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A GENERATOR OF THE LANCZOS POTENTIAL IN KERR SPACETIME VIA COMPLEXIFICATION OF REAL COORDINATES

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Abstract. Here, we consider the Kerr metric in Boyer-Lindquist coordinates (t, r, θ, φ), and realize the complexification of t and φ to obtain a generator of the Lanczos potential for the corresponding Weyl tensor.

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1. Introduction

Complexification of real coordinates to construct new solutions of partial differential equations is an old idea, it already appeared in the work of Appell [1, 2] on potential theory. For example, Schiffer et al [3] applied complexification to the Schwarzschild geometry to deduce the Kerr spacetime [4, 5, 6, 7, 8, 9]; besides, Barrera et al [10] used complexification of Minkowskian coordinates to obtain real scalar wave functions without singularities anywhere in special relativity.

Here, we consider the Kerr metric in Boyer-Lindquist coordinates (t, r, θ, φ) [7, 8, 11, 12, 13, 14, 15]:

$$ds^2 =$$

(1)
$$\left(1-\frac{2Mr}{\Sigma}\right)dt^2-\frac{\Sigma}{C}dr^2-\Sigma\, d\theta^2+\frac{4Mar}{\Sigma}sin^2\theta dt d\varphi-\left(r^2+a^2+\frac{2Ma^2r}{\Sigma}sin^2\theta\right)sin^2\theta d\varphi^2,$$

$$A = r - iacos\theta,\ \ \Sigma = A\bar{A} = r^2 + a^2cos^2\theta,\ \ C = r^2 - 2Mr + a^2,$$

where a and M are the rotation and the mass parameters, respectively. In Sec. 2, we shall make the changes $t \to it$ and $\varphi \to i\varphi$ to construct a generator of the Lanczos potential for the conformal tensor corresponding to the metric in eqn. (1).

2. Lanczos potential via complexification

If in the Kerr metric given by (1), we realize the complexification $t \to it$ and $\varphi \to i\varphi$, we obtain the following transformation:

(2) $$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad \to \quad 4P_{\mu\nu}dx^\mu dx^\nu,$$

such that:

(3) $$\Big(P_{\mu\nu}\Big) = \Big(P_{\nu\mu}\Big) = \frac{1}{4}\begin{pmatrix} -g_{00} & 0 & 0 & -g_{03} \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ -g_{03} & 0 & 0 & -g_{33} \end{pmatrix},$$

and the amazing thing is that this second-order symmetric tensor generates a Lanczos potential $K_{\mu\nu\alpha}$ [16, 17, 18, 19, 20, 21, 22, 23] for the Weyl tensor $C_{\mu\nu\alpha\beta}$ [7,12,13]:
(4)
$$K_{\mu\nu\alpha} = P_{\alpha\nu;\mu} - P_{\alpha\mu;\nu} + g_{\alpha\nu}A_\mu - g_{\alpha\mu}A_\nu, \quad \Big(A_\mu\Big) = \Big(\frac{1}{3}P^{\nu}{}_{\mu;\nu}\Big) = \Big(0, \frac{r-M}{6C}, \frac{ctg\theta}{6}, 0\Big),$$
$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}, \;\; K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0,$$

verifying the Lanczos gauges [16]:

(5) $$K^{\mu\nu}{}_{\nu} = 0, \;\; K^{\mu\nu\alpha}{}_{;\alpha} = 0,$$

hence our potential defined in eq.(4) satisfies the Lanczos-Illge wave equation [24, 25, 26, 27, 28, 29] in this vacuum spacetime:

(6) $$\Box K_{\mu\nu\alpha} = 0.$$

In the above equation the box denotes two successive covariant derivatives with the contracted indices.
The corresponding conformal tensor is generated by the expression
(7)
$$C_{\mu\nu\alpha\beta} = K_{\mu\nu\alpha;\beta} - K_{\mu\nu\beta;\alpha} + K_{\alpha\beta\mu;\nu} - K_{\alpha\beta\nu;\mu} + K_{\nu\alpha}g_{\mu\beta} - K_{\nu\beta}g_{\mu\alpha} + K_{\mu\beta}g_{\nu\alpha} - K_{\mu\alpha}g_{\nu\beta},$$
$$K_{\mu\nu} = K_{\nu\mu} = K_{\mu\alpha\nu}{}^{;\alpha}$$
which has type D in the Petrov classification [7, 12, 13, 20, 30].

Hence, we can use (3), (4) and the Christoffel symbols for the Kerr metric [14] to obtain the non-zero components of the Lanczos potential:

(8) $$K_{abc} = \frac{1}{2}\Big(g_{oc}\,\Gamma^{0}{}_{ab} + g_{3c}\,\Gamma^{3}{}_{ab}\Big) - g_{ac}A_b, \;\; a, c = 0, 3, \;\; b = 1, 2,$$
$$K_{121} = \frac{\Sigma}{6C}\,ctg\theta, \;\; K_{122} = \frac{\Sigma(M-r)}{6C},$$
that is:
(9)
$$K_{b00} = \frac{-1}{4}\,g_{00,b} + g_{00}A_b, \;\; K_{b03} = K_{b30} = \frac{-1}{4}\,g_{03,b} + g_{03}A_b, \;\; K_{b33} = \frac{-1}{4}g_{33,b} + g_{33}A_b,$$
$$b = 1, 2, \qquad K_{211} = g_{11}A_2, \;\; K_{122} = g_{22}A_1,$$
and we observe that $K_{03\mu} = 0, \; \forall \mu$.

The possible physical meaning of the Lanczos potential in general relativity remains an open problem. We believe that $K_{\mu\nu\alpha}$ behaves in the asymptotic region as the density for the angular momentum of a rotating black hole.

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