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REPRESENTATIONS BY DEGENERATE BERNOULLI POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. In this paper, we consider the problem of representing any polynomial in terms of the degenerate Bernoulli polynomials of the second kind and more generally of the higher-order degenerate Bernoulli polynomials of the second kind. We derive explicit formulas with the help of umbral calculus and illustrate our results with some examples.

1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to derive formulas (see Theorem 3.1) expressing any polynomial in terms of the degenerate Bernoulli polynomials of the second (see (1.11)) with the help of umbral calculus and to illustrate our results with some examples (see Chapter 6). This can be generalized to the higher-order degenerate Bernoulli polynomials (see (1.12)). Indeed, we deduce formulas (see Theorems 4.2) of representing any polynomial in terms of the higher-order degenerate Bernoulli polynomials again by using umbral calculus. Letting $\lambda \to 0$, we obtain formulas of expressing any polynomial in terms of the Bernoulli polynomials of the second and of the higher-order Bernoulli polynomials of the second kind. These formulas are also illustrated in Chapters 5. The contribution of this paper is the derivation of such formulas which, we think, have many potential applications.

Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Write $p(x) = \sum_{k=0}^{n} a_k B_k(x)$, where $B_n(x)$ are the Bernoulli polynomials (see (1.3)). Then it is known (see [11]) that

$$a_k = \frac{1}{k!}\int_0^1 p^{(k)}(x)dx, \text{ for } k = 0, 1, \dots, n. \tag{1.1}$$

We can obtain the following identity (see [11,17]) by applying the formula in (1.1) to the polynomial $p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x)$ and after slight modification:

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{1}{2k\,(2n-2k)} B_{2k}\,(x)\, B_{2n-2k}\,(x) + \frac{2}{2n-1} B_1\,(x)\, B_{2n-1}\,(x) \\ =&\frac{1}{n}\sum_{k=1}^{n} \frac{1}{2k}\binom{2n}{2k} B_{2k} B_{2n-2k}\,(x) + \frac{1}{n} H_{2n-1} B_{2n}\,(x) + \frac{2}{2n-1} B_1\,(x)\, B_{2n-1}, \end{aligned} \tag{1.2}$$

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where $n \geq 2$, and $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Letting $x = 0$ and $x = \frac{1}{2}$ in (1.2) respectively give a slight variant of the Miki's identity and the Faber-Pandharipande-Zagier (FPZ) identity. Here it should be emphasized that the other proofs of Miki's (see [6,20,23]) and FPZ identities (see [4,5]) are quite involved, while our proofs of Miki's and FPZ identities follow from the simple formula in (1.1) involving only derivatives and integrals of the given polynomials.

Analogous formulas to (1.1) can be obtained for the representations by Euler, Frobenius-Euler, ordered Bell and Genocchi polynomials. Many interesting identities have been derived by using these formulas (see [7-12] and references therein). The list in the References are far from being exhaustive. However, the interested reader can easily find more related papers in the literature. Also, we should mention here that there are other ways of obtaining the same result as the one in (1.2). One of them is to use Fourier series expansion of the function obtained by extending by periodicity 1 of the polynomial function restricted to the interval $[0, 1)$ (see [15-23]).

The outline of this paper is as follows. In Section 1, we recall some necessary facts that are needed throughout this paper. In Section 2, we go over umbral calculus briefly. In Section 3, we derive formulas expressing any polynomial in terms of the degenerate Bernoulli polynomials of the second. In Section 4, we derive formulas representing any polynomial in terms of the higher-order degenerate Bernoulli polynomials of the second kind. In Section 5, we illustrate our results with some examples on representations by Bernoulli polynomials of the second kind. In Section 6, we illustrate our results with some examples on representations by degenerate Bernoulli polynomials of the second kind. Finally, we conclude our paper in Section 7.

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.3}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. We observe that $B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} x^j$, $\frac{d}{dx} B_n(x) = n B_{n-1}(x)$. The first few terms of B_n are given by:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},$$
$$B_{12} = -\frac{691}{2730}, \ldots; B_{2k+1} = 0,\ (k \geq 1).$$

More generally, for any nonnegative integer r, the Bernoulli polynomials $B_n^{(r)}(x)$ of order r are given by

$$\left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}. \tag{1.4}$$

The Euler polynomials $E_n(x)$ are defined by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.5}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers. We observe that $E_n(x) = \sum_{j=0}^{n} \binom{n}{j} E_{n-j}x^j$, $\frac{d}{dx}E_n(x) = nE_{n-1}(x)$. The first few terms of E_n are given by:

$$E_0 = 1,\ E_1 = -\frac{1}{2},\ E_3 = \frac{1}{4},\ E_5 = -\frac{1}{2},\ E_7 = \frac{17}{8},\ E_9 = -\frac{31}{2}, \dots;$$
$$E_{2k} = 0,\ \ (k \geq 1).$$

The Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x = 0$, $G_n = G_n(0)$ are called the Genocchi numbers. We observe that $G_n(x) = \sum_{j=0}^{n} \binom{n}{j} G_{n-j}x^j$, $\frac{d}{dx}G_n(x) = nG_{n-1}(x)$, and $\deg G_n(x) = n - 1$, for $n \geq 1$. The first few terms of G_n are given by:

$$G_0 = 0,\ G_1 = 1,\ G_2 = -1,\ G_4 = 1,\ G_6 = -3,\ G_8 = 17,\ G_{10} = -155$$
$$G_{12} = 2073, \dots; G_{2k+1} = 0,\ \ (k \geq 1).$$

For any nonzero real number λ, the degenerate exponentials are given by

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!}, \tag{1.7}$$
$$e_\lambda(t) = e_\lambda^1(t) = (1 + \lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} (1)_{n,\lambda}\frac{t^n}{n!}.$$

The compositional inverse of $e_\lambda(t)$ is called the degenerate logarithm and given by

$$\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1),\ \ (\text{see [13]}), \tag{1.8}$$

which satisfies $e_\lambda(\log_\lambda(t)) = \log_\lambda(e_\lambda(t)) = t$.

Note here that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$, $\quad \lim_{\lambda\to 0} \log_\lambda(t) = \log(t)$.

Recall that the Bernoulli polynomials of the second $b_n(x)$ are given by

$$\frac{t}{\log(1 + t)}(1 + t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{1.9}$$

When $x = 0$, $b_n = b_n(0)$ are the Bernoulli numbers of the second kind.

More generally, for any nonnegative integer r, the Bernoulli polynomials of the second $b_n^{(r)}(x)$ of order r are given by

$$\left(\frac{t}{\log(1 + t)}\right)^r (1 + t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.10}$$

When $x = 0$, $b_n^{(r)} = b_n^{(r)}(0)$ are the Bernoulli numbers of the second kind of order r.

The degenerate Bernoulli polynomials of the second kind $b_{n,\lambda}(x)$ are defined by

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.11}$$

which are degenerate versions of the Bernoulli polynomials of the second kind in (1.9), For $x = 0$, $b_{n,\lambda} = b_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers of the second kind and introduced in [14].

More generally, for any nonnegative integer r, the degenerate Bernoulli polynomials of the second $b_{n,\lambda}^{(r)}(x)$ of order r are defined in [13] by

$$\left(\frac{t}{\log_\lambda(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{1.12}$$

We remark that $b_{n,\lambda}(x) \to b_n(x)$, and $b_{n,\lambda}^{(r)}(x) \to b_n^{(r)}(x)$, as λ tends to 0.

We recall some notations and facts about forward differences. Let f be any complex-valued function of the real variable x. Then, for any real number a, the forward difference Δ_a is given by

$$\Delta_a f(x) = f(x+a) - f(x). \tag{1.13}$$

If $a = 1$, then we let

$$\Delta f(x) = \Delta_1 f(x) = f(x+1) - f(x). \tag{1.14}$$

In general, the nth oder forward differences are given by

$$\Delta_a^n f(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(x+ia). \tag{1.15}$$

For $a = 1$, we have

$$\Delta^n f(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(x+i). \tag{1.16}$$

Finally, we recall that the Stirling numbers of the second kind $S_2(n,k)$ can be given by means of

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{1.17}$$

2. Review of umbral calculus

Here we will briefly go over very basic facts about umbral calculus. For more details on this, we recommend the reader to refer to [3,22,24]. Let $\mathbb{C}$ be the field of complex numbers. Then $\mathcal{F}$ denotes the algebra of formal power series in t over $\mathbb{C}$, given by

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \;\middle|\; a_k \in \mathbb{C} \right\},$$

and $\mathbb{P} = \mathbb{C}[x]$ indicates the algebra of polynomials in x with coefficients in $\mathbb{C}$.

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. If $\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, then the vector space operations on $\mathbb{P}^*$ are defined by

$$\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \quad \langle cL|p(x)\rangle = c\langle L|p(x)\rangle,$$

where c is a complex number.

For $f(t) \in \mathcal{F}$ with $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$, we define the linear functional on $\mathbb{P}$ by

$$\langle f(t)|x^k\rangle = a_k. \tag{2.1}$$

From (2.1), we note that

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \quad (n, k \geq 0),$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Some remarkable linear functionals are as follows:

$$\begin{aligned} &\langle e^{yt}|p(x)\rangle = p(y), \\ &\langle e^{yt}-1|p(x)\rangle = p(y)-p(0), \\ &\left\langle \frac{e^{yt}-1}{t}\middle| p(x)\right\rangle = \int_0^y p(u)du. \end{aligned} \tag{2.2}$$

Let

$$f_L(t) = \sum_{k=0}^{\infty} \langle L|x^k\rangle \frac{t^k}{k!}. \tag{2.3}$$

Then, by (2.1) and (2.3), we get

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle.$$

That is, $f_L(t) = L$. Additionally, the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$.

Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$. $\mathcal{F}$ is called the umbral algebra and the umbral calculus is the study of umbral algebra. For each nonnegative integer k, the differential operator t^k on $\mathbb{P}$ is defined by

$$t^k x^n = \begin{cases} (n)_k x^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases} \tag{2.4}$$

Extending (2.4) linearly, any power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F}$$

gives the differential operator on $\mathbb{P}$ defined by

$$f(t)x^n = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}, \quad (n \geq 0). \tag{2.5}$$

It should be observed that, for any formal power series $f(t)$ and any polynomial $p(x)$, we have

$$\langle f(t)|p(x)\rangle = \langle 1|f(t)p(x)\rangle = f(t)p(x)|_{x=0}. \tag{2.6}$$

Here we note that an element $f(t)$ of $\mathcal{F}$ is a formal power series, a linear functional and a differential operator. Some notable differential operators are as follows:

$$\begin{aligned} &e^{yt}p(x) = p(x+y), \\ &(e^{yt}-1)p(x) = p(x+y) - p(x), \\ &\frac{e^{yt}-1}{t}p(x) = \int_x^{x+y} p(u)du. \end{aligned} \tag{2.7}$$

The order $o(f(t))$ of the power series $f(t)(\neq 0)$ is the smallest integer for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series. If $o(f(t)) = 1$, then $f(t)$ is called a delta series.

For $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) of polynomials such that

$$\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}, \quad (n, k \geq 0). \tag{2.8}$$

The sequence $s_n(x)$ is said to be the Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$. We observe from (2.8) that

$$s_n(x) = \frac{1}{g(t)}p_n(x), \tag{2.9}$$

where $p_n(x) = g(t)s_n(x) \sim (1, f(t))$.

In particular, if $s_n(x) \sim (g(t), t)$, then $p_n(x) = x^n$, and hence

$$s_n(x) = \frac{1}{g(t)}x^n. \tag{2.10}$$

It is well known that $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)}e^{x\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!}t^k, \tag{2.11}$$

for all $x \in \mathbb{C}$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

The following equations (2.12), (2.13), and (2.14) are equivalent to the fact that $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$:

$$f(t)\, s_n(x) = n s_{n-1}(x), \quad (n \geq 0), \tag{2.12}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x)\, p_{n-j}(y), \tag{2.13}$$

with $p_n(x) = g(t)\, s_n(x)$,

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!}\langle g\left(\overline{f}(t)\right)^{-1} \overline{f}(t)^j \big| x^n\rangle x^j. \tag{2.14}$$

Let $p_n(x)$, $q_n(x)=\sum_{k=0}^{n} q_{n,k}x^k$ be sequences of polynomials. Then the umbral composition of $q_n(x)$ with $p_n(x)$ is defined to be the sequence

$$q_n(\mathbf{p}(x))=\sum_{k=0}^{n} q_{n,k}p_k(x). \tag{2.15}$$

3. Representation by degenerate Bernoulli polynomials of the second

Our interest here is to derive formulas expressing any polynomial in terms of the degenerate Bernoulli polynomials of the second kind (see (1.11)). From these formulas, by letting $\lambda \to 0$, we get formulas representing any polynomial in terms of the Bernoulli polynomials of the second kind (see (1.9)).

From (1.11) and (1.7), we first observe that

$$b_{n,\lambda}(x) \sim \big(g(t)=\frac{e^{\lambda t}-1}{\lambda f(t)}=\frac{e^{\lambda t}-1}{\lambda(e^t-1)}, f(t)=e^t-1\big), \tag{3.1}$$

$$(x)_n \sim (1, f(t)=e^t-1). \tag{3.2}$$

From (1.14), (2.7), (2.8), (2.12), (3.1) and (3.2), we note that

$$f(t)b_{n,\lambda}(x)=nb_{n-1,\lambda}(x)=(e^t-1)b_{n,\lambda}(x)=\Delta b_{n,\lambda}(x), \tag{3.3}$$

$$f(t)(x)_n=n(x)_{n-1}, \tag{3.4}$$

$$g(t)b_{n,\lambda}(x)=(x)_n. \tag{3.5}$$

It is immediate to see from (1.11) and (1.13) that

$$\Delta_\lambda b_{n,\lambda}(x)=b_{n,\lambda}(x+\lambda)-b_{n,\lambda}(x)=\lambda n(x)_{n-1}, \tag{3.6}$$

$$b_{n,\lambda}(\lambda)-b_{n,\lambda}=\lambda\delta_{n,1}, \tag{3.7}$$

where $\delta_{n,1}$ is the Kronecker's delta.

Now, we assume that $p(x)\in\mathbb{C}[x]$ has degree n, and write $p(x)=\sum_{k=0}^{n} a_k b_{k,\lambda}(x)$. Let $h(x)=p(x+\lambda)-p(x)=\Delta_\lambda p(x)$. Then, from (3.4) and (3.6), we have

$$\begin{aligned} h(x)&=\sum_{l=0}^{n} a_l(b_{l,\lambda}(x+\lambda)-b_{l,\lambda}(x))\\ &=\lambda\sum_{l=0}^{n} a_l l(x)_{l-1}\\ &=\lambda f(t)\sum_{l=0}^{n} a_l(x)_l. \end{aligned} \tag{3.8}$$

For $k\geq 1$, from (3.8) and (3.4) we obtain

$$\begin{aligned} (f(t))^{k-1}h(x)&=\lambda(f(t))^k\sum_{l=0}^{n} a_l(x)_l\\ &=\lambda\sum_{l=k}^{n} l(l-1)\cdots(l-k+1)a_l(x)_{l-k}. \end{aligned} \tag{3.9}$$

Letting $x=0$ in (3.9), we finally get

$$a_k=\frac{1}{\lambda k!}(f(t))^{k-1}h(x)|_{x=0}=\frac{1}{\lambda k!}\langle (f(t))^{k-1}|h(x)\rangle,\ (k\geq 1), \tag{3.10}$$

An alternative expression of (3.10) is given by
As $f(t)h(x)=(e^t-1)h(x)=\Delta h(x)$,

$$\begin{aligned} a_k &= \frac{1}{\lambda k!}\Delta^{k-1}h(x)|_{x=0} \\ &= \frac{1}{\lambda k!}(\Delta^{k-1}p(\lambda)-\Delta^{k-1}p(0)) \\ &= \frac{1}{\lambda k!}\Delta^{k-1}\Delta_\lambda p(0). \end{aligned} \tag{3.11}$$

From (1.16), we have another alternative expression of (3.10) which is given by

$$\begin{aligned} a_k &= \frac{1}{\lambda k!}\Delta^{k-1}h(x)|_{x=0} \\ &= \frac{1}{\lambda k!}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}h(x+l)|_{x=0} \\ &= \frac{1}{\lambda k!}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}(p(\lambda+l)-p(l)). \end{aligned} \tag{3.12}$$

By using (1.17), we obtain yet another expression of (3.10) that is given by

$$\begin{aligned} a_k &= \frac{1}{\lambda k!}\langle (f(t))^{k-1}|h(x)\rangle \\ &= \frac{1}{\lambda k}\left\langle \frac{1}{(k-1)!}(e^t-1)^{k-1}\middle| h(x)\right\rangle \\ &= \frac{1}{\lambda k}\left\langle \sum_{l=k-1}^{\infty}S_2(l,k-1)\frac{t^l}{l!}\middle| h(x)\right\rangle \\ &= \frac{1}{\lambda k}\sum_{l=k-1}^{n}S_2(l,k-1)\frac{1}{l!}\big(p^{(l)}(\lambda)-p^{(l)}(0)\big), \end{aligned} \tag{3.13}$$

where $p^{(l)}(x)=(\frac{d}{dx})^l p(x)$.
By making use of (2.7), we have still another expression of (3.10) which is given by

$$\begin{aligned} a_k &= \frac{1}{\lambda k!}(f(t))^{k-1}h(x)|_{x=0} \\ &= \frac{1}{\lambda k!}\left(\frac{e^t-1}{t}\right)^{k-1}t^{k-1}h(x)|_{x=0} \\ &= \frac{1}{\lambda k!}I^{k-1}h^{(k-1)}(x)|_{x=0}, \end{aligned} \tag{3.14}$$

where I is the linear integral operator given by $q(x)\to\int_x^{x+1}q(u)du$.

Now, it remains to determine a_0. We first note from (2.2), (2.6), (3.3) and (3.6) that

$$
\begin{aligned}
g(t)p(x)|_{x=0} &= \langle g(t)|p(x)\rangle \\
&= \left\langle \frac{e^{\lambda t}-1}{\lambda f(t)} \middle| p(x) \right\rangle \\
&= \sum_{k=0}^{n} a_k \left\langle \frac{e^{\lambda t}-1}{\lambda f(t)} \middle| b_{k,\lambda}(x) \right\rangle \\
&= \sum_{k=0}^{n} a_k \left\langle \frac{e^{\lambda t}-1}{\lambda f(t)} \middle| f(t)\frac{1}{k+1} b_{k+1,\lambda}(x) \right\rangle \\
&= \frac{1}{\lambda}\sum_{k=0}^{n} \frac{a_k}{k+1} \left\langle e^{\lambda t}-1 \middle| b_{k+1,\lambda}(x) \right\rangle \\
&= \sum_{k=0}^{n} \frac{a_k}{k+1}\delta_{k+1,1} = a_0.
\end{aligned} \tag{3.15}
$$

We want to find more explicit expression for (3.15). To proceed further, we let $p(x) = \sum_{i=0}^{n} b_i x^i$. From (2.2), (2.15) and (3.15), noting that $g(t) = \frac{e^{\lambda t}-1}{\lambda t}\frac{t}{e^t-1}$, we have

$$
\begin{aligned}
a_0 &= \frac{1}{\lambda}\left\langle \frac{e^{\lambda t}-1}{t} \middle| \frac{t}{e^t-1} p(x) \right\rangle \\
&= \frac{1}{\lambda}\left\langle \frac{e^{\lambda t}-1}{t} \middle| \sum_{i=0}^{n} b_i B_i(x) \right\rangle \\
&= \frac{1}{\lambda}\left\langle \frac{e^{\lambda t}-1}{t} \middle| p\big(\mathbf{B}(x)\big) \right\rangle \\
&= \frac{1}{\lambda}\int_0^{\lambda} p\big(\mathbf{B}(u)\big)\, du,
\end{aligned} \tag{3.16}
$$

where $p\big(\mathbf{B}(x)\big)$ denotes the umbral composition of $p(x)$ with $B_i(x)$, that is, it is given by $p\big(\mathbf{B}(x)\big) = \sum_{i=0}^{n} b_i B_i(x)$.

We would like to find yet another expression for a_0. For this, we note that $\frac{t}{e^t-1}p(x) = \sum_{l=0}^{\infty} B_l \frac{t^l}{l!} p(x) = \sum_{l=0}^{n} \frac{B_l}{l!} p^{(l)}(x)$. Thus from (3.15) we have

$$
\begin{aligned}
a_0 = g(t)p(x)|_{x=0} &= \sum_{l=0}^{n} \frac{B_l}{l!\lambda}\frac{e^{\lambda t}-1}{t} p^{(l)}(x)\big|_{x=0} = \sum_{l=0}^{n} \frac{B_l}{l!\lambda}\int_0^{\lambda} p^{(l)}(u)du \\
&= \frac{1}{\lambda}\int_0^{\lambda} p(u)du + \sum_{l=1}^{n} \frac{B_l}{l!\lambda}\big(p^{(l-1)}(\lambda) - p^{(l-1)}(0)\big).
\end{aligned} \tag{3.17}
$$

We still want to find another expression of a_k, which is valid for all $0 \le k \le n$. From (2.8) and (3.1), we have

$$\begin{aligned} a_k &= \frac{1}{k!}\langle g(t)f(t)^k|p(x)\rangle = \frac{1}{k!}g(t)\Delta^k p(x)|_{x=0} \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}g(t)p(x+l)|_{x=0} \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt}g(t)p(x)|_{x=0} \\ &= \frac{1}{k!\lambda}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_l^{l+\lambda}p\big(\mathbf{B}(u)\big)du, \quad (0 \le k \le n). \end{aligned} \tag{3.18}$$

Finally, from (3.10)–(3.18), we get the following theorem.

Theorem 3.1. *Let $p(x) \in \mathbb{C}[x], \deg p(x) = n$. Then we have $p(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}(x)$, where*

$$\begin{aligned} a_k &= \frac{1}{k!\lambda}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_l^{l+\lambda}p\big(\mathbf{B}(u)\big)du, \quad (0 \le k \le n), \\ a_0 &= g(t)p(x)|_{x=0} = \frac{1}{\lambda}\int_0^{\lambda}p\big(\mathbf{B}(u)\big)\,du \\ &= \frac{1}{\lambda}\int_0^{\lambda}p(u)du + \sum_{l=1}^{n}\frac{B_l}{l!\lambda}\big(p^{(l-1)}(\lambda) - p^{(l-1)}(0)\big), \\ a_k &= \frac{1}{\lambda k!}(f(t))^{k-1}(p(x+\lambda) - p(x))|_{x=0} \\ &= \frac{1}{\lambda k!}\langle (e^t-1)^{k-1}|p(x+\lambda) - p(x)\rangle \\ &= \frac{1}{\lambda k!}\Delta^{k-1}\Delta_\lambda p(0) \\ &= \frac{1}{\lambda k!}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}(p(\lambda+l) - p(l)) \\ &= \frac{1}{\lambda k}\sum_{l=k-1}^{n}S_2(l,k-1)\frac{1}{l!}(p^{(l)}(\lambda) - p^{(l)}(0)) \\ &= \frac{1}{\lambda k!}I^{k-1}(p^{(k-1)}(x+\lambda) - p^{(k-1)}(x))|_{x=0}, \quad (1 \le k \le n), \end{aligned}$$

where $g(t) = \frac{e^{\lambda t}-1}{\lambda(e^t-1)}$, $f(t) = e^t - 1$, I is the linear integral operator given by $q(x) \to \int_x^{x+1} q(u)du$, and $p\big(\mathbf{B}(x)\big)$ denotes the umbral composition of $p(x)$ with $B_i(x)$.

Remark 3.2. *Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Write $p(x) = \sum_{k=0}^{n} a_k b_k(x)$. As λ tends to 0, $g(t) \to \frac{t}{e^t-1}$. Thus we obtain the following result:*

$$
\begin{aligned}
a_k &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}p\big(\mathbf{B}(l)\big), \quad (0 \le k \le n),\\
a_0 &= p\big(\mathbf{B}\big),\\
a_k &= \frac{1}{k!}\Delta^{k-1}p'(x)|_{x=0} = \frac{1}{k!}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}p'(l)\\
&= \frac{1}{k!}\sum_{l=k-1}^{n-1}\frac{1}{l!}S_2(l,k-1)p^{(l+1)}(0), \quad (1 \le k \le n),
\end{aligned}
$$

where $p\big(\mathbf{B}(l)\big)$ and $p\big(\mathbf{B}\big)$ denote respectively the umbral composition of $p(x)$ with $B_i(l)$ and that with B_i.

4. Representation by higher-order degenerate Bernoulli polynomials of the second kind

Our concern here is to derive formulas expressing any polynomial in terms of the higher-order degenerate Bernoulli polynomials of the second kind (see (1.12)). Also, letting $\lambda \to 0$ gives formulas representing any polynomial in terms of the higher order Bernoulli polynomials of the second kind (see (1.10)).

With $g(t) = \frac{e^{\lambda t}-1}{\lambda f(t)} = \frac{e^{\lambda t}-1}{\lambda(e^t-1)}$, $f(t) = e^t - 1$, from (1.12) we note that

$$b_{n,\lambda}^{(r)}(x) \sim (g(t)^r, f(t)). \tag{4.1}$$

Also, from (2.12) and (4.1) we have

$$f(t)b_{n,\lambda}^{(r)}(x) = nb_{n-1,\lambda}^{(r)}(x) = (e^t-1)b_{n,\lambda}^{(r)}(x) = \Delta b_{n,\lambda}^{(r)}(x), \tag{4.2}$$

and from (1.12), it is immediate to see that

$$
\begin{aligned}
\Delta_\lambda b_{n,\lambda}^{(r)}(x) = b_{n,\lambda}^{(r)}(x+\lambda) - b_{n,\lambda}^{(r)}(x) = \lambda n b_{n-1,\lambda}^{(r-1)}(x),\\
b_{n,\lambda}^{(r)}(\lambda) - b_{n,\lambda}^{(r)} = \lambda n b_{n-1,\lambda}^{(r-1)},
\end{aligned} \tag{4.3}
$$

where we understand that $b_{n,\lambda}^{(0)}(x) = (x)_n$.

Now, we assume that $p(x) \in \mathbb{C}[x]$ has degree n, and write $p(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}^{(r)}(x)$. It is important to observe from (4.2) and (4.3) that

$$\begin{aligned} g(t)b_{n,\lambda}^{(r)}(x) &= \frac{e^{\lambda t}-1}{\lambda f(t)} b_{n,\lambda}^{(r)}(x) \\ &= \frac{e^{\lambda t}-1}{\lambda f(t)} f(t) \frac{b_{n+1,\lambda}^{(r)}(x)}{n+1} \\ &= \frac{1}{(n+1)\lambda}(b_{n+1,\lambda}^{(r)}(x+\lambda) - b_{n+1,\lambda}^{(r)}(x)) \\ &= \frac{1}{(n+1)\lambda}(n+1)\lambda\, b_{n,\lambda}^{(r-1)}(x) \\ &= b_{n,\lambda}^{(r-1)}(x). \end{aligned} \tag{4.4}$$

Thus, from (4.4) we have $g(t)^r b_{n,\lambda}^{(r)}(x) = b_{n,\lambda}^{(0)}(x) = (x)_n$, and hence

$$g(t)^r p(x) = \sum_{l=0}^{n} a_l\, g(t)^r b_{l,\lambda}^{(r)}(x) = \sum_{l=0}^{n} a_l (x)_l. \tag{4.5}$$

By using (4.5) and (3.4), we observe that

$$\begin{aligned} f(t)^k g(t)^r p(x) &= \sum_{l=0}^{n} a_l f(t)^k (x)_l \\ &= \sum_{l=k}^{n} a_l\, l(l-1)\cdots(l-k+1)(x)_{l-k}. \end{aligned} \tag{4.6}$$

By evaluating (4.6) at $x = 0$, we obtain

$$a_k = \frac{1}{k!} f(t)^k g(t)^r p(x)|_{x=0} = \frac{1}{k!}\langle f(t)^k g(t)^r | p(x)\rangle. \tag{4.7}$$

This also follows from the observation $\langle g(t)^r f(t)^k | b_{l,\lambda}^{(r)}(x)\rangle = l!\,\delta_{l,k}$, (see (2.8)).

To proceed further, we note that $f(t)g(t) = f(t)\frac{e^{\lambda t}-1}{\lambda f(t)} = \frac{1}{\lambda}(e^{\lambda t}-1)$.

Assume first that $r > n$. Then $r > k$, for all $k = 0, 1, \dots, n$.

$$\begin{aligned} a_k &= \frac{1}{k!}\langle f(t)^k g(t)^r | p(x)\rangle \\ &= \frac{1}{k!\lambda^k}\langle (e^{\lambda t}-1)^k g(t)^{r-k} | p(x)\rangle \\ &= \frac{1}{k!\lambda^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\langle e^{j\lambda t} | g(t)^{r-k} p(x)\rangle \\ &= \frac{1}{k!\lambda^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} g(t)^{r-k} p(x)|_{x=j\lambda} \\ &= \frac{1}{k!\lambda^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} g(t)^{r-k} p(j\lambda). \end{aligned} \tag{4.8}$$

Next, we assume that $r \leq n$. If further $0 \leq k < r$, then a_k is the same as the expression in (4.8). Let $r \leq k \leq n$. Then we have

$$
\begin{aligned}
a_k &= \frac{1}{k!}\langle f(t)^k g(t)^r | p(x)\rangle \\
&= \frac{1}{k!\lambda^r}\langle f(t)^{k-r}(e^{\lambda t}-1)^r | p(x)\rangle \\
&= \frac{1}{k!\lambda^r}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\langle e^{j\lambda t} | f(t)^{k-r}p(x)\rangle \\
&= \frac{1}{k!\lambda^r}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(t)^{k-r}p(x)|_{x=j\lambda} \\
&= \frac{1}{k!\lambda^r}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(t)^{k-r}p(j\lambda).
\end{aligned}
\tag{4.9}
$$

Summarizing the results so far, from (4.8) and (4.9) we obtain the next lemma.

Lemma 4.1. *Let $p(x) \in \mathbb{C}[x], \deg p(x) = n$. Let $g(t) = \frac{e^{\lambda t}-1}{\lambda(e^t-1)}, f(t) = e^t - 1$. Then we have the following:*
(a) For $r > n$, we have

$$
p(x) = \sum_{k=0}^{n}\frac{1}{k!\lambda^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} g(t)^{r-k}p(j\lambda)b_{k,\lambda}^{(r)}(x).
$$

(b) For $r \leq n$, we have

$$
\begin{aligned}
p(x) = &\sum_{k=0}^{r-1}\frac{1}{k!\lambda^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} g(t)^{r-k}p(j\lambda)b_{k,\lambda}^{(r)}(x) \\
&+ \sum_{k=r}^{n}\frac{1}{k!\lambda^r}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(t)^{k-r}p(j\lambda)b_{k,\lambda}^{(r)}(x).
\end{aligned}
$$

We would like to find more explicit expressions for the results in Lemma 4.1.

Let I be the linear integral operator defined on $\mathbb{P}$, which is given by $Iq(x) = \frac{e^t-1}{t}q(x) = \int_x^{x+1} q(u)\,du$.

First, we note from (2.7), (1.16) and (1.17) that

$$
\begin{aligned}
f(t)^{k-r}p(j\lambda) &= f(t)^{k-r}p(x)|_{x=j\lambda} \\
&= I^{k-r}p^{(k-r)}(x)|_{x=j\lambda} \\
&= \Delta^{k-r}p(x)|_{x=j\lambda} \\
&= \sum_{l=0}^{k-r}\binom{k-r}{l}(-1)^{k-r-l}p(j\lambda+l) \\
&= (k-r)!\sum_{l=k-r}^{n} S_2(l,k-r)\frac{1}{l!}p^{(l)}(j\lambda).
\end{aligned}
\tag{4.10}
$$

From (1.4) and (2.10), we note that

$$B_i^{(r-k)}(x) = \left(\frac{t}{e^t-1}\right)^{r-k} x^i, \tag{4.11}$$

and hence, from (4.11), we see that

$$\left(\frac{t}{e^t-1}\right)^{r-k} p(x) = p\big(\mathbf{B}^{(r-k)}(x)\big), \tag{4.12}$$

where $p\big(\mathbf{B}^{(r-k)}(x)\big)$ denotes the umbral composition of $p(x)$ with $B_i^{(r-k)}(x)$.

Now, noting that $g(t) = \frac{1}{\lambda}\frac{e^{\lambda t}-1}{t}\frac{t}{e^t-1}$, from (2.7) and (4.12), we have

$$\begin{aligned} g(t)^{r-k}p(j\lambda) &= g(t)^{r-k}p(x)|_{x=j\lambda} \\ &= \frac{1}{\lambda^{r-k}}\left(\frac{e^{\lambda t}-1}{t}\right)^{r-k}\left(\frac{t}{e^t-1}\right)^{r-k} p(x)\Big|_{x=j\lambda} \\ &= \frac{1}{\lambda^{r-k}}\left(\frac{e^{\lambda t}-1}{t}\right)^{r-k} p\big(\mathbf{B}^{(r-k)}(x)\big)|_{x=j\lambda} \\ &= \frac{1}{\lambda^{r-k}} I_\lambda^{r-k} p\big(\mathbf{B}^{(r-k)}(x)\big)|_{x=j\lambda}, \end{aligned} \tag{4.13}$$

where I_λ is the linear integral operator given by $I_\lambda q(x) = \frac{e^{\lambda t}-1}{t}q(x) = \int_x^{x+\lambda} q(u)\,du$.

From (1.17), we see that $\left(\frac{e^{\lambda t}-1}{t}\right)^m = \sum_{l=0}^\infty S_2(l+m,m)\lambda^{l+m}\frac{m!}{(l+m)!}t^l$, and hence from (4.12) and (4.13) we get another expression for $g(t)^{r-k}p(j\lambda)$ in the following:

$$\begin{aligned} g(t)^{r-k}p(j\lambda) &= \frac{1}{\lambda^{r-k}}\left(\frac{e^{\lambda t}-1}{t}\right)^{r-k}\left(\frac{t}{e^t-1}\right)^{r-k} p(x)\Big|_{x=j\lambda} \\ &= \sum_{l=0}^n S_2(l+r-k,r-k)\frac{(r-k)!}{(l+r-k)!}\lambda^l\left(\frac{d}{dx}\right)^l p\big(\mathbf{B}^{(r-k)}(x)\big)\big|_{x=j\lambda} \\ &= \sum_{l=0}^n S_2(l+r-k,r-k)\frac{(r-k)!}{(l+r-k)!}\lambda^l p^{(l)}\big(\mathbf{B}^{(r-k)}(x)\big)\big)\big|_{x=j\lambda}. \end{aligned} \tag{4.14}$$

Now, from Lemma 4.1, (4.10), (4.13) and (4.14), we finally arrive at the next theorem.

Theorem 4.2. *Let $p(x) \in \mathbb{C}[x], \deg p(x) = n$. Let $g(t) = \frac{e^{\lambda t}-1}{\lambda(e^t-1)}, f(t) = e^t - 1$, and let I and I_λ be the linear integral operators defined on $\mathbb{P}$, which are given by $Iq(x) = \frac{e^t-1}{t}q(x) = \int_x^{x+1} q(u)\,du, I_\lambda q(x) = \frac{e^{\lambda t}-1}{t}q(x) = \int_x^{x+\lambda} q(u)\,du$. Then we have the following:*
(a) For $r > n$, we have

$$p(x) = \sum_{k=0}^n \frac{1}{k!\lambda^k}\sum_{j=0}^k (-1)^{k-j}\binom{k}{j} g(t)^{r-k}p(j\lambda)b_{k,\lambda}^{(r)}(x),$$

where

$$g(t)^{r-k}p(j\lambda) = \frac{1}{\lambda^{r-k}} I_{\lambda}^{r-k} p\big(\mathbf{B}^{(r-k)}(x)\big)|_{x=j\lambda} \tag{4.15}$$
$$= \sum_{l=0}^{n} S_2(l+r-k, r-k)\frac{(r-k)!}{(l+r-k)!}\lambda^l p^{(l)}\big(\mathbf{B}^{(r-k)}(x)\big)\big)\big|_{x=j\lambda}.$$

(b) For $r \le n$, we have

$$p(x) = \sum_{k=0}^{r-1}\frac{1}{k!\lambda^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} g(t)^{r-k}p(j\lambda) b_{k,\lambda}^{(r)}(x)$$
$$+ \sum_{k=r}^{n}\frac{1}{k!\lambda^r}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(t)^{k-r}p(j\lambda) b_{k,\lambda}^{(r)}(x),$$

where $g(t)^{r-k}p(j\lambda)$ is the same as the one in (4.15), and

$$f(t)^{k-r}p(j\lambda) = I^{k-r}p^{(k-r)}(x)|_{x=j\lambda}$$
$$= \Delta^{k-r}p(x)|_{x=j\lambda}$$
$$= \sum_{l=0}^{k-r}\binom{k-r}{l}(-1)^{k-r-l}p(j\lambda+l)$$
$$= (k-r)!\sum_{l=k-r}^{n} S_2(l, k-r)\frac{1}{l!}p^{(l)}(j\lambda).$$

Remark 4.3. *Let $p(x) \in \mathbb{C}[x]$, with $\deg p(x) = n$. Write $p(x) = \sum_{k=0}^{n} a_k b_k^{(r)}(x)$. As λ tends to 0, $g(t) \to \frac{t}{e^t-1}$. Thus, from Theorem 4.2, we get the following result:*

(a) For $r > n$, we have

$$p(x) = \sum_{k=0}^{n}\frac{1}{k!}p^{(k)}\big(\mathbf{B}^{(r-k)}(x)\big)|_{x=0}\, b_k^{(r)}(x).$$

(b) For $r \le n$, we have

$$p(x) = \sum_{k=0}^{r-1}\frac{1}{k!}p^{(k)}\big(\mathbf{B}^{(r-k)}(x)\big)|_{x=0}\, b_k^{(r)}(x) + \sum_{k=r}^{n}\frac{1}{k!}I^{k-r}p^{(k)}(x)|_{x=0}\, b_k^{(r)}(x)$$
$$= \sum_{k=0}^{r-1}\frac{1}{k!}p^{(k)}\big(\mathbf{B}^{(r-k)}(x)\big)|_{x=0}\, b_k^{(r)}(x) + \sum_{k=r}^{n}\frac{1}{k!}\Delta^{k-r}p^{(r)}(x)|_{x=0}\, b_k^{(r)}(x)$$
$$= \sum_{k=0}^{r-1}\frac{1}{k!}p^{(k)}\big(\mathbf{B}^{(r-k)}(x)\big)|_{x=0}\, b_k^{(r)}(x)$$
$$+ \sum_{k=r}^{n}\frac{(k-r)!}{k!}\sum_{l=k-r}^{n-r}\frac{S_2(l, k-r)}{l!}p^{(r+l)}(x)|_{x=0}\, b_k^{(r)}(x).$$

Here I is the linear integral operator given by $Iq(x) = \int_x^{x+1} q(u)\,du$.

5. Examples on representations by Bernoulli polynomials of the second kind

Here we illustrate our formulas in Remarks 3.2 and 4.3 with some examples.

(a) Let $p(x) = B_n(x) = \sum_{k=0}^{n} a_k b_k(x)$. By Remark 3.2, we have

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n} \left\{ \frac{1}{k!} \sum_{l=0}^{k} \sum_{j=0}^{n} (-1)^{k-l} \binom{k}{l} \binom{n}{j} B_{n-j} B_j(l) \right\} b_k(x) \\ &= \sum_{j=0}^{n} \binom{n}{j} B_{n-j} B_j + \sum_{k=1}^{n} \left\{ \frac{1}{k!} \sum_{l=k-1}^{n-1} (l+1) \binom{n}{l+1} S_2(l, k-1) B_{n-l-1} \right\} b_k(x) \\ &= \sum_{j=0}^{n} \binom{n}{j} B_{n-j} B_j + \sum_{k=1}^{n} \left\{ \frac{n}{k!} \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} B_{n-1}(l) \right\} b_k(x). \end{aligned}$$

Next, we let $B_n(x) = \sum_{k=0}^{n} a_k b_k^{(r)}(x)$. By Remark 4.3, we have

$$B_n(x) = \sum_{k=0}^{n} \left\{ \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} B_l^{(r-k)} \right\} b_k^{(r)}(x), \ (r > n);$$

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{r-1} \left\{ \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} B_l^{(r-k)} \right\} b_k^{(r)}(x) \\ &\quad + \sum_{k=r}^{n} \left\{ \frac{(k-r)!}{k!} \sum_{l=k-r}^{n-r} \frac{S_2(l, k-r)}{l!} (n)_{r+l} B_{n-r-l} \right\} b_k^{(r)}(x), \ (r \le n). \end{aligned}$$

(b) Here we illustrate Remark 3.2, for $p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x), \ (n \ge 2)$. For this, we first recall from [11] that

$$p(x) = \frac{2}{n} \sum_{m=0}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} B_m(x) + \frac{2}{n} H_{n-1} B_n(x), \tag{5.1}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the harmonic number and a slight modification of (5.1) gives the identity in (1.2). Let $p(x) = \sum_{k=0}^{n} a_k b_k(x)$. Then we have

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x) &= \frac{2}{n} \sum_{k=0}^{n} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\{ \sum_{m=0}^{n-2} \sum_{j=0}^{m} \frac{1}{n-m} \right. \\ &\times \left. \binom{n}{n-m, j, m-j} B_{n-m} B_{m-j} B_j(l) + H_{n-1} \sum_{j=0}^{n} \binom{n}{j} B_{n-j} B_j(l) \right\} b_k(x). \end{aligned}$$

(c) In [17], it is proved that the following identity is valid for $n \geq 2$:

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)} G_k(x) G_{n-k}(x) = -\frac{4}{n} \sum_{m=0}^{n-2} \binom{n}{m} \frac{G_{n-m}}{n-m} B_m(x). \tag{5.2}$$

Again, by proceeding analogously to (b), we get the following identity:

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{1}{k(n-k)} G_k(x) G_{n-k}(x) \\ &= -\frac{4}{n} \sum_{k=0}^{n} \left\{ \frac{1}{k!} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \sum_{m=0}^{n-2} \sum_{j=0}^{m} \binom{n}{n-m, j, m-j} \frac{G_{n-m}}{n-m} B_{m-j} B_j(l) \right\} b_k(x). \end{aligned}$$

Further, we have

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{1}{k(n-k)} G_k(x) G_{n-k}(x) \\ &= -\frac{4}{n} \sum_{k=0}^{n} \left\{ \sum_{m=0}^{n-2} \sum_{j=0}^{m-k} \binom{n}{n-m, k, j, m-k-j} \frac{G_{n-m}}{n-m} B_{m-k-j} B_j^{(r-k)} \right\} b_k^{(r)}(x), \ (r > n); \\ &\sum_{k=1}^{n-1} \frac{1}{k(n-k)} G_k(x) G_{n-k}(x) = -\frac{4}{n} \sum_{k=0}^{r-1} \left\{ \sum_{m=0}^{n-2} \sum_{j=0}^{m-k} \binom{n}{n-m, k, j, m-k-j} \right. \\ &\quad \left. \times \frac{G_{n-m}}{n-m} B_{m-k-j} B_j^{(r-k)} \right\} b_k^{(r)}(x) \\ &- \frac{4}{n} \sum_{k=r}^{n} \left\{ \frac{(k-r)!}{k!} \sum_{l=k-r}^{n-r} \frac{S_2(l, k-r)}{l!} \sum_{m=0}^{n-2} \binom{n}{m} \frac{G_{n-m}}{n-m} (m)_{r+l} B_{m-r-l} \right\} b_k^{(r)}(x), \ (r \leq n). \end{aligned}$$

(d) In [11], it is shown that the following identity holds for $n \geq 2$:

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x) E_{n-k}(x) = -\frac{4}{n} \sum_{m=0}^{n} \frac{\binom{n}{m}(H_{n-1} - H_{n-m})}{n-m+1} E_{n-m+1} B_m(x), \tag{5.3}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the harmonic number.

Write $\sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x) E_{n-k}(x) = \sum_{k=0}^{n} a_k b_k(x)$.

By proceeding similarly to (b), we obtain the following identity:

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x) E_{n-k}(x) &= -\frac{4}{n} \sum_{k=0}^{n} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \\ &\times \left\{ \sum_{m=0}^{n} \sum_{j=0}^{m} \binom{n}{m} \binom{m}{j} \frac{H_{n-1} - H_{n-m}}{n-m+1} E_{n-m+1} B_{m-j} B_j(l) \right\} b_k(x). \end{aligned}$$

(e) Nielsen [19,2] also represented products of two Euler polynomials in terms of Bernoulli polynomials as follows:

$$\begin{aligned}E_m(x)E_n(x) = &-2\sum_{r=1}^{m}\binom{m}{r}E_r\frac{B_{m+n-r+1}(x)}{m+n-r+1}\\&-2\sum_{s=1}^{n}\binom{n}{s}E_s\frac{B_{m+n-s+1}(x)}{m+n-s+1}\\&+2(-1)^{n+1}\frac{m!\,n!}{(m+n+1)!}E_{m+n+1}.\end{aligned}$$

In the same way as (b), we can show that

$$\begin{aligned}&E_m(x)E_n(x) = -2\sum_{k=0}^{m+n}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\\&\times\Bigg\{\sum_{r=1}^{m}\sum_{i=0}^{m+n-r+1}\frac{\binom{m}{r}E_r}{m+n-r+1}\binom{m+n-r+1}{i}B_{m+n-r+1-i}B_i(l)\\&\quad+\sum_{s=1}^{n}\sum_{j=0}^{m+n-s+1}\frac{\binom{n}{s}E_s}{m+n-s+1}\binom{m+n-s+1}{j}B_{m+n-s+1-j}B_j(l)\\&\quad+(-1)^n\frac{m!\,n!}{(m+n+1)!}E_{m+n+1}\Bigg\}b_k(x).\end{aligned}$$

6. Examples on representations by degenerate Bernoulli polynomials of the second kind

(a) Here we illustrate Theorem 3.1, with $p(x) = B_n(x)$. Let $B_n(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}(x)$. Then, as $B_n(x) = \sum_{j=0}^{n}\binom{n}{j}B_{n-j}x^j$ and $\frac{d}{dx}\frac{1}{j+1}B_{j+1}(x) = B_j(x)$, we have

$$\begin{aligned}&\int_{l}^{l+\lambda} B_n\big(\mathbf{B}(u)\big)du\\&=\sum_{j=0}^{n}\binom{n}{j}B_{n-j}\int_{l}^{l+\lambda}B_j(u)du \qquad (6.1)\\&=\sum_{j=0}^{n}\binom{n}{j}B_{n-j}\frac{1}{j+1}\big(B_{j+1}(l+\lambda)-B_{j+1}(l)\big).\end{aligned}$$

Thus we obtain the following various expressions from Theorem 3.1.

$$\begin{aligned}
B_n(x) &= \sum_{k=0}^{n} \frac{1}{\lambda k!}\left\{\sum_{l=0}^{k}\sum_{j=0}^{n}(-1)^{k-l}\binom{k}{l}\binom{n}{j}\frac{B_{n-j}}{j+1}\big(B_{j+1}(l+\lambda)-B_{j+1}(l)\big)\right\} b_{k,\lambda}(x)\\
&= \frac{1}{\lambda}\sum_{j=0}^{n}\binom{n}{j}\frac{B_{n-j}}{j+1}\big(B_{j+1}(\lambda)-B_{j+1}\big)b_{0,\lambda}(x)\\
&\quad + \sum_{k=1}^{n}\frac{1}{\lambda k!}\left\{\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}\big(B_n(\lambda+l)-B_n(l)\big)\right\} b_{k,\lambda}(x)\\
&= \frac{1}{\lambda}\sum_{j=0}^{n}\binom{n}{j}\frac{B_{n-j}}{j+1}\big(B_{j+1}(\lambda)-B_{j+1}\big)b_{0,\lambda}(x)\\
&\quad + \sum_{k=1}^{n}\left\{\frac{1}{\lambda k}\sum_{l=k-1}^{n}\binom{n}{l}S_2(l,k-1)\big(B_{n-l}(\lambda)-B_{n-l}\big)\right\} b_{k,\lambda}(x).
\end{aligned}$$

In addition, we have the next result from Theorem 4.2.

$$\begin{aligned}
B_n(x) = \sum_{k=0}^{n}\frac{1}{k!\lambda^k}&\left\{\sum_{j=0}^{k}\sum_{l=0}^{n}\sum_{i=0}^{n-l}(-1)^{k-j}\binom{k}{j}\frac{(r-k)!}{(l+r-k)!}\lambda^l(n)_l\binom{n-l}{i}\right.\\
&\left.\times S_2(l+r-k,r-k)B_{n-l-i}B_i^{(r-k)}(j\lambda)\right\} b_{k,\lambda}^{(r)}(x),\ \ (r>n).
\end{aligned}$$

$$\begin{aligned}
B_n(x) &= \sum_{k=0}^{r-1}\frac{1}{k!\lambda^k}\left\{\sum_{j=0}^{k}\sum_{l=0}^{n}\sum_{i=0}^{n-l}(-1)^{k-j}\binom{k}{j}\frac{(r-k)!}{(l+r-k)!}\lambda^l(n)_l\binom{n-l}{i}\right.\\
&\qquad\left.\times S_2(l+r-k,r-k)B_{n-l-i}B_i^{(r-k)}(j\lambda)\right\} b_{k,\lambda}^{(r)}(x)\\
&\quad + \sum_{k=r}^{n}\frac{1}{k!\lambda^k}\left\{\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\sum_{l=0}^{k-r}\binom{k-r}{l}(-1)^{k-r-l}B_n(j\lambda+l)\right\} b_{k,\lambda}^{(r)}(x)\\
&= \sum_{k=0}^{r-1}\frac{1}{k!\lambda^k}\left\{\sum_{j=0}^{k}\sum_{l=0}^{n}\sum_{i=0}^{n-l}(-1)^{k-j}\binom{k}{j}\frac{(r-k)!}{(l+r-k)!}\lambda^l(n)_l\binom{n-l}{i}\right.\\
&\qquad\left.\times S_2(l+r-k,r-k)B_{n-l-i}B_i^{(r-k)}(j\lambda)\right\} b_{k,\lambda}^{(r)}(x)\\
&\quad + \sum_{k=r}^{n}\frac{1}{k!\lambda^k}\left\{\sum_{j=0}^{r}\sum_{l=k-r}^{n}(-1)^{r-j}\binom{r}{j}\binom{n}{l}(k-r)!S_2(l,k-r)B_{n-l}(j\lambda)\right\} b_{k,\lambda}^{(r)}(x),\\
&\qquad (r\le n).
\end{aligned}$$

(b) Here we illustrate Theorem 3.1, for $p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x)B_{n-k}(x)$, $(n \geq 2)$. Let $p(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}(x)$. Then, by making use of the identity in (5.1), we obtain

$$\begin{aligned}
&\sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x)B_{n-k}(x) = \sum_{k=0}^{n} \frac{1}{k!\lambda} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \\
&\times \left\{ \frac{2}{n} \sum_{m=0}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} \int_{l}^{l+\lambda} B_m(\mathbf{B}(u))du + \frac{2}{n} H_{n-1} \int_{l}^{l+\lambda} B_n(\mathbf{B}(u))du \right\} b_{k,\lambda}(x) \\
&= \frac{2}{n\lambda} \left\{ \sum_{m=0}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} \int_{0}^{\lambda} B_m(\mathbf{B}(u))du + H_{n-1} \int_{0}^{\lambda} B_n(\mathbf{B}(u))du \right\} b_{0,\lambda}(x) \\
&+ \frac{2}{n\lambda} \sum_{k=1}^{n} \frac{1}{k!} \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} \left\{ \sum_{m=0}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} \big(B_m(\lambda + l) - B_m(l) \big) \right. \\
&\left. + H_{n-1} \big(B_n(\lambda + l) - B_n(l) \big) \right\} b_{k,\lambda}(x) \\
&= \frac{2}{n\lambda} \left\{ \sum_{m=0}^{n-2} \frac{1}{n-m} \binom{n}{m} B_{n-m} \int_{0}^{\lambda} B_m(\mathbf{B}(u))du + H_{n-1} \int_{0}^{\lambda} B_n(\mathbf{B}(u))du \right\} b_{0,\lambda}(x) \\
&+ \frac{2}{n\lambda} \sum_{k=1}^{n} \frac{1}{k} \sum_{l=k-1}^{n} S_2(l, k-1) \left\{ \sum_{m=0}^{n-2} \frac{1}{n-m} \binom{n}{m} \binom{m}{l} B_{n-m} \big(B_{m-l}(\lambda) - B_{m-l} \big) \right. \\
&\left. + H_{n-1} \binom{n}{l} \big(B_{n-l}(\lambda) - B_{n-l} \big) \right\} b_{k,\lambda}(x),
\end{aligned}$$

where $\int_{l}^{l+\lambda} B_n(\mathbf{B}(u))du$ is given by (6.1) and $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the harmonic number.

(c) Here we illustrate Theorem 3.1, for $p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x)E_{n-k}(x)$, $(n \geq 2)$. Let $p(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}(x)$. Write $\sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x)E_{n-k}(x) = \sum_{k=0}^{n} a_k b_{k,\lambda}(x)$. Then, by making use of the identity in (5.3), and proceeding similarly to (b), we obtain the following identity:

$$\begin{aligned}
&\sum_{k=1}^{n-1} \frac{1}{k(n-k)} E_k(x)E_{n-k}(x) = -\frac{4}{n\lambda} \sum_{k=0}^{n} \frac{1}{k!} \left\{ \sum_{l=0}^{k} \sum_{m=0}^{n} \binom{k}{l} (-1)^{k-l} \right. \\
&\qquad \left. \times \frac{\binom{n}{m}(H_{n-1} - H_{n-m})}{n-m+1} E_{n-m+1} \int_{l}^{l+\lambda} B_m(\mathbf{B}(u))du \right\} b_{k,\lambda}(x) \\
&= -\frac{4}{n\lambda} \left\{ \sum_{m=0}^{n} \frac{\binom{n}{m}(H_{n-1} - H_{n-m})}{n-m+1} E_{n-m+1} \int_{0}^{\lambda} B_m(\mathbf{B}(u))du \right\} b_{0,\lambda}(x) \\
&\quad - \frac{4}{n\lambda} \sum_{k=1}^{n} \frac{1}{k!} \left\{ \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} \sum_{m=0}^{n} \frac{\binom{n}{m}(H_{n-1} - H_{n-m})}{n-m+1} \right. \\
&\qquad \left. \times E_{n-m+1} \big(B_m(\lambda + l) - B_m(l) \big) \right\} b_{k,\lambda}(x)
\end{aligned}$$

$$= -\frac{4}{n\lambda}\left\{\sum_{m=0}^{n}\frac{\binom{n}{m}(H_{n-1}-H_{n-m})}{n-m+1}E_{n-m+1}\int_0^\lambda B_m(\mathbf{B}(u))du\right\}b_{0,\lambda}(x)$$
$$-\frac{4}{n\lambda}\sum_{k=1}^{n}\frac{1}{k}\left\{\sum_{l=k-1}^{n}S_2(l,k-1)\sum_{m=0}^{n}\frac{\binom{n}{m}(H_{n-1}-H_{n-m})}{n-m+1}\right.$$
$$\left.\times E_{n-m+1}\binom{m}{l}\big(B_{m-l}(\lambda)-B_{m-l}\big)\right\}b_{k,\lambda}(x).$$

(d) Here we illustrate Theorem 3.1, for $p(x)=\sum_{k=1}^{n-1}\frac{1}{k(n-k)}G_k(x)G_{n-k}(x)$, $(n\geq 2)$. Write $p(x)=\sum_{k=0}^{n}a_kb_{k,\lambda}(x)$. Then, by making use of the identity in (5.2), and proceeding analogously to (b), we get the following identity:

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)}G_k(x)G_{n-k}(x)$$
$$= -\frac{4}{n\lambda}\sum_{k=0}^{n-2}\frac{1}{k!}\left\{\sum_{l=0}^{k}\sum_{m=0}^{n-2}(-1)^{k-l}\binom{k}{l}\binom{n}{m}\frac{G_{n-m}}{n-m}\int_l^{l+\lambda}B_m(\mathbf{B}(u))du\right\}b_{k,\lambda}(x)$$
$$= -\frac{4}{n\lambda}\left\{\sum_{m=0}^{n-2}\binom{n}{m}\frac{G_{n-m}}{n-m}\int_0^\lambda B_m(\mathbf{B}(u))du\right\}b_{0,\lambda}(x)$$
$$-\frac{4}{n\lambda}\sum_{k=1}^{n-2}\frac{1}{k!}\left\{\sum_{l=0}^{k-1}\sum_{m=0}^{n-2}(-1)^{k-1-l}\binom{k-1}{l}\binom{n}{m}\frac{G_{n-m}}{n-m}\big(B_m(\lambda+l)-B_m(l)\big)\right\}b_{k,\lambda}(x)$$
$$= -\frac{4}{n\lambda}\left\{\sum_{m=0}^{n-2}\binom{n}{m}\frac{G_{n-m}}{n-m}\int_0^\lambda B_m(\mathbf{B}(u))du\right\}b_{0,\lambda}(x)$$
$$-\frac{4}{n\lambda}\sum_{k=1}^{n-2}\frac{1}{k}\left\{\sum_{l=k-1}^{n}\sum_{m=0}^{n-2}\binom{m}{l}\binom{n}{m}S_2(l,k-1)\frac{G_{n-m}}{n-m}\big(B_{m-l}(\lambda)-B_{m-l}\big)\right\}b_{k,\lambda}(x).$$

(e) Nielsen [19,2] expressed products of two Bernoulli polynomials in terms of Bernoulli polynomials. Namely, for positive integers m and n, with $m+n\geq 2$,

$$B_m(x)B_n(x)=\sum_s\left(\binom{m}{2s}n+\binom{n}{2s}m\right)\frac{B_{2s}B_{m+n-2s}(x)}{m+n-2s}+(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}.$$

Again, in a similar way to (b), we can show that

$$B_m(x)B_n(x)=\sum_{k=0}^{m+n}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left\{\frac{1}{\lambda}\sum_s\left(\binom{m}{2s}n+\binom{n}{2s}m\right)\right.$$
$$\left.\times\frac{B_{2s}}{m+n-2s}\int_l^{l+\lambda}B_{m+n-2s}(\mathbf{B}(u))du+(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}\right\}b_{k,\lambda}(x)$$

$$
\begin{aligned}
&= \frac{1}{\lambda}\sum_{s}\left(\binom{m}{2s}n+\binom{n}{2s}m\right)\frac{B_{2s}}{m+n-2s}\int_0^{\lambda}B_{m+n-2s}(\mathbf{B}(u))du+(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}\\
&+\frac{1}{\lambda}\sum_{k=1}^{m+n}\frac{1}{k!}\left\{\sum_{l=0}^{k-1}\sum_{s}(-1)^{k-1-l}\binom{k-1}{l}\left(\binom{m}{2s}n+\binom{n}{2s}m\right)\frac{B_{2s}}{m+n-2s}\right.\\
&\left.\times\big(B_{m+n-2s}(\lambda+l)-B_{m+n-2s}(l)\big)\right\}b_{k,\lambda}(x)\\
&= \frac{1}{\lambda}\sum_{s}\left(\binom{m}{2s}n+\binom{n}{2s}m\right)\frac{B_{2s}}{m+n-2s}\int_0^{\lambda}B_{m+n-2s}(\mathbf{B}(u))du+(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{m}}\\
&+\frac{1}{\lambda}\sum_{k=1}^{m+n}\frac{1}{k}\left\{\sum_{l=k-1}^{n}\sum_{s}S_2(l,k-1)\left(\binom{m}{2s}n+\binom{n}{2s}m\right)\frac{B_{2s}}{m+n-2s}\right.\\
&\left.\times\binom{m+n-2s}{l}\big(B_{m+n-2s-l}(\lambda)-B_{m+n-2s-l}\big)\right\}b_{k,\lambda}(x).
\end{aligned}
$$

7. Conclusion

Carlitz [1] is the first one who studied degenerate versions of some special numbers and polynomials, namely degenerate Bernoulli and degenerate Euler polynomials. In recent years, we have witnessed that studying degenerate versions of some special numbers and polynomials regained the interests of some mathematicians and yielded quite a few interesting results (see [7-19] and references therein).

In this paper, we were interested in representing any polynomial in terms of the degenerate Bernoulli polynomials of the second kind and of the higher-order degenerate Bernoulli polynomials of the second kind. We were able to derive formulas for such representations with the help of umbral calculus. Further, by letting λ tend to zero, we derived formulas for representations by the Bernoulli polynomials of the second kind and by the higher-order Bernoulli polynomials of the second kind. In addition, we illustrated such formulas with some examples.

Even though the method adopted in this paper is elementary, they are very useful and powerful. Indeed, as we mentioned in the Section 1, both a variant of Miki's identity and Faber-Pandharipande-Zagier (FPZ) identity follow from the one identity (see (1.2)) that can be derived from a formula involving only derivatives and integrals of the given polynomial (see (1.1)), while all the other proofs are quite involved. We recall here that the FPZ identity was a conjectural relations between Hodge integrals in Gromov-Witten theory.

It is one of our future research projects to continue to find formulas representing polynomials in terms of some special polynomials and numbers and to use those in discovering some interesting results in mathematics and in finding their applications to physics, science and engineering.

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