\frac{N_{\lambda\min}}{N_{\lambda_1}}}\right)^2 \left(\sum_{i=1}^{n} |N_{\lambda_i}|\right)^2$$

$$\implies (S + 2SSS(G) + 4S_2(G))n \leq \frac{1}{4}\left(\frac{(N_{\lambda_1} + N_{\lambda\min})^2}{N_{\lambda_1}N_{\lambda\min}}\right)(E_{NSS}(G))^2$$

$$\implies E_{NSS}(G) \geq \frac{2\sqrt{N_{\lambda_1}N_{\lambda\min}}\sqrt{(S + 2SSS(G) + 4S_2(G))n}}{N_{\lambda_1} + N_{\lambda\min}}.$$

□

Theorem 4.15. *Let G be a graph of order n. Let $N_{\lambda_1} \geq N_{\lambda_2} \geq \ldots \geq N_{\lambda_n}$ be the non zero eigenvalues of $NSS(G)$. Then*

$$E_{NSS}(G) \geq \frac{(S + 2SSS(G) + 4S_2(G)) + nN_{\lambda_1}N_{\lambda\min}}{N_{\lambda_1} + N_{\lambda\min}},$$

where $N_{\lambda_1} = N_{\lambda\max} = \max_{1\leq i\leq n}|N_{\lambda_i}|$ and $N_{\lambda\min} = \min_{1\leq i\leq n}|N_{\lambda_i}|$.

Proof. Assigning $d_i = |N_{\lambda_i}|, c_i = 1, R = |N_{\lambda_1}|$ and $r = |N_{\lambda\min}|$. Then by theorem 3.4, we get

$$\sum_{i=1}^{n} |N_{\lambda_i}|^2 + N_{\lambda_1}N_{\lambda\min}\sum_{i=1}^{n} 1^2 \leq (N_{\lambda_1} + N_{\lambda\min})\sum_{i=1}^{n} |N_{\lambda_i}|$$

$$\implies (S + 2SSS(G) + 4S_2(G)) + nN_{\lambda_1}N_{\lambda\min} \leq (N_{\lambda_1} + N_{\lambda\min})\, E_{NSS}(G)$$

After simplifying the expression and using the definition of $E_{NSS}(G)$, we obtain

$$E_{NSS}(G) \geq \frac{(S+2SSS(G)+4S_2(G))+nN_{\lambda_1}N_{\lambda\min}}{N_{\lambda_1}+N_{\lambda\min}}.$$

□

Theorem 4.16. *Let G be a graph of order n. Let $N_{\lambda_1} \geq N_{\lambda_2} \geq \ldots \geq N_{\lambda_n}$ be the eigenvalues of $NSS(G)$. Then*

$$E_{NSS}(G) \geq \sqrt{(S+2SSS(G)+4S_2(G))n-\alpha(n)\left(N_{\lambda_1}-N_{\lambda\min}\right)^2},$$

where $N_{\lambda_1} = N_{\lambda\max} = \max_{1\leq i\leq n}|N_{\lambda_i}|$ and $N_{\lambda\min} = \min_{1\leq i\leq n}|N_{\lambda_i}|$ and $\alpha(n) = n\left\lceil \frac{n}{2}\right\rceil\left(1-\frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)$.

Proof. Setting $c_i = |N_{\lambda_i}| = d_i, A \leq |N_{\lambda_i}| \leq B$ and $a \leq |N_{\lambda_n}| \leq b$, then by theorem 3.3 we get

$$\left|n\sum_{i=1}^{n}|N_{\lambda_i}|^2-\left(\sum_{i=1}^{n}|N_{\lambda_i}|\right)^2\right| \leq \alpha(n)(N_{\lambda_1}-N_{\lambda\min})^2$$

$$\Longrightarrow \left|(S+2SSS(G)+4S_2(G))n-(E_{NSS}(G))^2\right| \leq \alpha(n)\left(N_{\lambda_1}-N_{\lambda\min}\right)^2$$

$$\Longrightarrow E_{NSS}(G) \geq \sqrt{(S+2SSS(G)+4S_2(G))n-\alpha(n)\left(N_{\lambda_1}-N_{\lambda\min}\right)^2}.$$

□

Lemma 4.17. *(Abel's inequality) Let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be real numbers such that $d_n \geq d_{n+1} \geq 0$ for all n, then*

$$|c_1d_1+c_2d_2+\cdots+c_nd_n| \leq Ad_1,$$

where

$$A = \max\left\{|c_1|, |c_1+c_2|, \ldots, |c_1+c_2+\cdots+c_n|\right\}.$$

Theorem 4.18. *Let G be any graph with $n \geq 2$ vertices. Then*

$$E_{NSS(G)} \geq \frac{S+2SSS(G)+4S_2(G)}{|N_{\lambda_1}|},$$

where $|N_{\lambda_1}| \geq |N_{\lambda_2}| \geq \ldots \geq |N_{\lambda_n}|$ be the sequence of modulus value of eigenvalues of $NSS(G)$. Equality holds if and only if $|N_{\lambda_1}| = |N_{\lambda_2}| = \cdots = |N_{\lambda_n}|$ or $|N_{\lambda_1}| = |N_{\lambda_l}|, 2 \leq l \leq n$ and $|N_{\lambda_k}| = 0$ where $k \neq l, 2 \leq k \leq n$.

Proof. Consider $c_i = |N_{\lambda_i}|$ and $d_i = |N_{\lambda_i}|$ for all $1 \leq i \leq n$. Clearly, $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$. Then by lemma 4.17, we have

$$||N_{\lambda_1}|^2+|N_{\lambda_2}|^2+\cdots+|N_{\lambda_n}|^2| \leq E_{NSS}(G)|\,|N_{\lambda_1}|.$$

Since,

$$A = \max\left\{|N_{\lambda_1}|, ||N_{\lambda_1}|+|N_{\lambda_2}||, \ldots, ||N_{\lambda_1}|+|N_{\lambda_2}|+\cdots+|N_{\lambda_n}||\right\} = E_{NSS}(G)$$

Further,

$$\mathrm{Tr}\left(NSS(G)^2\right) \leq E_{NSS}(G)\,|N_{\lambda_1}|$$

$$\Longrightarrow \frac{\mathrm{Tr}\left(NSSS(G)^2\right)}{|N_{\lambda_1}|} \leq E_{NSS}(G).$$

From theorem 4.3, the above inequality becomes

$$E_{NSS(G)} \geq \frac{S + 2SSS(G) + 4S_2(G)}{|N_{\lambda_1}|}$$

Further,

$$\begin{aligned} E_{NSS}(G) =& \frac{\operatorname{Tr}\left(NSS(G)^2\right)}{|N_{\lambda_1}|} \\ & \iff |N_{\lambda_1}|\left(|N_{\lambda_1}| + |N_{\lambda_2}| + \cdots + |N_{\lambda_n}|\right) = \sum_{i=1}^{n} |N_{\lambda_i}|^2 \\ & \iff |N_{\lambda_1}|\left(|N_{\lambda_2}| + \cdots + |N_{\lambda_n}|\right) = \sum_{i=2}^{n} |N_{\lambda_i}|^2 . \end{aligned}$$

This is possible if and only if

$$\begin{aligned} |N_{\lambda_1}| = |N_{\lambda_2}| = \cdots = |N_{\lambda_n}| \text{ or } |N_{\lambda_1}| = |N_{\lambda_l}|, 2 \leq l \leq n \\ \text{and } |N_{\lambda_k}| = 0 \text{ where } k \neq l, 2 \leq k \leq n. \end{aligned}$$

□

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