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# On one approach to solution of the Vidal-Wolf's model

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ABSTRACT. We consider the model of optimal advertising strategy of the company and discuss one approach to its solution from the view of the optimal control theory. We formulate conditions of the existence of the optimal solution and, on the base of Pontryagin's maximum principle, find its structure.

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KEYWORDS AND PHRASES. Optimal control problem, Pontryagin's maximum principle, Vidal-Wolf's model.

## 1. Introduction

Optimal control theory began to take shape as a mathematical discipline in the 1950s. The motivation for its development were the actual problems of automatic control, satellite navigation, aircraft control, chemical engineering and a number of other engineering problems.

Optimal control is regarded as a modern branch of the classical calculus of variations, which is the branch of mathematics that emerged about three centuries ago at the junction of mechanics, mathematical analysis and the theory of differential equations. The calculus of variations studies problems of extreme in which it is necessary to find the maximum or the minimum of some numerical characteristic (functional) defined on the set of curves, surfaces, or other mathematical objects of a complex nature.

The development of the calculus of variations is associated with the names of some famous scientists: Bernoulli, Euler, Newton, Lagrange, Weierstrass, Hamilton and others. Optimal control problems differ from variation problems by the additional requirements imposed on sought solution, and these requirements are sometimes difficult and even impossible to fit applying for solving the methods of the calculus of variations. The need for practical methods resulted in further development of variation calculus, which ultimately led to the formation of the modern theory of optimal control. This theory, absorbed all previous achievements in the calculus of variations, and it was enriched with new results and new content. The central results of the theory – the Pontryagin's maximum principle and the dynamic programming method of Bellman – became well known in the scientific and engineering community, and these are now widely used in various academic fields [1, 2, 3, 5, 6].

In this paper, we consider one very interesting application of the methods of optimal control theory for solution of some economical problem known as Vidal-Wolf's model of optimal advertising strategy of some company [4, 7, 8] This model, a significant framework in advertising strategy optimization, has its roots in the evolving landscape of marketing theory and practice during the late 20th century. As businesses began to recognize the importance of data-driven decision-making and consumer-centric approaches, the need for a structured model became evident.

The Vidal-Wolf's model is one of the first models of this type and in this model change of the goods sales volume at time $t$ is the function of four factors: advertising expenses; constants expressing sales reaction on advertising; saturation level of the market with advertised goods and constants expressing the reduction of sales volume. This model serves as a powerful tool for analyzing the dynamics of consumption on the advertising and uses in real-word business.

One of the groundbreaking aspects of the Vidal-Wolf's model was its focus on feedback mechanisms. This innovation allowed marketers to continuously assess the performance of their campaigns and make real-time adjustments based on consumer responses. This iterative process was particularly relevant in the context of digital marketing, where consumer preferences could shift rapidly.

As the model gained recognition, it was adopted by various industries, from retail to technology. Businesses began to see tangible improvements in their advertising effectiveness and return on investment (ROI), reinforcing the model's credibility. Over time, the Vidal-Wolf's model became a cornerstone of modern marketing education, featured in numerous textbooks, academic papers, and marketing courses around the world.

The Vidal-Wolf model emerged from the collaborative efforts of Dr. Jean Vidal and Dr. Anna Wolf, who sought to create a comprehensive framework for optimizing advertising strategies. Their vision, combined with the technological advancements of the time, laid the foundation for a model that continues to influence marketing practices today. As businesses strive to adapt to ever-changing consumer behaviors and market dynamics, the principles of the Vidal-Wolf's model remain relevant and valuable in guiding advertising strategies towards success.

In this paper, we formulate the problem of optimal advertising strategy of the company as the simplest optimal control problem, derive the basic conditions for existence of the optimal solution and, using Pontryagin's maximum principle, find its structure.

## 2. Specifying of the problem

Optimal control problems are classified on several types: the simplest problem, the two point minimum time problem, the general problem, the problem with intermediate states, the common problem, etc. [1, 5, 6]

We consider one of them – the simplest problem (*S-problem*). According to [1], it consists of minimizing a terminal functional on a set of processes $x(t)$, $u(t)$ of a controlled system with fixed left end of a trajectory and fixed initial $t_0$ and terminal $t_1$ times. This problem has the form

$$J = \Phi(x(t_1)) \to \min ,$$

$$\dot{x} = f(x,u,t),\ x(t_0) = x^0,\ u \in U,\ t \in [t_0,t_1] , \tag{1}$$

where a scalar function $\Phi(x)$ belongs to the class $C_1(R^n \to R)$. Here $x \in R^n$ is state variable, $u \in U \subset R^r$ is control variable. Time interval $I = [t_0,t_1]$ is fixed. We assume that the function $f(x,u,t)$ is defined and continuous on $R^n \times U \times I$ and has continuous partial derivatives $f_x(x,u,t)$ on that set. We consider a set of piecewise-continuous functions $u(t)$ with the range in compact $U$ as the class of controls. A process $x(t),\ u(t)$ is regarded to be *optimal* if for any other process $\tilde{x}(t),\ \tilde{u}(t)$, the following inequality is true

$$\Phi(x(t_1)) \le \Phi(\tilde{x}(t_1)) .$$

The *S*-problem consists in determining of the optimal process.

Necessary conditions of optimality in S-problem are given by Pontryagin's maximum principle [1, 7].

**Theorem.** Let process $x(t)$, $u(t)$ be optimal in S-problem. Then there exists a solution $\psi(t)$ of conjugate initial-value problem

$$\dot{\psi} = -H(\psi, x(t), u(t), t), \ \psi(t_1) = -\Phi_{x(t_1)}(x(t_1)) \tag{2}$$

such that for any $t_0 \le t \le t_1$

$$H(\psi(t), x(t), u(t), t) = \max_{u \in U} H(\psi(t), x(t), u, t). \tag{3}$$

Here $H(\psi, x, u, t) = \psi^T f(x, u, t)$ is Hamilton function (Hamiltonian). We will consider the application of the maximum principle for solution of one important problem known as Vidal-Wolf's model of advertising strategy of a company.

Let us consider a company supplying a certain product to the market. Let $M$ be the market capacity of this product, i.e. the saturation level, depending on the number of potential buyers. In other words, $M$ is the maximum possible volume of demand for the company's products. The company's income depends on the sales volume $S$, and two factors influence the change in sales volume:

1. The increase in sales $\dot{S} = \dfrac{dS}{dt}$ is proportional to the flow of advertising expenses $u(t)$ and the portion of potential demand $\left(1 - \dfrac{S}{M}\right)$ that is not yet satisfied by the firm's supply. Therefore, the closer the sales volume is to the saturation level, the less effective the advertising expenses become.

2. The sales volume decreases at a constant rate $b$, since part of the population forgets about the products of this firm or begins to buy similar products of other companies. This assumption means that in the absence of advertising, when $u(t) = 0$, the dynamics of the variable $S$ is described by the equation $S = -bS$ and, therefore, the sales volume falls exponentially.

Combining these hypotheses, we obtain the following equation for the dynamics of sales volume taking into account investments in advertising:

$$\dot{S} = Au\left(1 - \frac{S}{M}\right) - bS, \ \ S(0) = S_0 \in (0, M), \tag{4}$$

where $b$ is «forgetting» coefficient, $A > 0$ is proportionality coefficient.

The company's advertising costs are described by the function $C(u)$ (so that, generally speaking, at any given moment in time they may not coincide with the flow of expenses $u(t)$).

Let the company's objective be to maximize discounted profit. Since the company's income is proportional to sales volume, then the current profit is equal to the difference $PS(t) - C(u(t))$, and the profit received over the period $[0, T]$ is determined by the integral

$$\Phi(u) = \int_0^T e^{-rt} \left( PS(t) - C(u(t)) \right) dt .$$

If current advertising expenses are limited by the sum $R$, then admissible controls must satisfy the constraint $0 \le u(t) \le R$.

Thus, the problem consists of determining the admissible advertising strategy $u(t)$, which provides the maximum value for the criterion $\Phi(u)$. In this case, the dynamics of sales volume is described by equation (4).

Let us consider the Vidal-Wolf model under the assumption that the company's advertising costs coincide with the flow of expenses $u(t)$, that is, $C(u) = u$. Then the optimality criterion will be written as

$$\Phi(u) = \int_0^T e^{-rt} \left( PS(t) - u(t) \right) dt .$$

It is convenient to replace the variable $S$ with a new variable $x = \frac{S}{M}$, which has the meaning of the market share covered by the sales of a given company. From the equation

$$\dot{S} = Au\left(1 - \frac{S}{M}\right) - bS , \quad S(0) = S_0$$

we obtain a dynamic equation and an initial condition for the variable $x$:

$$\dot{x} = au\left(1 - x\right) - bx , \; x(0) = x_0 ,$$

where $a = \dfrac{A}{M}$ and $x_0 = \dfrac{S_0}{M}$. Taking into account the change of variables, the Vidal-Wolf model will take the following form:

$$J = \int_0^T e^{-rt}\left(px(t) - u(t)\right)dt \to \max ,$$

$$\dot{x} = au\left(1 - x\right) - bx,\ x(0) = x_0 \in (0,1) ,$$

$$0 \le u(t) \le R .$$

Here $p = PM$ . Introducing variables $x_1 = x$ and $\dot{x}_2 = e^{-rt}\left(px_1 - u\right)$ with initial condition $x_2(0) = 0$, we arrive at the simplest problem of optimal control

$$J = -x_2(T) \to \min ,$$

$$\begin{cases} \dot{x}_1 = au\left(1 - x_1\right) - bx_1 \\ \dot{x}_2 = e^{-rt}(px_1 - u) \end{cases} \tag{5}$$

$$x_1(0) = x_{10},\ x_2(0) = 0$$

$$0 \le u(t) \le R .$$

Since $x_{10} \in (0,1)$ note that from equation $\dot{x}_1 = au\left(1 - x_1\right) - bx_1$ it follows that $0 < x_1(t) < 1$ on any admissible trajectory.

## 3. Solution of the simplest problem

We write necessary conditions of optimality. We form Hamiltonian $H(\psi, x, u, t) = \left(au(1 - x_1) - bx_1\right)\psi_1 + e^{-rt}\left(px_1 - u\right)\psi_2$ and conjugate initial-value problem

$$\begin{cases} \dot{\psi}_1 = (au + b)\psi_1 - e^{-rt}p\psi_2 \\ \dot{\psi}_2 = 0 \end{cases};\ \begin{cases} \psi_1(T) = 0 \\ \psi_2(T) = 1 \end{cases}.$$

From the latter, we get $\psi_2(t) = 1,\ 0 \le t \le T$ . Therefore, the first differential equation in the conjugate system becomes

$$\dot{\psi}_1 = (au - b)\psi_1 - e^{-rt}p \tag{6}$$

with initial condition $\psi_1(T)=0$.

According to the maximum principle, optimal control $u^{opt}(t)$ in any time $t\in[0,T]$ is a solution of the extreme problem

$$\left(a(1-x_1)\psi_1-e^{-rt}\right)u\to\max_{0\le u\le R}. \tag{7}$$

For convenience, we introduce auxiliary variable $\lambda(t)=e^{-rt}\psi_1$. Then differential equation (6) becomes

$$\dot{\lambda}=(b+r+au)\lambda-p \tag{8}$$

and $\lambda(T)=0$.

Then extreme problem (7) gets the form

$$\left(a(1-x_1)\lambda-1\right)u\to\max_{0\le u\le R},\quad t\in[0,T]. \tag{9}$$

Its solution, due to linearity of Hamiltonian by control variable, is

$$u^*(t)=\begin{cases} R, & if\ a(1-x_1)\lambda-1>0,\\ 0, & if\ a(1-x_1)\lambda-1<0,\\ [0,R], & if\ a(1-x_1)\lambda-1=0.\end{cases} \tag{10}$$

Taking into account the rule (10), we begin to construct optimal control in backwards. When $t=T$ we have

$$\left(a(1-x_1(T))\lambda(T)-1\right)u(T)\to\max_{0\le u\le R}.$$

From here and from the fact that $\lambda(T)=e^{-rT}\psi_1(T)=0$ we obtain $u^{opt}(T)=0$.

We introduce a switching function $h(t)=a(1-x_1(t))\lambda(t)-1$. Then (9) can be written as

$$h(t)u\to\max_{0\le u\le R},\quad t\in[0,T]. \tag{11}$$

Due to continuity of the function $h(t)$ we conclude [9] that $u^{opt}(t)=0$ on $t\in(\tau,T]$ (Fig. 1).

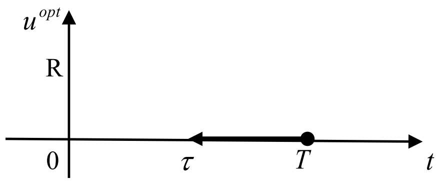

Fig.1. Optimal control on interval $t \in (\tau, T]$.

Optimal control $u^{opt}(t)$ to the left from $\tau$ can be especial or take the boundaries 0 or $R$.

First we consider the case of especial control. Let on some interval $\Delta$ a switching function $h(t)=0$ ($\Delta$ is called especial interval of time). Then $\dot{h}(t)=0$ on $\Delta$ as well. Observe that

$$\dot{h}(t)=a(1-x_1)\dot{\lambda}-a\lambda\dot{x}_1=a(1-x_1)\left[(b+r+au)\lambda-p\right]-a\lambda(au(1-x_1)-bx_1)=$$
$$=a\left[(b+r)\lambda-r\lambda x_1-p(1-x_1)\right].$$

Note that on especial interval $\lambda=\dfrac{1}{a(1-x_1)}$. Consequently,

$$\dot{h}(t)=a\left[\frac{b+r}{a(1-x_1)}-\frac{rx_1}{a(1-x_1)}-p(1-x_1)\right]=0$$

or

$$ap(x_1)^2+(r-2ap)x_1+(ap-b-r)=0,\quad t\in\Delta. \tag{12}$$

Thus, on especial interval optimal trajectory must satisfy the equation (12). Its roots are

$$x_1^1=1-\frac{r+\sqrt{r^2+4apb}}{2ap} \text{ and } x_1^2=1-\frac{r-\sqrt{r^2+4apb}}{2ap}. \tag{13}$$

If

$$ap>b+r \tag{14}$$

then

$$x_1^1=1-\frac{r+\sqrt{r^2+4apb}}{2ap}\in(0,1). \tag{15}$$

We denote it by $\bar{x}=x_1^1$. Note that if condition (14) doesn't hold then both roots $x_1^1$ and $x_1^2$ of (12) don't belong to $(0,1)$ and, therefore, optimal control cannot have especial interval $\Delta$.

Assume further that (14) holds. This means that on especial interval optimal trajectory is

$$x^{opt}(t) = \bar{x}\ .$$

Since on $\Delta\ \dot{x} = 0$, therefore $0 = au(1-\bar{x}) - b\bar{x}$ and, taking into account (15), we get especial control

$$\bar{u} = \frac{b}{a}\left(\frac{2ap}{r+\sqrt{r^2+4apb}} - 1\right). \qquad (16)$$

If $\bar{u} \le R$ then $u^{opt}(t) = \bar{u}$ on especial interval $\Delta$.

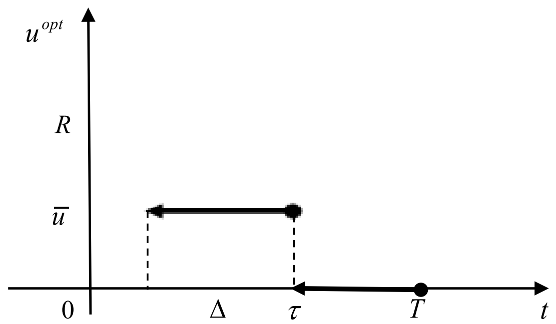


Fig.2. Optimal control on interval $t \in \Delta \cup (\tau, T]$.

Thus, if optimal control has areas of especial one, then in these areas investments in advertising are constant and aimed at maintaining the most profitable level of sales volume $\bar{x}$ (main level).

Now we will show that a control satisfying the maximum principle has at most two switching points. First, we note that if there is an interval of especial control, then there is only one. Therefore, if the control has more than two switching points, then it must take boundary values on some section located between the two switching points. We will show that such a control does not satisfy the maximum principle and, therefore, cannot be optimal.

We assume the opposite. Let on some interval $(t_1, t_2) \subset [0, T]$ $u^{opt}(t) = 0$, $h(t_1) = 0$ and $h(t_2) = 0$. From (10) it follows $h(t) < 0$ on $t \in (t_1, t_2)$. Then, obviously, the function

$h(t)$ does not increase at the point $t_1$ $\left(\dot{h}(t_1) \le 0\right)$ and does not decrease at the point $t_2$ $\left(\dot{h}(t_1) \ge 0\right)$.

Previously we calculated $\dot{h}(t) = a\left[(b+r)\lambda - r\lambda x_1 - p(1-x_1)\right]$. Since $t_1$ and $t_2$ are switching points then

$$\lambda(t_1) = \frac{1}{a(1-x_1(t_1))} \text{ and } \lambda(t_2) = \frac{1}{a(1-x_1(t_2))}.$$

Consequently,

$$\dot{h}(t_1) = \frac{1}{1-x_1(t_1)}\left[-ap\left(x_1(t_1)\right)^2 + (2ap-r)x_1(t_1) + (b+r-ap)\right],$$

$$\dot{h}(t_2) = \frac{1}{1-x_1(t_2)}\left[-ap\left(x_1(t_2)\right)^2 + (2ap-r)x_1(t_2) + (b+r-ap)\right].$$

Since $x_1(t_1) \in (0,1)$ and $x_1(t_2) \in (0,1)$ then

$$\dot{h}(t_1) = 0 \Leftrightarrow \bar{x} = x(t_1) \text{ and } \dot{h}(t_2) = 0 \Leftrightarrow \bar{x} = x(t_2),$$

$$\dot{h}(t_1) < 0 \Leftrightarrow 0 < x(t_1) < \bar{x} \text{ and } \dot{h}(t_2) < 0 \Leftrightarrow 0 < x(t_2) < \bar{x},$$

$$\dot{h}(t_1) > 0 \Leftrightarrow \bar{x} < x(t_1) < 1 \text{ and } \dot{h}(t_2) > 0 \Leftrightarrow \bar{x} < x(t_2) < 1.$$

From $\dot{h}(t_1) \le 0$ and $\dot{h}(t_2) \ge 0$ it follows $x_1(t_1) \le \bar{x}$ and $x_1(t_2) \ge \bar{x}$. Consequently,

$$x_1(t_1) \le x_1(t_2). \tag{17}$$

But from differential equation $\dot{x}_1 = au\left(1-x_1\right) - bx_1$ for $u(t) = 0$ (that is, $\dot{x}_1 = -bx_1$) we get that function $x_1(t)$ is strictly decreasing on $(t_1, t_2)$. Therefore,

$$x_1(t_1) > x_1(t_2).$$

The latter contradicts to (17). This means that control $u(t) = 0$ in the interval located between the switching points cannot be optimal.

It is similarly proved that the control taking the value $u(t) = R$ on the interval $(t_1, t_2) \subset [0, T]$ does not satisfy the maximum principle.

Now the structure of control satisfying the maximum principle is clear. If there is an especial interval to the left of $\tau$ then the control has a main level $\bar{x}$, that is, advertising costs are such that for some time the sales volume is maintained at the trunk level $\bar{x}$. At the same time, if the initial sales volume is $x_{01} \neq \bar{x}$ then the interval of main level must be preceded by an interval of enter to the trunk $\bar{x}$.

If $x_{10} > \bar{x}$ then the enter to the trunk is provided by control $u(t) = 0$ acting until the moment $\tau_1$ such that $x_1(\tau_1) = \bar{x}$. Sales volume is declining exponentially $x_1(t) = x_{10}e^{-bt}$, $t \le \tau_1$. From here we find

$$\tau_1 = \frac{1}{b}\ln\frac{x_{10}}{\bar{x}}.$$

Note that since $x_{10} > \bar{x}$, then $\tau_1 > 0$.

We will determine the moment $\tau$ of exit from the trunk $\bar{x}$. From (8) we get that on interval $(\tau, T]$ function $\lambda(t)$ is defined by

$$\lambda(t) = \frac{p}{b+r}\left(1 - e^{-(b+r)(T-t)}\right).$$

Putting $t = \tau$ and taking into account that $x_1(\tau) = \bar{x}$ we obtain

$$\tau = T + \frac{1}{b+r}\ln\left(1 - \frac{b+r}{pa(1-\bar{x})}\right). \quad (18)$$

Note that if optimal control has trunk interval then $\tau_1 < \tau$, that is the following condition holds:

$$\frac{1}{b}\ln\frac{x_{10}}{\bar{x}} < T + \frac{1}{b+r}\ln\left(1 - \frac{b+r}{pa(1-\bar{x})}\right). \quad (19)$$

If (19) doesn't hold then investments in advertising are not profitable and $u^{opt}(t) = 0,\ t \in [0, T]$. Thus, optimal control in case $x_{10} > \bar{x}$ and fulfilling condition (19) is

$$u^{opt}(t) = \begin{cases} 0, & 0 \le t < \tau_1, \\ \bar{u}, & \tau_1 \le t \le \tau, \\ 0, & \tau < t \le T. \end{cases} \quad (20)$$

Consider the case $x_{10} < \bar{x}$. Here the enter on the trunk is provided by control $u(t) = R$. From the first differential equation in (5) we get

$$x_1(t) = \frac{aR}{b+aR} + e^{-(b+aR)t}\left(x_{10} - \frac{aR}{b+aR}\right).$$

Putting $t=\tau_2$ into the last expression we find the moment of enter of the trunk

$$\tau_2 = \frac{1}{b+aR}\ln\left(\frac{aR-(b+aR)x_{10}}{aR-(b+aR)\overline{x}}\right).$$

Thus, for $x_{10}<\overline{x}$ and $\tau_2<\tau$ optimal control has the form

$$u^{opt}(t)=\begin{cases} R, & 0\le t<\tau_2, \\ \overline{u}, & \tau_2\le t\le\tau, \\ 0, & \tau<t\le T. \end{cases} \quad (21)$$

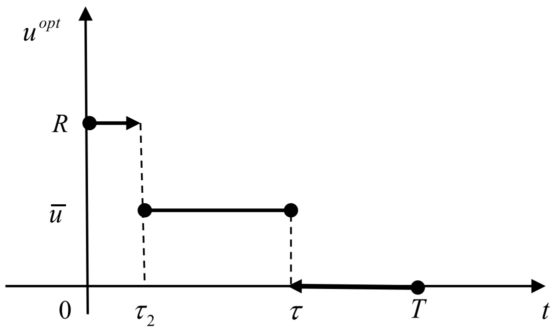

Fig.3. Optimal control (21)

If $T$ is small and $\tau_2>\tau$ then trunk interval doesn't exist and

$$u^{opt}(t)=\begin{cases} R, & 0\le t\le\tau_3, \\ 0, & \tau_3<t\le T, \end{cases} \quad (22)$$

where $\tau_3$ is determined from the condition $h(\tau_3)=0$.

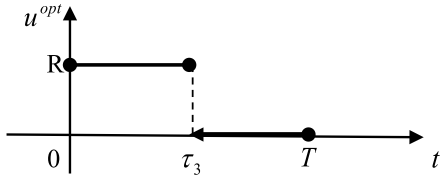

Fig.4. Optimal control (22)

## Conclusion

We discussed very important question of the theory of optimal control - the application in the economic problems. In particular, we considered the problem of optimal advertising strategy in the Vidal-Wolf's model, reduced it to the simplest problem of optimal control and, using Pontryagin's maximum principle, derived the optimal solution.

We showed the structure of the most effective distribution of the investment in advertising policy of a company guaranteeing maximum return on investment (ROI) for various input conditions. Despite its limitations, Vidal-Wolf's model remains an essential component of economic theory and practice, offering insights into optimal resource allocation over time.

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