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ON RAMANUJAN'S CONSTANT

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ABSTRACT. The name "Ramanujan's constant" was coined by Simon Plouffe and derives from an April Fool's joke played by Martin Gardner (Apr. 1975) on the readers of Scientific American. In his column, Gardner claimed that $e^{\pi\sqrt{163}}$ was exactly an integer, and that Ramanujan had conjectured this in his 1914 paper. Gardner admitted his hoax a few months later (Gardner, July 1975). Although, Ramanujan (1913-1914) gave few rather spectacular examples of almost integers (such $e^{\pi\sqrt{58}}$), he did not actually mention the particular near-identity given above. In fact,

$$e^{\pi\sqrt{163}} = 262,537,412,640,768,743.9999999999992500\ldots$$

Why this is so close to integer? In this article, I will give the reason proving the Schneider-Lang theorem. Also, in this article, I prove the 'Theorem of 6 exponentials' that old, first published accounts due to Ramachandra and Lang.

1. INTRODUCTION

We start the following question.

Question 1. *Which are the real number t for which* 2^t *is a rational integer ?*

For any $a \in \mathbb{N}$, $a \neq 0$, if we set $t = \frac{\log a}{\log 2}$, then $2^t = \exp(t \log 2) = a \in \mathbb{N}$. Hence,

$$\{t \in \mathbb{R} \; ; \; 2^t \in \mathbb{N}\} = \left\{ \frac{\log a}{\log 2} \; ; \; a \in \mathbb{N}, \; a > 0 \right\}.$$

If we denote this set by E_1, then $E_1 \cap \mathbb{Q} = \mathbb{N}$. Consider now the set $E_2 = \{t \in \mathbb{R} \; : \; 2^t \in \mathbb{N} \text{ and } 3^t \in \mathbb{N}\}$. Naturally, $\mathbb{N} \subset E_2 \subset E_1$. Also, $E_2 \cap \mathbb{Q} = \mathbb{N}$. The following problem is still open.

Question 2. *Is it true that* $E_2 = \mathbb{N}$ *?*

This means

Question 3. *Dose there exists an irrational number which belongs to* E_2*?*

Proof. This question amount to ask whether there exist two positive integers a and b such that $\frac{\log a}{\log 2} = \frac{\log b}{\log 3}$ and at the same this quotient is irrational.

Another equivalent formulation is to ask whether a 2×2 matrix $\begin{pmatrix} \log a & \log b \\ \log 2 & \log 3 \end{pmatrix}$ (with positive integers a and b) can be singular without a being a power of 2.
We will consider the this quotient in a more general setting. □

Finally, introduce a third set $E_3 = \{t \in \mathbb{R} : 2^t \in \mathbb{N}, \; 3^t \in \mathbb{N}, \text{ and } 5^t \in \mathbb{N}\}$. Of course we have $\mathbb{N} \subset E_3 \subset E_2 \subset E_1$. In [6], The six exponential theorem implies $E_3 = \mathbb{N}$.

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- Replace $\{2,3,5\}$ be any set of distinct primes. More generally, if we consider three multiplicatively independent algebraic numbers, then there is no need to restrict the discussion of real value of t.

In [1-6], this result is known that the six exponentials theorems as follows.

Theorem 1.1 (old, first published accounts due to Ramachandra and Lang)**.** *Let* $\alpha_1,\alpha_2 \in \mathbb{C}$*, linearly independent over* $\mathbb{Q}$*. And* $\beta_1,\beta_2,\beta_3 \in \mathbb{C}$*, also linearly independent over* $\mathbb{Q}$*. Then one of* $\exp(\alpha_i\beta_j)$ *is transcendental.*

It is the corollary of the above theorem.

Theorem 1.2. *If* $\alpha \notin \mathbb{Q}$ *then one of* $2^\alpha, 3^\alpha$ *and* 5^α *is transcendental.*

The Six Exponentials Theorem occurs for the first time in a paper by L. Alaogu and P. Erdös, when these authors try to prove Ramanujan's assertion that the quotient of two consecutive *superior highly composite number* is a prime, they need to know that if x is a real number such that p_1^x and p_2^x are both rational numbers, with p_1 and p_2 distinct prime numbers, then x is an integer.

Definition 1.3 ([7])**.** *An integer n said to be a 'superior highly composite number' if there exists* $\varepsilon > 0$ *such that the divisor function* $d(n)$ *satisfies*

$$d(m)m^{-\varepsilon} < d(n)n^{-\varepsilon} \quad \text{for} \quad m \neq n$$

Conjecture 1.4 (Four Exponentials Conjecture)**.** *Let* x_1,x_2 *be two* $\mathbb{Q}$*-linearly independent complex numbers, and* y_1,y_2 *also two* $\mathbb{Q}$*-linearly independent complex numbers. Then one at least of the four numbers*

$$\exp(x_i y_j), \quad (i = 1,2,\ j = 1,2)$$

is transcendental.

However, this statement (special case of the above conjecture =) is yet unproven. They quote C.L. Siegel and claim that x indeed is an integer if one assume p_i^x to be rational for three distinct primes p_i. This is just a special case of the Theorem 1. They deduce that the quotient of two consecutive superior highly composite numbers is either a prime or else a product of primes. Theorem 1 can be deduced from a very general result of Th. Schneider conjectures is equivalent to the first of eight problems at the end of Schneider's book. An explicit statement of the six exponentials theorem , together with a proof, has been published and at the same time by S. Lang's book, Chapter 2 and K. Ramachandra's book Chapter 2, see [8, 9]. They both formulated the Four Exponentials Conjecture.

2. Basic Definitions and Lemmas

Lemma 2.1 (Thúe-Siegel)**.** *Suppose* $u_{ij},\ 1 \le i \le M,\ 1 \le j \le N$ *are integers with* $|u_{ij}| \le U$*. Want to solve*

$$\sum_{j=1}^{N} u_{ij}x_j = 0,$$

$x_1,\ldots,x_N \in \mathbb{Z}$, $N > M$*. Then there is a non-trivial solution with* $|x_j| \le (NU)^{\frac{M}{N-M}}$*.*

Proof. Essentially, the Thue-Siegel lemma is a glorified version of the pigeon hole principle. Say $0 \le x_j \le X$. Then we have $(X+1)^N$ possblities for the x_j. Consider the M-tuple

$$\left\{ \sum_{j=1}^{N} u_{ij} x_j \right\}_{i=1,\ldots,M}.$$

So there are $(NUX)^M$ possible choices for this M-tuple in $\mathbb{Z}^M$. Then if $(X+1)^N > (NUX)^M$ there exists a by the PHP a non-trivial solution to the system of equations. So there exists a solution with $|x_j| \le (NU)^{\frac{M}{N-M}}$. □

Lemma 2.2 (Thue-Siegel for a number field)**.** *Let F be a number field, and consider M variables. Suppose we have a homogeneous linear equation*

$$\sum_{j=1}^{N} a_{ij} x_j = 0,$$

with $N > M$ and $\alpha_{ij} \in \mathcal{O}_F$. Assume that $\overline{|\alpha_{ij}|} \le A$. Then there exists a non-trivial solution with

$$\overline{|x_j|} \le (CNA)^{\frac{M}{N-M}},$$

where the constants only depend on F and nothing else.

Proof. The proof is same as in the rational case. Let $w_1, \ldots, w_d$ be an integral basis for $\mathcal{O}_F$. Write α_{ij} and x_j in terms of $w_1, \ldots, w_d$. Then we have Md equations and Nd variables, and the sizer of the w_i are fixed in terms of F, so we can bound them by CA. Now, apply the Thue-Siegel lemma. □

Definition 2.3 (Entire functions of finite order)**.**
An entire function f is 'finite order' if there exist ρ_0, R_0 such that

$$|f(z)| < \exp\left(|z|^{\rho_0}\right) \text{ whenever } |z| \ge R_0$$

The infimum of ρ_0 is called the 'order' of f and is denoted by $\rho = \rho(f)$.

Example 1. *Here are some functions and their orders:*

(1) e^z, $\rho = 1$
(2) $\sin(z)$, $\rho = 1$
(3) $\cos(\sqrt{z})$, $\rho = 1/2$
(4) e^{e^z}, $\rho = \infty$
(5) e^{z^2}, $\rho = 2$.

Definition 2.4 (Meromorphic function)**.** *A 'meromorphic function' on an open subset D of the complex plane is a function that is holomorphic on all of D except for a set of isolated points, which are poles of the function.*

Definition 2.5. *Let ω_1, ω_2 be two complex numbers which are linearly independent over the reals. Let L be the* ***lattice*** *spanned by ω_1, ω_2. That is,*

$$L = \{m\omega_1 + n\omega_2 \,:\, m, n \in \mathbb{Z}\}.$$

Proposition 2.6 (The Maximum Modulus Principle)**.** *If f is a non-constant analytic function in a region R, then the function $|f|$ does not attain tis maximum in R. In other words if from some $z_0 \in R$, $|f(z)| \le |f(z_0)|$ for all points $z \in R$, then f is constant.*

Definition 2.7. *An **elliptic function** (relative to the lattice L) is a meromorphic function f on $\mathbb{C}$ (thus an analytic map $f : \mathbb{C} \to \mathbb{CP}_1$) which satisfies*

$$f(z+\omega) = f(z)$$

for all $\omega \in L$ and $z \in \mathbb{C}$.

The vaule of such a function can be determined by its value on the *fundamental parallelogram*:

$$D = \{s\omega_1 + t\omega_2 \ : \ 0 \le s,t < 1\}.$$

Any translate of D is referred to as a *fundamental domain* for the elliptic functions relative to L. The set of all elliptic functions (relative to L) forms a field and L is called the *period lattice* or the *lattice of periods*.

Definition 2.8. *The **Weierstrass $\wp$-function** associated with L is defined by the series*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L'} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\},$$

*where L' denotes the set of non-zero periods. The associated **Eisenstein series** of **weight** $2k$ is*

$$G_{2k}(L) := \sum_{\omega \in L'} \omega^{-2k}.$$

For a complex number z with imaginary part $\Im(z) > 0$, let L_z denote the lattice spanned by z and 1. We will denote the corresponding g_2, g_3 associated with L_z by $g_2(z)$ and $g_3(z)$. Thus,

$$g_2(z) = 60 \sum_{(m,n)\neq(0,0)} (mz+n)^{-4},$$

and

$$g_3(z) = 140 \sum_{(m,n)\neq(0,0)} (mz+n)^{-6}.$$

We set

$$\Delta(z) = g_2(z)^3 - 27g_3(z)^3$$

which is the discriminant of the cubic defined by the corresponding Weierstrass equation. We now introduce the important j-function defined as

$$j(z) := 1728 \frac{g_2(z)^3}{g_2(z)^3 - 27g_3(z)^2}$$

which by the previous lemma is well defined for every z in the upper half-plane. the definition of j and using the q expansions for Δ and G_4, we have the following q expansions for the j function:

$$j(z) = \frac{1}{q} + 744 + 196884q + \cdots.$$

Thus the j function has a simple pole at $i\infty$. Since a meromorphic function on a compact Riemann surface has an equal number of zeros as poles, we see that the equation $j(z) = c$ has exactly one solution since j has only a simple pole at $i\infty$. In other words, the j function defines an analytic isomorphism between the compact Riemann surface $\widehat{\mathbb{H}/\Gamma}$ and the Riemann sphere $\mathbb{CP}_1$.

3. Proof of 6-Exponentials Theorem

Proof. We construct an auxiliary function

$$\phi(z) = \sum_{k_1=1}^{K}\sum_{k_2=1}^{K} p(k_1,k_2)e^{k_1\alpha_1 z}e^{k_2\alpha_2 z}.$$

We want to pick the $p(k_1,k_2)$ to be not all zero, lie in $\mathcal{O}_F$, and $\overline{|p(k_1,k_2)|}$ small. So we want $l_1\beta_1 + l_2\beta_2 + l_3\beta_3$, for $1 \le l_1, l_2, l_3 \le L$ to have

$$\phi(l_1\beta_1 + l_2\beta_2 + l_3\beta_3) = 0.$$

We have L^3 equations, and K^2 free variables, so we'll eventually take $K^2 \ge 2L^3$. Evaluating ϕ we will get powers of $e^{\alpha_j\beta_j}$ are algebraic. Clear denominators by multiplying thorugh by, say, D^{6KL}. So $D^{6KL}\phi(z)$ vanishes at the L^3 points $l_1\beta_1 + l_2\beta_2 + l_3\beta_3$. The size of coefficients is C^{KL}, so by the Thue-Siegel lemma for number fields, we can find $p(k_1.k_2)$ with

$$\overline{|p(k_1,k_2)|} \le (C^{KL})^{\frac{L^3}{K^2-L^3}}.$$

Let $K^2 = 2L^3$, then $\overline{|p(k_1,k_2)|} \le C^{KL}$.

Fact. ϕ is not identically zero, since α_1, α_2 are linearly independent over $\mathbb{Q}$.

Fact. ϕ does not vanish on all linear combinations $l_1\beta_1 + l_2\beta_2 + l_3\beta_3$ with $l_1, l_2, l_3 \in \mathbb{N}$. Why? Since $\phi(z)$ is order 1m and can only have about R zeros in a circle of radius R, but it has at least R^3 zeros. So there is a number $s \ge L$ such that ϕ vanishes at all $l_1\beta_1 + l_2\beta_2 + l_3\beta_3$ with $l_j < s$ but doesn't vanish for some chosen $W = s_1\beta_1 + s_2\beta_2 + s_3\beta_3$, with $\max(s_1, s_2, s_3) = s$.

Now look at

$$\frac{\phi(z)}{\prod_{l_1,l_2,l_3<s}(z - l_1\beta_1 - l_2\beta_2 - l_3\beta_3)};$$

let $z = s_1\beta_1 + s_2\beta_2 + s_3\beta_3$, and use maximum modulus principle on some circle $|z| = R$. Then we have

$$|\phi(s_1\beta_1 + s_2\beta_2 + s_3\beta_3)| \le (Cs)^{s^3} \max_{|z|=R} \frac{\phi(z)}{\prod_{l_1,l_2,l_3<s}(z - l_1\beta_1 - l_2\beta_2 - l_3\beta_3)}$$
$$\le \frac{(Cs)^{s^3}}{(R/2)^{s^3}} \max_{|z|=R}|\phi(z)| \le \frac{(Cs)^{s^3}}{(R/2)^{s^3}} C^{KL}\exp(CRK).$$

Choose $R = s^3/K$. Then the above is

$$\le C^{KL}\left(\frac{10CK}{s^2}\right)^{s^3} \le \exp(-cs^3\log s),$$

where we've used $s > L$, $K = 2^{1/2}L^{3/2}$. So if all of it's conjugates are not too big, the usual norm argument will show that it is actually zero. After multiplying bt D^{6KL}, $D^{6KL}\phi(s_1\beta_1 + s_2\beta_2 + s_3\beta_3)$ is an algebraic integer, and by our estimate on $\overline{|p(k_1,k_2)|}$, we have all it's conjugates are $\le C^{KL}\exp(CKs)D^{6sK}$. So ϕ is zero, but not zero. Contradiction. □

Remark. What about 4 exponentials? Then we'd have K^2 free variables, and L^2 equations. So we'd have to take $K = 2L$ in the end, and $s > L$ which would give $\left(\frac{(\cdots)K}{s}\right)^{s^2}$, and barely fail to give the 4 exponentials conjecture.

4. The Schneider-Lang Theorem

Theorem 4.1 (Schneider-Lang). *Let K be a number field, and $f_1, \ldots, f_N$ meromorphic functions of order $< \rho$. Let $f_i = g_i/h_i$, where g,h are holomorphic functions, and their orders are $< \rho$. Consider the ring $K[f_1, \ldots, f_N]$, and assume it satisfies two properties,*

(1) This ring has transcendence degree ≥ 2
(2) $\frac{d}{dz}$ preserves this ring.

Then there are only finitely many $w_1, \ldots, w_m$ where the f_j are simultaneously algebraic. We have $m \leq 20\rho\,[K:\mathbb{Q}]$.

Before the proof, we give some corollaries.

- Take $f_1 = z$, and $f_2 = e^z$. Then there are only finitely many $\alpha \in K$ with $e^\alpha \in K$. But if α has e^α algebraic, then $n\alpha$ is also algebraic for any $n \in \mathbb{N}$. So $e^\alpha \notin \overline{\mathbb{Q}}$ if $\alpha(\neq 0) \in \overline{\mathbb{Q}}$. So we recover a special case of Lindemann-Weierstrass Theorem.
- Let $f_1 = e^z$, $f_2 = e^{\beta z}$, $\beta \in \overline{\mathbb{Q}} - \mathbb{Q}$ with $\beta \in K$. Then we get that there are only finitely many $\alpha \in K$ for which $\alpha^\beta \in K$. But if there is one, there are infinitely many: $\alpha, \alpha^2, \alpha^3, \ldots$, except if $\alpha = 0, 1$, i.e. we have recovered Gelfond-Schneider.
- Let Λ be a lattice, say $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, $\omega_2/\omega_1 \notin \mathbb{R}$. We have the doubly periodic function
$$\wp(z) = \frac{1}{z^2} + \sum_{0\neq\lambda\in\Lambda}\left[\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right].$$
It is meromorphic, and has pole of order 2 at the points of Λ. Then
$$\wp'(z) = \frac{-2}{z^3} - \sum_{0\neq\lambda\in\Lambda}\frac{2}{(z+\lambda)^3}$$
is also meromorphic of order two. We have the relation
$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$
where
$$g_2 = 60G_4 = 60\sum_{\lambda\neq 0}\frac{1}{\lambda^4}, \quad g_3 = 140G_6 = 140\sum_{\lambda\neq 0}\frac{1}{\lambda^6}.$$
Suppose we have a lattice with g_2 and g_3 are algebraic, in some K. Then $K[\wp(z), \wp'(z), z]$ satisfies the conditions of the Schneider-Lang theorem. So there are only finitely many $\alpha \in K$ with $\wp(\alpha), \wp'(\alpha)$ both in K. Suppose we have periods ω_1, ω_2. Then consider $\wp(\omega_1/2)$ and $\wp'(\omega_1/2)$, and suppose that they are algebraic. Siegel prove that the at least one of the two periods are transcendental. Schneider proved both are. If $\omega_1/2$, $\wp(\omega_1/2)$ and $\wp'(\omega_1/2)$ are all algebraic, then $n\omega_1/2$ is also, contradiction Schneider

-Lang. As a consequence, we know that if α is algebraic, then $\wp(\alpha)$ is transcendental.

- The modular j-function.

$$j(\tau) := 1728\frac{g_2^3}{g_2^3 - 27g_3^2},$$

where $\tau = \omega_2/\omega_1$, and thus the lattice is generated by 1 and τ. Consequence: if τ is algebraic and τ is not a quadratic irrationality, then $j(\tau)$ is transcendental.

5. Proof of Schneider-Lang

Proof. Out of $f_1, \dots, f_N$ there are 2 functions which are algebraically independent, say f and g. Use these to construct an auxiliary function

$$\varphi(z) = \sum_{k_1=1}^{K}\sum_{k_2=1}^{K} p(k_1,k_2) f(z)^{k_1} g(z)^{k_2}.$$

The algebraic independence shows that this φ is no identically zero unless all $p(k_1,k_2)$ are zero. We will pick $p(k_1,k_2)$ to be algebraic integers in K with smallish size. Say $z = \omega_1, \dots, \omega_m$ are points where $f_j(\omega_k) \in K$. We want that $\varphi^{(j)}(\omega_j) = 0$ for all $0 \le l \le L$. By the second condition, for any j, f_j' is expressible as a polynomial in the other meromorphic functions, say, $f_j'(z) = P_j(f_1, \dots, f_N)$. There are Lm equations to be satisfied, and K^2 free variables. What happens to the sizer of these quantities when we differentiate a bunch of time? Pick B large which kills all denominators of $f(\omega_j), g(\omega_j), f_j'(\omega_k), \dots$ etc. We want $B^{2K+L}\phi^{(l)}(\omega_j) = 0$. The size of the coefficients is $\le B^K(CK)^L$. So choose $K^2 = 2Lm$. The Thue-Siegel lemma applies, and we find $p(k_1,k_2)$ with $\overline{|p(k_1,k_2)|} \le \exp(L\log L)$. Pick s to be the smallest number such that $\varphi^{(s+1)}(\omega) \neq 0$ for some $\omega = \omega_1, \dots, \omega_n$, but all smaller derivatives are zero. By construction, $s \ge L$.

Look at

$$\frac{\varphi(z)\Theta(z)^{2k}}{\left((z-\omega_1)\cdots(z-\omega_s)\right)^{s+j}}$$

where $\Theta(z)$ is a holomorphic function of order $< \rho$ such that $f(z)\Theta(z)$ and $g(z)\Theta(z)$ are holomorphic. Then above fraction is an entire function of order $< \rho$. Apply the maximum modulus principle using a circle of big radius R to be chosen later. Evaluate at $z = w$. Then

$$\left|\frac{\varphi^{(s+1)}(\omega)\Theta(\omega)^{2k}}{\prod_{\omega_j\neq\omega}(\omega-\omega_j)^{s+1}(s+1)!}\right| \le \max_{|z|=R} \frac{\varphi(z)\Theta(z)^{2k}}{\left((z-\omega_1)\cdots(z-\omega_s)\right)^{s+j}}$$
$$\le \exp(CKR^{\rho} + L\log L - sm\log R/2),$$

where in the last equality, the three terms come from Θ, φ and the denominator, respectively.

Recall we have $K^2 = 2Lm$ and $s \ge L$, so the optimal value of R is $C\rho KR^{\rho-1} = \frac{sm}{R}$. So $R = \left(\frac{sm}{k}\right)^{\frac{1}{\rho}}$. So the bound is

$$\le \exp\left(L\log L - \frac{sm}{\rho}\log\frac{sm}{10K}\right)$$

. Conclusion:

$$|\phi^{(s+1)}(\omega)| \le \exp\left(2s\log s - \frac{sm}{\rho}\log\frac{sm}{10K}\right)$$

By multiplying $\varphi^{(s+1)}(\omega)$ by a suitable B^{s+2K}, we get an algebraic integer which is $\le \exp(L\log L + Cs)$, and we derive a contradiction by a norm calculation. The norm calculation implies $|\phi^{(s+1)}(\omega)| \ge \exp(-L\log L - dCs)$, where $d = [K:\mathbb{Q}]$. So if $m = 20\rho[K:\mathbb{Q}]$, we get the desired contradiction. □

6. Conclusion

Let us consider the following interesting example by letting

$$\alpha = \frac{1+\sqrt{-163}}{2}.$$

The field $\mathbb{Q}(\sqrt{-163})$ has class number one. In fact it is the “largest” imaginary quadratic field with class number one. More precisely, there exists no square-free integer $d > 163$ such that $\mathbb{Q}(\sqrt{-d})$ has class number one.
Now for any z in the upper half-plane, the j-function has the following expansion

$$j(z) = \frac{1}{q} + 744 + 196884q + \cdots$$

where $q = e^{2\pi iz}$. In the case $z = \alpha$, we have

$$j(\alpha) = -e^{-\pi\sqrt{163}} + 744 - 196,884e^{-\pi\sqrt{163}} + \cdots$$

Now $j(\alpha)$ must be an ordinary integer as $\mathbb{Q}(\sqrt{-163})$ has class number one.

Consequently, we have the following expression

$$\begin{aligned} e^{\pi\sqrt{163}} &= 262,537,412,640,768,743.99999999999925 \\ &= (640,320)^3 + 744 + O\left(e^{-\pi\sqrt{163}}\right) \end{aligned}$$

and that $j(\alpha) = -(640,320)^3$. Note that $e^{\pi\sqrt{163}}$ is a transcendental number by the Gelfond-Schneider Theorem.

The observation that $\mathbb{Q}(\sqrt{-163})$ is the largest imaginary quadratic field with class number one is deeply connected to the Stark-Heegner theorem, sometimes referred to as the Stark-Heegner-Baker theorem after the work of Stark (1967), Heegner (1952), and Baker (1971).

- Heegner (1952) was the first to prove that there are precisely nine values of d for which $\mathbb{Q}(\sqrt{-d})$ has class number one, although his proof was not widely accepted at first.
- Stark (1967) later provided a rigorous verification of Heegner’s argument.
- Baker (1971), using his celebrated theory of linear forms in logarithms, independently proved the result while simultaneously developing powerful methods for proving effective results in transcendence theory.

The case $d = 163$ is particularly striking because of its connection to the near-integer property of $e^{\pi\sqrt{163}}$, as seen above. This near-integer phenomenon is explained by the rapid decay of the exponential term $e^{-\pi\sqrt{163}}$ in the expansion of $j(\alpha)$.

The theorem states that the only square-free values of d for which $\mathbb{Q}(\sqrt{-d})$ has class number one are:

$$d \in \{1,2,3,7,11,19,43,67,163\}.$$

The case $d = 163$ is particularly striking because of its connection to the near-integer property of $e^{\pi\sqrt{163}}$, as seen above. This near-integer phenomenon is explained by the rapid decay of the exponential term $e^{-\pi\sqrt{163}}$ in the expansion of $j(\alpha)$.

Remark. The interplay between algebraic number theory, modular functions, and transcendence theory-exemplified by the Stark-Heegner-Baker theorem and the near-integer property of $e^{\pi\sqrt{163}}$-demonstrates the deep and beautiful structure underlying class field theory and the theory of modular forms. Moreover, Baker's work in transcendence theory has had profound implications, leading to explicit diophantine approximations and results concerning the linear independence of logarithms of algebraic numbers.

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