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New two-mode of second and third order Clannish Random Walker's Parabolic equations: $\left(\frac{G''}{G'+A}\right)$-expansion method

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Abstract

We develop two new equations which describe propagation of two different wave modes simultaneously. The first equation is a two-mode Clannish Random Walker's Parabolic equation (CRWPE), and the second is a two-mode third-order Clannish Random Walker's Parabolic equation (TCRWPE). We will use a new method, namely, the $\left(\frac{G''}{G'+A}\right)$-expansion method to conduct this analysis.

2000 Mathematics Subject Classification. Primary 65L05,65D25; Secondary 49M25, 90C05.

Key Words: $\left(\frac{G''}{G'+A}\right)$-expansion method,two-mode Clannish Random Walker's Parabolic equation, two-mode third-order Clannish Random Walker's Parabolic equation, soliton solution, traveling wave solutions.

1 Introduction

Two-mode type is a new family of nonlinear partial differential equations (PDEs) which fall in the following form: [1, 2]

$$u_{tt}-s^2u_{xx}+\left(\frac{\partial}{\partial t}-\alpha s\frac{\partial}{\partial x}\right)N\left(u,u_x,...\right)+\left(\frac{\partial}{\partial t}-\beta s\frac{\partial}{\partial x}\right)L\left(u_{rx},...\right)=0, \quad (1)$$

where $N(u, u_x, ...)$ and $L(u_{rx}, r \geq 2)$ represent the nonlinear terms and the linear terms of the examined equation. $u(x,t)$ is the unknown field-function, $s > 0$ is the phase velocity, $|\beta| \leq 1, |\alpha| \leq 1, \beta$ is the dispersion parameter and α is the parameter of nonlinearity. With $s = 0$ and integrating with respect to t, the dual-mode problem is reduced to a (PDE) of the first order in time t.

In [3, 4, 5], the focusing (CRWPE) hierarchy in $(1+1)$-dimensions was given in the form

$$u_t + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + u - 1\right)^n u = 0, n = 0, 1, 2, ... \quad (2)$$

The first few elements of the hierarchies Eq. (2) are given by

$$u_t + u_x = 0, \quad (3)$$

$$u_t + u_{2x} + 2uu_x - u_x = 0, \quad (4)$$

$$u_t + u_{3x} - 2u_{2x} + u_x + 3(u_x)^2 - 4uu_x + 3uu_{2x} + 3u^2u_x = 0, \quad (5)$$

$$\begin{aligned} u_t + u_{4x} - 3u_{3x} + 3u_{2x} - u_x - 9(u_x)^2 + 6uu_x - 9uu_{2x} + 4uu_{3x} + \\ 10u_xu_{2x} + 12u(u_x)^2 - 9u^2u_x + 6u^2u_{2x} + 4u^3u_x = 0, \end{aligned} \quad (6)$$

$$\begin{aligned} u_t + u_{5x} - 4u_{4x} + 6u_{3x} - 4u_{xx} + u_x + 15(u_x)^3 + 18(u_x)^2 + 10(u_{xx})^2 \\ -8uu_x + 18uu_{2x} - 16uu_{3x} + 5uu_{4x} - 40u_xu_{2x} + 15u_xu_{3x} \\ +50uu_xu_{2x} - 48u(u_x)^2 + 30u^2(u_x)^2 + 18u^2u_x - 24u^2u_{2x} \\ +10u^2u_{3x} - 16u^3u_x + 10u^3u_{2x} + 5u^4u_x = 0, \end{aligned} \quad (7)$$

obtained by substituting $n = 0, 1, 2, 3, 4$, where $u_{rx} = \frac{\partial^r u}{\partial x^r}$, $u(x,t)$ denotes the unknown function depending on the temporal variable t and the spatial variable x. The resulting (PDEs) are of first order, second order, third order, fourth order and fifth order respectively, where other equations of higher order can be obtained by substituting $n \geq 5$.

A few novel of nonlinear (PDEs) known as two-mode or dual-mode has just been described. Researchers have been exploring this problem and have discovered two-mode nonlinear (PDEs), such as: two-mode (tm) mKdV [6, 7], tm KdV [8], tm Sharma–Tasso–Olver [9], tm fifth order KdV [10, 11], two-mode Burger equation (tmBE) [12], tm perturbed Burger (tmPB) [13], tm KdV Burgers (tmKdVB) [14], tm Kadomtsev Petviashvili (tmKP) [15], two-mode dispersive Fisher (tmdF) [16], tm Kuramoto–Sivashinsky (tmKS) [17], tm Boussinesq Burgers (tmBB) [18], two-mode coupled KdV (tmKdV) and mKdV (tm- CmKdV) [19, 20], two-mode nonlinear Schrodinger (tmNLS) [21], and tm Hirota Satsuma coupled KdV(tmHSKdV) [22], equations and different analytical methodologies are used to create the dual-wave solutions. Among these techniques are: Exp-function method [23], the $\left(\frac{G'}{G}\right)$-expansion method [24, 25], the $\left(\frac{1}{G'}\right)$-expansion method [26, 27], the $\left(\frac{G'}{G^2}\right)$-expansion method [28], the extended hyperbolic function method [29], the tanh-coth method [30, 31], the double $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method [32]-[34], the Jacobi elliptic function expansion method [35], the mapping method [36], the sine-cosine method [37].

Our aim is to study the systems proposed above. Moreover, we will determine travelling wave solutions by using the new $\left(\frac{G''}{G'+A}\right)$-expansion method. The computer symbolic system Maple will be used to perform the computational work.

2 The New $\left(\frac{G''}{G'+A}\right)$-Expansion Method

In this section, we describe our new method, namely, the $\left(\frac{G''}{G'+A}\right)$ expansion method for finding travelling wave solutions of nonlinear (PDEs). Consider the general nonlinear (PDEs), say, in two variables,

$$P\left(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, ...\right) = 0, \tag{8}$$

where $v = v(x,t)$ is an unknown function, P is a polynomial in $v(x,t)$ and the subscripts stand for the partial derivatives.

We suppose that the combination of real variables x and t by a complex variable ξ

$$v(x,t) = v(\xi), \quad \xi = ax - ct, \tag{9}$$

where a is the wave number and c is the speed of the traveling wave. Now using Eq. (9), Eq. (8) is converted into an ordinary differential equation (ODE) for $v = v(\xi)$:

$$F\left(v, -cv', av', c^2v'', -cav'', a^2v'', ...\right) = 0,' \equiv \frac{d}{d\xi}. \tag{10}$$

Suppose that the traveling wave solution of Eq. (10) can be expressed as follows:

$$v\left(\xi\right) = \sum_{i=0}^{N} a_i \left(\frac{G''}{G' + A}\right)^i, \tag{11}$$

where the coefficients $a_i(i = 0, 1, 2, ..., N), a$ and c are arbitrary constants, and $G = G(\xi)$ satisfies the following auxiliary (ODE)

$$G''' + \mu G' + \lambda = 0, \tag{12}$$

then by the help of Eq. (12) we get

$$\left(\frac{G''}{G' + A}\right)' = -\left(\frac{G''}{G' + A}\right)^2 - \mu, \tag{13}$$

where $\lambda = A\mu$; and A are constants, the positive integer N can be determined by using homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (10).

Substituting Eq. (11) into Eq. (10), using Eq. (13) repeatedly, and setting the coefficients of the each order of $\left(\frac{G''}{G'+A}\right)^i$ to zero, we obtain a set of nonlinear algebraic equations for $a_i(i = 0, 1, 2, ..., N), a, c$ and μ. With the aid of the computer program Maple, we can solve the set of nonlinear algebraic equations and obtain all the constants $a_i(i = 0, 1, 2, ..., N), a$ and c.

Using the general solution of Eq. (12), we have the following solutions

Family 1. When $\mu < 0$,

$$\left(\frac{G''}{G' + A}\right) = \frac{\sqrt{-\mu}\left(C_1 \cosh\left(\sqrt{-\mu}\left(\xi + h\right)\right) + C_2 \sinh\left(\sqrt{-\mu}\left(\xi + h\right)\right)\right)}{C_1 \sinh\left(\sqrt{-\mu}\left(\xi + h\right)\right) + C_2 \cosh\left(\sqrt{-\mu}\left(\xi + h\right)\right)},$$

case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$\left(\frac{G''}{G' + A}\right) = \sqrt{-\mu} \tanh\left(\sqrt{-\mu}\left(\xi + h\right)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

$$\left(\frac{G''}{G'+A}\right) = \sqrt{-\mu}\coth\left(\sqrt{-\mu}\left(\xi+h\right)\right).$$

Family 2. When $\mu > 0$,

$$\left(\frac{G''}{G'+A}\right) = \frac{\sqrt{\mu}\left(C_1\cos\left(\sqrt{\mu}\left(\xi+h\right)\right) + C_2\sin\left(\sqrt{\mu}\left(\xi+h\right)\right)\right)}{C_1\sin\left(\sqrt{\mu}\left(\xi+h\right)\right) - C_2\cos\left(\sqrt{\mu}\left(\xi+h\right)\right)},$$

case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$\left(\frac{G''}{G'+A}\right) = -\sqrt{\mu}\tan\left(\sqrt{\mu}\left(\xi+h\right)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

$$\left(\frac{G''}{G'+A}\right) = \sqrt{\mu}\cot\left(\sqrt{\mu}\left(\xi+h\right)\right).$$

Where h is constant of integration.

3 Formulation of the two-mode equations

To establish the two-mode (CRWPE), we first rewrite the (CRWPE) Eq. (4) with dual nonlinear terms as

$$u_t + \left(u^2 - u\right)_x + u_{2x} = 0, \tag{14}$$

where

$$\begin{aligned} N\left(u, u_x, ...\right) &= \left(u^2 - u\right)_x, \\ L\left(u_{rx}\right) &= u_{2x}. \end{aligned} \tag{15}$$

We next combine the sense of Korsunsky [1], as proposed in Eq. (1), and the structure of the (CRWPE) Eq. (14), to propose the nonlinear dispersive equation,

$$u_{tt} - s^2 u_{xx} + \left(\frac{\partial}{\partial t} - \alpha s\frac{\partial}{\partial x}\right)\left\{\left(u^2 - u\right)_x\right\} + \left(\frac{\partial}{\partial t} - \beta s\frac{\partial}{\partial x}\right)\left(u_{2x}\right) = 0, \tag{16}$$

α is the nonlinearity parameter, β is the dispersive parameter, and s is the phase velocity. We next proceed to establish a two wave mode (TCRWPE). In a manner parallel to the analysis presented earlier, we formulate a two-mode (TCRWPE). We first rewrite Eq. (5) with four nonlinear terms and one linear term as

$$u_t + \left(u^3 - 2u^2 + u + 3uu_x\right)_x - 2u_{2x} + u_{3x} = 0, \tag{17}$$

where

$$\begin{aligned} N\left(u, u_x, ...\right) &= \left(u^3 - 2u^2 + u + 3uu_x\right)_x, \\ L\left(u_{rx}\right) &= u_{3x} - 2u_{2x}. \end{aligned} \tag{18}$$

We next combine the sense of Korsunsky [1], as proposed in Eq. (1), and the structure of the (CRWPE) Eq. (17), to propose the nonlinear dispersive equation,

$$u_{tt} - s^2 u_{xx} + \left(\frac{\partial}{\partial t} - \alpha s\frac{\partial}{\partial x}\right)\left(u^3 - 2u^2 + u + 3uu_x\right)_x + \left(\frac{\partial}{\partial t} - \beta s\frac{\partial}{\partial x}\right)\left(u_{3x} - 2u_{2x}\right) = 0, \tag{19}$$

α is the nonlinearity parameter, β is the dispersive parameter, and s is the phase velocity.

3.1 The two-mode (CRWPE)

In this section, we aim first to apply our new method for solve Eq. (16).

Substituting $u(x,t) = u(\xi), \xi = ax - ct$ in Eq. (16) and integrating twice gives

$$\left(c^2 - a^2s^2 + ac + a^2s\alpha\right)u(\xi) - \left(ac + a^2s\alpha\right)\left(u(\xi)\right)^2 - a^2\left(c + as\beta\right)\frac{d}{d\xi}u(\xi) + k = 0. \tag{20}$$

Balancing the order of the nonlinear term u^2 with the highest derivative u' gives $2N = N + 1$ that gives $N = 1$.

By the use of Eq. (11), we present the solution of Eq. (20) as:

$$u(\xi) = a_0 + a_1\left(\frac{G''}{G' + A}\right). \tag{21}$$

Substituting Eq. (21) in Eq. (20) and using Eq. (13), collecting the coefficients of each power of $\left(\frac{G''}{G'+A}\right)^i$, $0 \leq i \leq 2$, setting each coefficient to zero, and solving the algebraic equations by Maple we get,

$$\begin{aligned}
a_0 &= \frac{a^2\left(-\beta^2 s-\alpha+\beta+s\right)+2aa_1\left(s\alpha\beta-s-\frac{1}{2}\beta+\frac{1}{2}\alpha\right)+sa_1^2\left(1-\alpha^2\right)}{2a\left(\beta-\alpha\right)\left(a-a_1\right)},\\
a_1 &= a_1, c=-\frac{as\left(a\beta-\alpha a_1\right)}{a-a_1}.
\end{aligned}$$

Using Eq. (21), in the solutions of Eq. (12), we get

Family 1. When $\mu < 0$,

$$u_1\left(\xi\right)=a_0+\frac{a_1\sqrt{-\mu}\left(C_1\cosh\left(\sqrt{-\mu}\left(\xi+h\right)\right)+C_2\sinh\left(\sqrt{-\mu}\left(\xi+h\right)\right)\right)}{C_1\sinh\left(\sqrt{-\mu}\left(\xi+h\right)\right)+C_2\cosh\left(\sqrt{-\mu}\left(\xi+h\right)\right)},$$

case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$u_2\left(\xi\right)=a_0+a_1\sqrt{-\mu}\tanh\left(\sqrt{-\mu}\left(\xi+h\right)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

$$u_3\left(\xi\right)=a_0+a_1\sqrt{-\mu}\coth\left(\sqrt{-\mu}\left(\xi+h\right)\right).$$

Family 2. When $\mu > 0$,

$$u_4\left(\xi\right)=a_0+\frac{a_1\sqrt{\mu}\left(C_1\cos\left(\sqrt{\mu}\left(\xi+h\right)\right)+C_2\sin\left(\sqrt{\mu}\left(\xi+h\right)\right)\right)}{C_1\sin\left(\sqrt{\mu}\left(\xi+h\right)\right)-C_2\cos\left(\sqrt{\mu}\left(\xi+h\right)\right)},$$

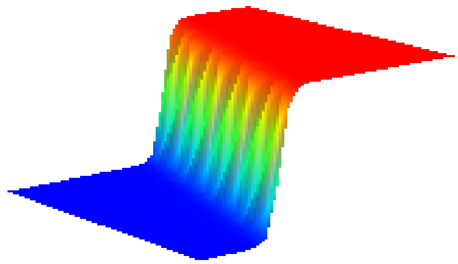
case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$u_5\left(\xi\right)=a_0-a_1\sqrt{\mu}\tan\left(\sqrt{\mu}\left(\xi+h\right)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

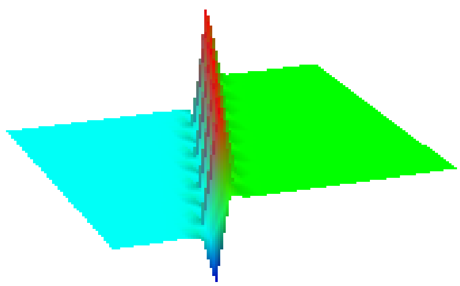
$$u_6\left(\xi\right)=a_0+a_1\sqrt{\mu}\cot\left(\sqrt{\mu}\left(\xi+h\right)\right),$$

where $a_0 = \frac{a^2\left(-\beta^2 s-\alpha+\beta+s\right)+2aa_1\left(s\alpha\beta-s-\frac{1}{2}\beta+\frac{1}{2}\alpha\right)+sa_1^2\left(1-\alpha^2\right)}{2a(\beta-\alpha)(a-a_1)}$,
$\xi = ax+\left(\frac{as(a\beta-\alpha a_1)}{a-a_1}\right)t$.



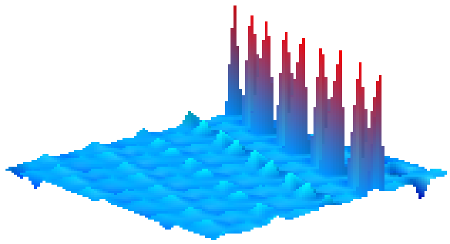
$\mu < 0$

Figure 1. Above of figure are represents kink soliton solution of the exact traveling wave solution of $u_2(x,t)$ for the parameter
$(h = \beta = s = a_1 = 1, c_1 = 1, c_2 = a = 2$ and $\alpha = 0.5)$.



$\mu < 0$

Figure 2. Above of figure are represents singular-kink soliton solution of the exact traveling wave solution of $u_3(x,t)$ for the parameter
$(h = \beta = s = a_1 = 1, c_1 = 1, c_2 = a = 2$ and $\alpha = 0.5)$.



$\mu > 0$

Figure 3. Above of figure are represents singular-kink soliton solution of the exact traveling wave solution of $u_5(x,t)$ for the parameter
$(h = \beta = s = a_1 = 1, c_1 = 1, c_2 = a = 2 \text{ and } \alpha = 0.5)$.

3.2 The two-mode (TCRWPE)

In this section, we apply the our new method for solve Eq. (19).

Substituting $u(x,t) = u(\xi), \xi = ax - ct$ in Eq. (19) and integrating twice gives

$$\left(c^2 - a^2 s^2 - ac - a^2 s\alpha\right) u(\xi) - \left(ac + a^2 s\alpha\right) \left((u(\xi))^3 - 2\,(u(\xi))^2 + 3au(\xi)\tfrac{d}{d\xi}u(\xi)\right) \\ +a^2\,(c + as\beta) \left(2\tfrac{d}{d\xi}u(\xi) - a\tfrac{d^2}{d\xi^2}u(\xi)\right) + k = 0 \tag{22}$$

Balancing the order of the nonlinear term u^3 with the highest derivative u'' gives $3N = N + 2$ that gives $N = 1$.

By the use of Eq. (11), we present the solution of Eq. (22) as:

$$u(\xi) = a_0 + a_1 \left(\frac{G''}{G' + A}\right). \tag{23}$$

Substituting Eq. (23) in Eq. (22) and using Eq. (13), collecting the coefficients of each power of $\left(\frac{G''}{G'+A}\right)^i$, $0 \leq i \leq 3$, setting each coefficient to zero, and solving the algebraic equations by Maple we find that solution exists if $\beta = \alpha$, *set* (**1**).

$$a_0 = \tfrac{2}{3}, a_1 = 2a,$$

$$c = a\left(-2\mu a^2 - \tfrac{1}{6} + \tfrac{1}{6}\sqrt{144\mu^2 a^4 - 144\alpha\mu s a^2 + 24a^2\mu - 12s\alpha + 36s^2 + 1}\right).$$

set (**2**).

$$a_0 = \tfrac{2}{3}, a_1 = a,$$

$$c = a\left(-\tfrac{1}{2}\mu a^2 - \tfrac{1}{6} + \tfrac{1}{6}\sqrt{9\mu^2 a^4 - 36\alpha\mu s a^2 + 6a^2\mu - 12s\alpha + 36s^2 + 1}\right).$$

set (**3**).

$$\begin{aligned} a_0 &= \frac{2s\alpha a^2 + 2ca}{3a(as\alpha + c)} \\ &\quad + \frac{\sqrt{3}\sqrt{a(as\alpha + c)\left(\alpha s\mu a^4 + c\mu a^3 - sa^2\left(s - \frac{1}{3}\alpha\right) + \frac{1}{3}ca + c^2\right)}}{3a(as\alpha + c)}, \\ a_1 &= a, c = c. \end{aligned}$$

Putting the *set* (**1**). into Eq. (23), we get

Family 1. When $\mu < 0$,

$$u_1(\xi) = \frac{2}{3} + \frac{2a\sqrt{-\mu}\left(C_1 \cosh\left(\sqrt{-\mu}(\xi + h)\right) + C_2 \sinh\left(\sqrt{-\mu}(\xi + h)\right)\right)}{C_1 \sinh\left(\sqrt{-\mu}(\xi + h)\right) + C_2 \cosh\left(\sqrt{-\mu}(\xi + h)\right)},$$

case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$u_2(\xi) = \frac{2}{3} + 2a\sqrt{-\mu}\tanh\left(\sqrt{-\mu}(\xi + h)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

$$u_3(\xi) = \frac{2}{3} + 2a\sqrt{-\mu}\coth\left(\sqrt{-\mu}(\xi + h)\right).$$

Family 2. When $\mu > 0$,

$$u_4(\xi) = \frac{2}{3} + \frac{2a\sqrt{\mu}\left(C_1 \cos\left(\sqrt{\mu}(\xi + h)\right) + C_2 \sin\left(\sqrt{\mu}(\xi + h)\right)\right)}{C_1 \sin\left(\sqrt{\mu}(\xi + h)\right) - C_2 \cos\left(\sqrt{\mu}(\xi + h)\right)},$$

case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$u_5(\xi) = \frac{2}{3} - 2a\sqrt{\mu}\tan\left(\sqrt{\mu}(\xi + h)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

$$u_6(\xi) = \frac{2}{3} + 2a\sqrt{\mu}\cot\left(\sqrt{\mu}(\xi + h)\right),$$

where $\xi = ax - a\left(-2\mu a^2 - \frac{1}{6} + \frac{1}{6}\sqrt{144\mu^2 a^4 - 144\alpha\mu s a^2 + 24a^2\mu - 12s\alpha + 36s^2 + 1}\right)t$.
Putting the *set* (**2**). into Eq. (23), we get

Family 1. When $\mu < 0$,

$$u_7(\xi) = \frac{2}{3} + \frac{a\sqrt{-\mu}\left(C_1\cosh\left(\sqrt{-\mu}(\xi + h)\right) + C_2\sinh\left(\sqrt{-\mu}(\xi + h)\right)\right)}{C_1\sinh\left(\sqrt{-\mu}(\xi + h)\right) + C_2\cosh\left(\sqrt{-\mu}(\xi + h)\right)},$$

case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$u_8(\xi) = \frac{2}{3} + a\sqrt{-\mu}\tanh\left(\sqrt{-\mu}(\xi + h)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

$$u_9(\xi) = \frac{2}{3} + a\sqrt{-\mu}\coth\left(\sqrt{-\mu}(\xi + h)\right).$$

Family 2. When $\mu > 0$,

$$u_{10}(\xi) = \frac{2}{3} + \frac{a\sqrt{\mu}\left(C_1\cos\left(\sqrt{\mu}(\xi + h)\right) + C_2\sin\left(\sqrt{\mu}(\xi + h)\right)\right)}{C_1\sin\left(\sqrt{\mu}(\xi + h)\right) - C_2\cos\left(\sqrt{\mu}(\xi + h)\right)},$$

case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$u_{11}(\xi) = \frac{2}{3} - a\sqrt{\mu}\tan\left(\sqrt{\mu}(\xi + h)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

$$u_{12}(\xi) = \frac{2}{3} + a\sqrt{\mu}\cot\left(\sqrt{\mu}(\xi + h)\right),$$

where $\xi = ax - a\left(-\frac{1}{2}\mu a^2 - \frac{1}{6} + \frac{1}{6}\sqrt{9\mu^2 a^4 - 36\alpha\mu s a^2 + 6a^2\mu - 12s\alpha + 36s^2 + 1}\right)t$.
Putting the *set* (**3**). into Eq. (23), we get

Family 1. When $\mu < 0$,

$$u_{13}(\xi) = a_0 + \frac{a\sqrt{-\mu}\left(C_1 \cosh\left(\sqrt{-\mu}(\xi+h)\right) + C_2 \sinh\left(\sqrt{-\mu}(\xi+h)\right)\right)}{C_1 \sinh\left(\sqrt{-\mu}(\xi+h)\right) + C_2 \cosh\left(\sqrt{-\mu}(\xi+h)\right)},$$

case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$u_{14}(\xi) = a_0 + a\sqrt{-\mu}\tanh\left(\sqrt{-\mu}(\xi+h)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

$$u_{15}(\xi) = a_0 + a\sqrt{-\mu}\coth\left(\sqrt{-\mu}(\xi+h)\right).$$

Family 2. When $\mu > 0$,

$$u_{16}(\xi) = a_0 + \frac{a\sqrt{\mu}\left(C_1 \cos\left(\sqrt{\mu}(\xi+h)\right) + C_2 \sin\left(\sqrt{\mu}(\xi+h)\right)\right)}{C_1 \sin\left(\sqrt{\mu}(\xi+h)\right) - C_2 \cos\left(\sqrt{\mu}(\xi+h)\right)},$$

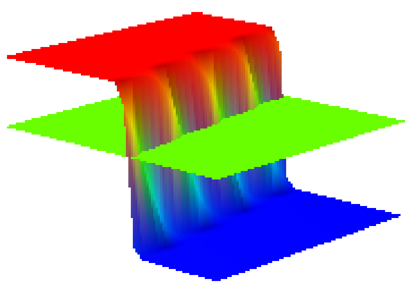
case (i). If $C_2 \neq 0$, $C_1 = 0$,

$$u_{17}(\xi) = a_0 - a\sqrt{\mu}\tan\left(\sqrt{\mu}(\xi+h)\right),$$

case (ii). If $C_1 \neq 0$, $C_2 = 0$,

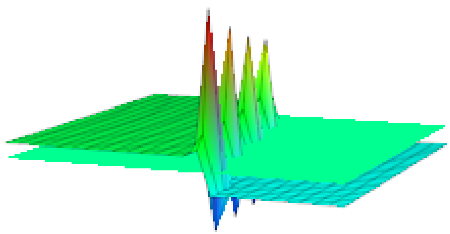
$$u_{18}(\xi) = a_0 + a\sqrt{\mu}\cot\left(\sqrt{\mu}(\xi+h)\right),$$

where $a_0 = \frac{2s\alpha a^2 + 2ca + \sqrt{3}\sqrt{a(as\alpha+c)\left(\alpha s\mu a^4 + c\mu a^3 - sa^2\left(s - \frac{1}{3}\alpha\right) + \frac{1}{3}ca + c^2\right)}}{3a(as\alpha+c)}$,
$\xi = ax - ct$.



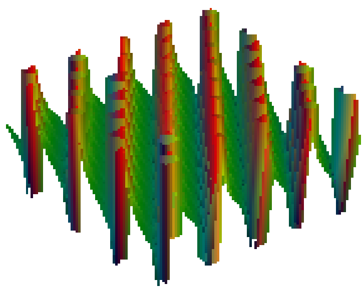
$\mu < 0$

Figure 4. Above of figure are represents kink soliton solution of the exact traveling wave solution of $u_2(x,t)$ for the parameter $(h=a=1, s=3$ and $\alpha=0.5)$.



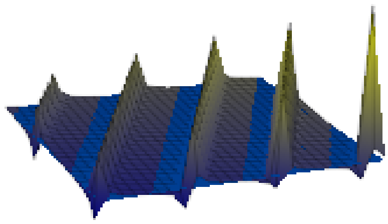
$\mu<0$

Figure 5. Above of figure are represents singular-kink soliton solution of the exact traveling wave solution of $u_3(x,t)$ for the parameter $(h=a=1, s=3$ and $\alpha=0.5)$.



$\mu>0$

Figure 6. Above of figure are represents singular-periodic soliton solution of the exact traveling wave solution of $u_6(x,t)$ for the parameter $(h=a=1, s=3$ and $\alpha=0.5)$.



$\mu > 0$

Figure 7. Above of figure are represents singular-periodic soliton solution of the exact traveling wave solution of $u_{17}(x,t)$ for the parameter $(h = a = \alpha = 1 \text{ and } s = 5)$.

4 Conclusion

In this work, we established two wave mode equations, namely the two-mode (CRWPE) and the two mode (TCRWPE), which we believe that these two equations are introduced to the first time.

The new $\left(\frac{G''}{G'+A}\right)$- expansion method has been successfully implemented to find the new two traveling waves solutions. The results show that this method is a powerful Mathematical tool for obtaining exact solutions for our equations. It is also a promising method to solve other nonlinear (PDEs).

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