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LANCZOS DERIVATIVE VIA RUFFA-TONI'S EXPRESSION FOR A DEFINITE INTEGRAL AND SOLUTION OF NON-HOMOGENEOUS DOUBLE FREDHOLM INTEGRAL EQUATIONS

M. A. PATHAN, H. KUMAR, T. KIM, AND J. LÓPEZ-BONILLA

Abstract. An exact series representation of a definite integral is studied in the literature. This is an important result for it may be used to provide the series representation of a typical definite integral which allows a deduction of the Lanczos derivative [classical differentiation via integration]. Further we use Lanczos derivative techniques to solve the nonhomogeneous single and double Fredholm integral equations.

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1. Introduction

Ruffa-Toni [12] has given an exact series representation of a definite integral. In this paper, we prove that it is applicable in a deduction of the Lanczos derivative [classical differentiation via integration].

Ruffa-Toni [12] obtained the following interesting formula for a definite integral:

$$\int_a^b g(t)dt = (b-a)\sum_{n=1}^{\infty} 2^{-n} \sum_{m=1}^{2^n-1} (-1)^{m+1} g\left(a + m(b-a)2^{-n}\right) \tag{1}$$

In (1), we apply for the case $g(t) = tf(t+x_0), b = -a = \epsilon$ to deduce the Lanczos derivative[1, 2, 3, 4, 8, 9, 10, 14, 16] :

$$f'(x_0) = \lim_{\epsilon\to 0} \frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} tf(t+x_0)\, dt \tag{2}$$

that is, classical differentiation via integration.

In addition, we present an evaluation of Lanczos derivative techniques for solving the Fredholm integral equations.

2. Lanczos formula

From (1):

(3)

$$\int_{-\epsilon}^{\epsilon} tf(t+x_0)\, dt = \epsilon^2 \sum_{n=1}^{\infty} 2^{1-n} \sum_{m=1}^{2^n-1} (-1)^{m+1} Q_{mn} f(x_0 + \epsilon Q_{mn}), \quad Q_{mn} = m2^{1-n} - 1,$$

and now we shall accept that $\epsilon \ll 1$ to apply in (3) the Taylor expansion:

$$f\left(x_0+\epsilon Q_{mn}\right)=f\left(x_0\right)+\epsilon Q_{mn} f'\left(x_0\right)+O\left(\epsilon^2\right) \tag{4}$$

then (3) implies the expression:

$$\int_{-\epsilon}^{\epsilon} t f\left(t+x_0\right) dt=\epsilon^2 A f\left(x_0\right)+\epsilon^3 B f'\left(x_0\right)+O\left(\epsilon^4\right), \tag{5}$$

such that:

$$A=\sum_{n=1}^{\infty} 2^{1-n} \sum_{m=1}^{2^n-1}(-1)^{m+1} Q_{mn}, \quad B=\sum_{n=1}^{\infty} 2^{1-n} \sum_{m=1}^{2^n-1}(-1)^{m+1} Q_{mn}^2. \tag{6}$$

It is easy to obtain the values:

(7)

$$\sum_{m=1}^{2^n-1}(-1)^{m+1}=1, \quad \sum_{m=1}^{2^n-1}(-1)^{m+1} m=2^{n-1}, \quad \sum_{m=1}^{2^n-1}(-1)^{m+1} m^2=2^{n-1}\left(2^n-1\right),$$

which gives $A=0, B=\frac{2}{3}$ and therefore (5) implies (2), q.e.d.

Remark: Rangarajan-Purushothaman [11] used the Legendre polynomials [13, 15] to study the Lanczos derivative for higher orders, thus the following relation generalizes the expression (2):

$$f^{(j)}\left(x_0\right)=\lim_{\epsilon \rightarrow 0} \frac{(2j+1)!}{2\epsilon^{j+1}} \int_{-\epsilon}^{\epsilon} P_j\left(\frac{t}{\epsilon}\right) f\left(t+x_0\right) dt, \quad j=1,2, \ldots \tag{8}$$

$\int_{-\epsilon}^{\epsilon} t f\left(t+x_0\right) dt=\epsilon^2 \sum_{n=1}^{\infty} 2^{1-n} \sum_{m=1}^{2^n-1}(-1)^{m+1} Q_{mn} f\left(x_0+\epsilon Q_{mn}\right), \quad Q_{mn}=m 2^{1-n}-1.$

3. Lanczos derivative and solution of the non-homogeneous Fredholm integral equations

The Fredholm integral equations are solved by the theory of separable kernels, method of successive approximations, classical Fredholm theory and symmetric kernels (see in [7]). Here we use the Lanczos theory to solve the nonhomogeneous Fredholm integral equations.

Theorem 3.1. *If there exists the constants*

(9)

$$C_q=\int_a^b \varphi_q(t) u(t) dt \forall q=1,2, \ldots, \text{ where, } \varphi_q(t)=\lim_{x \rightarrow 0} \frac{\partial^q}{\partial x^q} K(x, t) \text{ and } \int_a^b \psi_q(t) dt=D_q.$$

Again consider that

$$\{u(t) \psi_1(t)\}_{t=a}^{t=b}=\sigma_1,\left\{\frac{\partial u}{\partial t} \psi_2(t)\right\}_{t=a}^{t=b}=\sigma_2, \ldots,\left\{\frac{\partial^{q-1} u}{\partial t^{q-1}} \psi_q(t)\right\}_{t=a}^{t=b}=\sigma_q,$$

and

$$\int \ldots \int \varphi_q(t) dt^q=\psi_q(t).$$

Then $\forall x \in[a, b]$ such that $b>a \geq 0$, the non-homogeneous Fredholm integral equation

$$u(x)=F(x)+\lambda \int_a^b K(x, t) u(t) dt, \tag{10}$$

has the solution due to the differential equations

$$\frac{d^q}{dx^q}u(x) = \frac{d^q}{dx^q}F(x) + \lambda C_q$$

and

$$\frac{d^{q+1}}{dx^{q+1}}u(x) = \frac{d^{q+1}}{dx^{q+1}}F(x). \tag{11}$$

where, for any $\lambda \neq \frac{1}{(-1)^q D_q} \forall q = 1, 2, \ldots,$ *there exists following constants*

$$C_q = \frac{\sigma_1 - \sigma_2 + \cdots + (-1)^{q-1}\sigma_q}{(1 - \lambda(-1)^q D_q)} + \frac{(-1)^q}{(1 - \lambda(-1)^q D_q)} \int_a^b \frac{d^q F}{dt^q} \psi_q(t) dt,$$

Proof. Consider the non-homogeneous Fredholm integral equation (10) and in both sides of it replace x by $x + s$ and again multiplying by n-degree polynomials $P_n\left(\frac{s}{\epsilon}\right)$ in these sides and thus integrating these sides with respect to s from $-\epsilon$ to ϵ, we write it in the form

$$\int_{-\epsilon}^{\epsilon} u(x+s) P_n\left(\frac{s}{\epsilon}\right) ds = \int_{-\epsilon}^{\epsilon} F(x+s) P_n\left(\frac{s}{\epsilon}\right) ds + \lambda \int_a^b \left\{ \int_{-\epsilon}^{\epsilon} K(x+s,t) P_n\left(\frac{s}{\epsilon}\right) ds \right\} u(t) dt. \tag{12}$$

On making some manipulations in the formula (12), we find that

$$\int_{-1}^{1} u(x+\epsilon s) P_n(s) ds = \int_{-1}^{1} F(x+\epsilon s) P_n(s) ds + \lambda \int_a^b \left\{ \int_{-1}^{1} K(x+\epsilon s,t) P_n(s) ds \right\} u(t) dt. \tag{13}$$

Then in the equation (13) on applying Taylor's formula, we get

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{d^k}{dx^k} u(x) \frac{\epsilon^k}{k!} \int_{-1}^{1} s^k P_n(s) ds &= \sum_{p=0}^{\infty} \frac{d^p}{dx^p} F(x) \frac{\epsilon^p}{p!} \int_{-1}^{1} s^p P_n(s) ds \\ &+ \lambda \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} \int_a^b \left\{ \int_{-1}^{1} s^m P_n(s) ds \frac{\partial^m}{\partial x^m} K(x,t) \right\} u(t) dt. \end{aligned} \tag{14}$$

It is remarked that the n-degree polynomials $P_n(s) \forall n = 0, 1, 2, 3, \ldots,$ defined by Gaussian hypergeometric function due to [5, p. 158], are given as

$$P_n(s) = \frac{(2n)! s^n}{2^n (n!)^2} {}_2F_1\left(-\frac{n}{2}, \frac{-n+1}{2}; \frac{1}{2} - n; \frac{1}{s^2}\right),$$

which follows a formula given by [5, p. 159]

$$\int_{-1}^{1} s^k P_n(s) ds = \begin{cases} \frac{2^{n+1}(n!)^2 \Gamma((1+k)/2)\Gamma((2+k)/2)\Gamma(n+3/2)}{(2n+1)!\Gamma((k-n+2)/2)\Gamma^2((n+1)/2)\Gamma((k+n+3)/2)}, & \text{when } n+k \text{ even}, \\ 0, & \text{when } n+k \text{ odd.} \end{cases} \tag{15}$$

Then on using (15) in (14), we get

$$\lim_{k\to\infty}\sum_{j=0}^{k}\frac{d^j}{dx^j}u(x)\frac{\Gamma((1+j)/2)\Gamma((2+j)/2)}{\Gamma((j-n+2)/2)\Gamma((j+n+3)/2)}\frac{\epsilon^j}{j!} \tag{16}$$
$$=\lim_{p\to\infty}\sum_{j=0}^{p}\frac{d^j}{dx^j}F(x)\frac{\Gamma((1+j)/2)\Gamma((2+j)/2)}{\Gamma((j-n+2)/2)\Gamma((j+n+3)/2)}\frac{\epsilon^j}{j!}$$
$$+\lambda\lim_{m\to\infty}\sum_{j=0}^{m}\frac{\epsilon^j}{j!}\frac{\Gamma((1+j)/2)\Gamma((2+j)/2)}{\Gamma((j-n+2)/2)\Gamma((j+n+3)/2)}\int_a^b\left\{\frac{\partial^j}{\partial x^j}K(x,t)\right\}u(t)dt.$$

Now in (16) consider that $k+n=p+n=m+n=q$ (even number), then on equating the coefficients of ϵ^q and find that

$$\frac{d^q}{dx^q}u(x)=\frac{d^q}{dx^q}F(x)+\lambda\int_a^b\left\{\frac{\partial^q}{\partial x^q}K(x,t)\right\}u(t)dt. \tag{17}$$

Again making an appeal to the conditions (9) in the equation (17) we find the differential equations in (11). Further making an appeal to conditions given in the Theorem 3.1 and by Eqns. (9), we obtain

$$C_q=\sigma_1-\sigma_2+\cdots+(-1)^{q-1}\sigma_q+(-1)^q\int_a^b\frac{d^qF}{dt^q}\psi_q(t)dt+\lambda(-1)^qC_qD_q. \tag{18}$$

The Eqn. (18) gives us the constants given in the Eqn. (11).

Finally, on successively integrating both sides of the first differential equation of (11), we obtain the required solution of the Fredholm integral equation (10). □

4. Examples on solution of non-homogeneous single Fredholm integral equations

By this theory and methods presented in Section 3, we introduce some examples of non-homogeneous Fredholm integral equations to get the solution of them:

Example 1. Consider the Fredholm integral equation

$$u(x)=F(x)+\lambda\int_a^b(1-xt)u(t)dt, \tag{19}$$

to solve it whenever $x\geq 0$.

Solution. Making an appeal to the Theorem 3.1 in (19), we find the differential equations

$$\frac{d}{dx}u(x)=\frac{d}{dx}F(x)+\lambda C \text{ and } \frac{d^2}{dx^2}u(x)=\frac{d^2}{dx^2}F(x). \tag{20}$$

Due to (19), (20) and $\int_a^b(-t)u(t)dt=C$, to get

$$C=\int_a^b(-t)u(t)dt=\left\{u(t)\left(-\frac{t^2}{2}\right)\right\}_{t=a}^{t=b}-\int_a^b u'(t)\left(-\frac{t^2}{2}\right)dt.$$

Here let the constant $\left\{u(t)\left(-\frac{t^2}{2}\right)\right\}_{t=a}^{t=b}=\mu$, then we find

$$C=\mu+\frac{1}{2}\int_a^b t^2F'(t)dt+\frac{\lambda C}{6}\left(b^3-a^3\right).$$

So that

$$C = \frac{\mu}{\left\{1 - \frac{\lambda}{6}\left(b^3 - a^3\right)\right\}} + \frac{1}{\left\{2 - \frac{\lambda}{3}\left(b^3 - a^3\right)\right\}} \int_a^b t^2 F'(t)dt,$$

provided that $1 - \frac{\lambda}{6}\left(b^3 - a^3\right) \neq 0$.

Integrating both sides of first equation in (20) with respect to x, we get

(21) $$u(x) = F(x) + \lambda C x + E,$$

E is an arbitrary constant.

Now in (20) putting $x = 0$, we find

$$E = u(0) - F(0).$$

Hence, the required solution is

$$u(x) = F(x) + \lambda C x + u(0) - F(0),$$

were, for $1 - \frac{\lambda}{6}\left(b^3 - a^3\right) \neq 0$,

(22)

$$C = \frac{\mu}{\left\{1 - \frac{\lambda}{6}\left(b^3 - a^3\right)\right\}} + \frac{1}{\left\{2 - \frac{\lambda}{3}\left(b^3 - a^3\right)\right\}} \int_a^b t^2 F'(t)dt \text{ and } \mu = \left\{u(a)\frac{a^2}{2} - u(b)\frac{b^2}{2}\right\}.$$

Example 2. Consider the Fredholm integral equation

(23) $$u(x) = F(x) + \lambda \int_a^b e^{x-t} u(t)dt,$$

to solve it, whenever $x \geq 0$.

Solution. On making an appeal to the Theorem 3.1 in (23), we get

(24)

$$\frac{d^q}{dx^q}u(x) = \frac{d^q}{dx^q}F(x) + \lambda C_q, \text{ where } C_q = \lim_{x \to 0} \frac{\partial^q}{\partial x^q} \int_a^b e^{x-t} u(t)dt = \int_a^b e^{-t} u(t)dt.$$

Letting the constants

$$\left\{-u(t)e^{-t}\right\}_{t=a}^{t=b} = \mu_1, \left\{-u^{(\prime)}(t)e^{-t}\right\}_{t=a}^{t=b} = \mu_2, \ldots, \left\{-u^{(q-1)}(t)e^{-t}\right\}_{t=a}^{t=b} = \mu_q,$$

then by (24) we obtain

$$C_q = \mu_1 + \mu_2 + \mu_3 + \cdots + \mu_q + \int_a^b e^{-t} \frac{d^q}{dt^q}F(t)dt - \lambda C_q \left\{e^{-b} - e^{-a}\right\}.$$

$$\Rightarrow C_q = \frac{\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_q}{\left\{1 + \lambda\left\{e^{-b} - e^{-a}\right\}\right\}} + \frac{1}{\left\{1 + \lambda\left\{e^{-b} - e^{-a}\right\}\right\}} \int_a^b e^{-t} \frac{d^q}{dt^q}F(t)dt,$$

provided that

(25) $$1 + \lambda\left\{e^{-b} - e^{-a}\right\} \neq 0.$$

Therefore due to (24) and (25), for $1 + \lambda\left\{e^{-b} - e^{-a}\right\} \neq 0$, we find an interesting equation

$$\frac{d^q}{dx^q}u(x) = \frac{d^q}{dx^q}F(x) + \lambda C_q,$$

where

$$C_q = \frac{\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_q}{\left\{1 + \lambda\left\{e^{-b} - e^{-a}\right\}\right\}} + \frac{1}{\left\{1 + \lambda\left\{e^{-b} - e^{-a}\right\}\right\}} \int_a^b e^{-t} \frac{d^q}{dt^q}F(t)dt,$$

provided that

$$1+\lambda\left\{e^{-b}-e^{-a}\right\}\neq 0. \tag{26}$$

Now integrating both sides of (26) successively with respect to x, we have
$\frac{d^{q-1}}{dx^{q-1}}u(x)=\frac{d^{q-1}}{dx^{q-1}}F(x)+C_q x+E_1$
$\Rightarrow \frac{d^{q-2}}{dx^{q-2}}u(x)=\frac{d^{q-2}}{dx^{q-2}}F(x)+C_q\frac{x^2}{2}+E_1x+E_2$
$\Rightarrow \frac{d^{q-3}}{dx^{q-3}}u(x)=\frac{d^{q-3}}{dx^{q-3}}F(x)+C_q\frac{x^3}{3!}+E_1\frac{x^2}{2!}+E_2x+E_3$
$\Longrightarrow u(x)=F(x)+C_q\frac{x^q}{q!}+E_1\frac{x^{q-1}}{(q-1)!}+E_2\frac{x^{q-2}}{(q-2)!}+\cdots+E_{q-1}x+E_q.$

(27) Here, $E_1, E_2, \ldots, E_q$, are arbitrary constants.

In both sides of equation (27) on putting $x=0$, we get

$$u(0)-F(0)=E_q. \tag{28}$$

Therefore under the condition, $1+\lambda\left\{e^{-b}-e^{-a}\right\}\neq 0$, we find the required solution in the form

$$u(x)=F(x)+\frac{\lambda}{1+\lambda\left\{e^{-b}-e^{-a}\right\}}\left[\mu_1+\mu_2+\mu_3+\cdots+\mu_q+\eta_q\right]\frac{x^q}{q!}$$
$$+E_1\frac{x^{q-1}}{(q-1)!}+E_2\frac{x^{q-2}}{(q-2)!}+\cdots+E_{q-1}x+u(0)-F(0),$$

where,

$$\eta_q=\int_a^b e^{-t}\frac{d^q}{dt^q}F(t)dt. \tag{29}$$

Example 3. Consider the Fredholm integral equation

$$u(x)=F(x)+\lambda\int_a^b \sin(x+t)u(t)dt, \tag{30}$$

to solve it whenever $x\geq 0$.

Solution. Applying above methods in (30), we obtain

$$\frac{d^q}{dx^q}u(x)=\frac{d^q}{dx^q}F(x)+\lambda C_q,$$

where,

$$C_q=\lim_{x\to 0}\frac{\partial^q}{\partial x^q}\int_a^b \sin(x+t)u(t)dt=\int_a^b \sin\left(t+\frac{q\pi}{2}\right)u(t)dt. \tag{31}$$

Again let

$$\left\{-u(t)\cos\left(t+\frac{q\pi}{2}\right)\right\}_{t=a}^{t=b}=\rho_1,\left\{-u^{(\prime)}(t)\cos\left(t+\frac{(q-1)\pi}{2}\right)\right\}_{t=a}^{t=b}=\rho_2,$$

(32)

$$\left\{-u^{(\prime\prime)}(t)\cos\left(t+\frac{(q-2)\pi}{2}\right)\right\}_{t=a}^{t=b}=\rho_3,\ldots,\left\{-u^{(q-1)}(t)\cos\left(t+\frac{\pi}{2}\right)\right\}_{t=a}^{t=b}=\rho_q.$$

Therefore by (31) and (32), we get

$$C_q=\rho_1-\rho_2+\cdots+(-1)^{q-1}\rho_q+(-1)^q\int_a^b u^{(q)}(t)\sin(t)dt$$

$$\Rightarrow C_q = \rho_1 - \rho_2 + \cdots + (-1)^{q-1}\rho_q + (-1)^q \int_a^b \left\{ \frac{d^q}{dx^q} F(x) + \lambda C_q \right\} \sin t dt.$$

$$C_q = \frac{\rho_1 - \rho_2 + \cdots + (-1)^{q-1}\rho_q}{\{1 + (-1)^q \lambda(\cos b - \cos a)\}} + \frac{(-1)^q}{\{1 + (-1)^q \lambda(\cos b - \cos a)\}} \int_a^b \sin t \frac{d^q}{dt^q} F(t) dt,$$

provided that

(33) $$\{1 + (-1)^q \lambda(\cos b - \cos a)\} \neq 0.$$

Again by (31), we have

$$\Rightarrow u(x) = F(x) + \lambda C_q \frac{x^q}{q!} + E_1 \frac{x^{q-1}}{(q-1)!} + E_2 \frac{x^{q-2}}{(q-2)!} + \cdots + E_{q-1}x + E_q.$$

On putting $x = 0$, we get

$$E_q = u(0) - F(0),$$

hence for $1 + (-1)^q\{\cos b - \cos a\}\lambda \neq 0$, the required solution is

$$u(x) = F(x) + \lambda C_q \frac{x^q}{q!} + E_1 \frac{x^{q-1}}{(q-1)!} + E_2 \frac{x^{q-2}}{(q-2)!} + \cdots + E_{q-1}x + u(0) - F(0),$$

where,

(34)

$$C_q = \frac{\rho_1 - \rho_2 + \cdots + (-1)^{q-1}\rho_q}{\{1 + (-1)^q \lambda(\cos b - \cos a)\}} + \frac{(-1)^q}{\{1 + (-1)^q \lambda(\cos b - \cos a)\}} \int_a^b \sin t \frac{d^q}{dt^q} F(t) dt.$$

5. Solution of non-homogeneous double Fredholm integral equations

In this section, we apply above theory and methods (see also in [1, 8, 16]) of Sections 3 and 4, and solve the nonhomogeneous double Fredholm integral equations which are consisting of double integrals [7].

Example 4. If $x_1, x_2; t_1$ and t_2 are all positive such that $a \leq x_1 + x_2 \leq b; a \leq t_1 + t_2 \leq b; b > a \geq 0$. Also $u : u(x_1, x_2) = u(x_1 + x_2)\, \forall a \leq x_1 + x_2 \leq b$; otherwise, zero and $F : F(x_1, x_2) = F(x_1 + x_2)\, \forall a \leq x_1 + x_2 \leq b$; otherwise zero. Then the double non-homogeneous Fredholm integral equation

(35) $$u(x_1, x_2) = F(x_1, x_2) + \lambda \iint e^{x_1 + x_2 - (t_1 + t_2)} u(t_1 + t_2)\, dt_1 dt_2,$$

is solvable, whenever $x_1 + x_2 \geq 0$.

Solution. Let $x_1 + x_2 = X \Rightarrow a \leq X \leq b$ and due to Liouville theorem studied in [7], the Eqn. (35) may be written by

(36) $$u(X) = F(X) + \lambda \int_a^b e^{X-t} t u(t) dt$$

Therefore on setting the constants

$$\mu_1' = \left\{-u(t)(1+t)e^{-t}\right\}_{t=a}^{t=b}, \mu_2' = \left\{-u^{(1)}(t)(2+t)e^{-t}\right\}_{t=a}^{t=b}, \ldots, \mu_q' = \left\{-u^{(q-1)}(t)(q+t)e^{-t}\right\}_{t=a}^{t=b}$$

and making an appeal to the Theorem 3.1, via (36) we find

(37) $$\frac{d^q}{dX^q} u(X) = \frac{d^q}{dX^q} F(X) + \lambda C_q, \text{ where } C_q = \int_a^b e^{-t} t u(t) dt.$$

Again due to (37), for $\left[1-\lambda\left\{(a+q+1)e^{-a}-(b+q+1)e^{-b}\right\}\right] \neq 0$, we obtain

$$(38)\quad C_q = \frac{\mu_1' + \mu_2' + \cdots + \mu_q'}{[1-\lambda\{(a+q+1)e^{-a}-(b+q+1)e^{-b}\}]} + \frac{1}{[1-\lambda\{(a+q+1)e^{-a}-(b+q+1)e^{-b}\}]}\int_a^b e^{-t}(t+q)\frac{d^q}{dt^q}F(t)dt.$$

Finally, integrating both sides of (37) successively with respect to X and then using the formula (38), we derive the solution of double Fredholm integral equation (35) in the form

$$u(x_1,x_2) = F(x_1+x_2) + \frac{\lambda}{[1-\lambda\{(a+q+1)e^{-a}-(b+q+1)e^{-b}\}]}\left[\mu_1'+\mu_2'+\cdots+\mu_q'+\eta_q'\right] \times\frac{(x_1+x_2)^q}{q!} + E_1\frac{(x_1+x_2)^{q-1}}{(q-1)!} + E_2\frac{(x_1+x_2)^{q-2}}{(q-2)!} + \cdots + E_{q-1}(x_1+x_2) + u(0) - F(0),$$

whenever $x_1+x_2=0$, subject to the conditions
$\left[1-\lambda\left\{(a+q+1)e^{-a}-(b+q+1)e^{-b}\right\}\right] \neq 0$,
and

$$(39)\qquad \eta_q' = \int_a^b e^{-t}(t+q)\frac{d^q}{dt^q}F(t)dt.$$

Example 5. If $x_1, x_2; t_1$ and t_2 are all positive such that $a \le x_1+x_2 \le b; a \le t_1+t_2 \le b; b > a \ge 0$. Also $u : u(x_1,x_2) = u(x_1+x_2)\, \forall a \le x_1+x_2 \le b$; otherwise zero, $F : F(x_1,x_2) = F(x_1+x_2)\, \forall a \le x_1+x_2 \le b$; otherwise zero. Then the double non-homogeneous Fredholm integral equation

$$(40)\qquad u(x_1,x_2) = F(x_1,x_2) + \lambda \iint \sin(x_1+x_2+(t_1+t_2))\, u(t_1+t_2)\, dt_1 dt_2,$$

is solvable, whenever $x_1+x_2 \ge 0$.
Solution. Suppose that $x_1+x_2 = X \Rightarrow a \le X \le b$, then the Eqn. (38) may be written by

$$(41)\qquad u(X) = F(X) + \lambda\int_a^b \sin(X+t)tu(t)dt.$$

Now letting the constants

$$\left\{u(t)\left(t\sin\left(t+\frac{(q-1)\pi}{2}\right)+\sin\left(t+\frac{q\pi}{2}\right)\right)\right\}_{t=a}^{t=b} = \rho_1',$$

$$\left\{u^{(1)}(t)\left(t\sin\left(t+\frac{(q-2)\pi}{2}\right)+2\sin\left(t+\frac{(q-1)\pi}{2}\right)\right)\right\}_{t=a}^{t=b} = \rho_2' \ldots,$$

$$(42)\qquad \left\{u^{(q-1)}(t)\left(t\sin t + q\sin\left(t+\frac{\pi}{2}\right)\right)\right\}_{t=a}^{t=b} = \rho_q',$$

then due to (40), there exists a constant

$$(43)\qquad C_q = \int_a^b \sin\left(t+\frac{q\pi}{2}\right)tu(t)dt,$$

whereon applying (40), we get

(44) $$C_q = \frac{\rho_1' - \rho_2' + \cdots + (-1)^{q-1}\rho_q'}{[1-(-1)^q\lambda\{a\cos a - \sin a + \sin b - b\cos b\}]}$$
$$+\frac{(-1)^q}{[1-(-1)^q\lambda\{a\cos a + (q+1)\sin b - b\cos b - (q+1)\sin a\}]}\int_a^b (t\sin t + q\,\mathrm{cost})\frac{d^q}{dt^q}F(t)dt.$$

Finally, we derive the solution of the double Fredholm integral equation (40) in the form

(45)
$$u(x_1,x_2) = F(x_1+x_2) + \lambda C_q\frac{(x_1+x_2)^q}{q!} + E_1\frac{(x_1+x_2)^{q-1}}{(q-1)!} + E_2\frac{(x_1+x_2)^{q-2}}{(q-2)!}$$
$$+ \cdots + E_{q-1}(x_1+x_2) + u(0) - F(0), \text{ whenever } x_1+x_2 \geq 0.$$

6. Special Cases

Special Cases 1. In example 4, setting $a=0, b=1, u(0)=0, u(1)=1, F(t)=t^2, u^{(1)}(0)=u^{(2)}(0)=\cdots=u^{(q-1)}(0)=0$ and $u^{(1)}(1)=u^{(2)}(1)=\cdots=u^{(q-1)}(1)=0$.

In Eqn. (11), it is said that $\frac{d^q}{dt^q}u(t)$ is proportional to $\frac{d^q}{dt^q}F(t)$.

Also $\frac{d^q}{dt^q}F(t)=\frac{d^q}{dt^q}t^2=0\forall q\geq 3$.

Then in this case we get $q=2$.

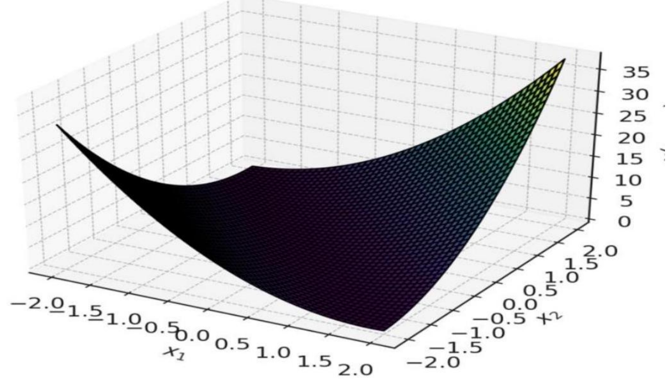
Again, we find $\eta_2'=\left[-8e^{-1}+6\right], \mu_1'=-2e^{-1}, \mu_q'=0\forall q\geq 2$, and $C_2=\left\{1+\frac{\lambda\left[6-10e^{-1}\right]}{[1-\lambda[3-4e^{-1}]]}\right\}$.

Therefore, the Eqn. (39) becomes

(46) $$u(x_1,x_2)=\left\{1+\frac{\lambda\left[3-5e^{-1}\right]}{[1-\lambda[3-4e^{-1}]]}\right\}(x_1+x_2)^2+(x_1+x_2)E_1.$$

Using MATLAB, we plot the graph of Eqn. (46):

3D Plot of $u(x_1,x_2)$



Graph 1 [Distribution of $u(x_1,x_2)$ when $\lambda=0.5$ and $E_1=0.7$]

Special Cases 2. In example 5, setting $a=0, b=\pi, u(0)=0, u(\pi)=1, F(t)=t^2, u^{(1)}(0)=u^{(2)}(0)=\cdots=u^{(q-1)}(0)=0$ and $u^{(1)}(\pi)=u^{(2)}(\pi)=\cdots=u^{(q-1)}(\pi)=0$.

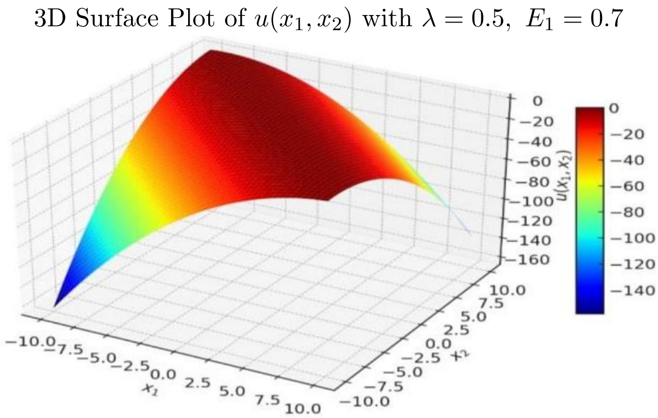
Also $\frac{d^q}{dt^q}F(t) = \frac{d^q}{dt^q}t^2 = 0 \forall q \geq 3$.

Then in this case we get $q = 2$. Again, we find $C_2 = \left[-\frac{\pi}{\pi\lambda - 1}\right], \rho_1' = -\pi, \rho_q' = 0 \forall q \geq 2$.

Therefore, the Eqn. (45) becomes

$$u(x_1, x_2) = \left\{1 - \frac{\pi\lambda}{2(\pi\lambda - 1)}\right\}(x_1 + x_2)^2 + (x_1 + x_2)E_1. \tag{47}$$

Using MATLAB, we plot the graph of Eqn. (47):



Graph 2 [Distribution of $u(x_1, x_2)$ when $\lambda = 0.5$ and $E_1 = 0.7$]

7. Conclusions

On making an appeal to Sections **3, 4 and 5,** it is concluded that via the Lanczos derivative techniques [1, 8, 16] any non-homogeneous Fredholm integral equation is solved, and the solution is found in terms of the polynomials. Hence any scientific problem converted into the non-homogeneous Fredholm integral equation is further transformed into polynomials via the techniques applied in Sections **3, 4 and 5,** for computational purposes. In Appendix A, we present algorithms of the MATLAB programs of graph 1 and graph 2 .

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Appendix A

1. The MATLAB program to draw graph 1 of the Special case I

```
clc; clear; close all;
% Define parameters
lambda = 0.5; % Given lambda value
E1 = 0.7;     % Given E1 value
% Define function
[X1, X2] = meshgrid(-10:0.5:10, -10:0.5:10);
factor = (1 − (pi * lambda) / (2 * (lambda * pi - 1) ));
U = factor * (X1 + X2). ∧ 2 + E1 * (X1 + X2);
% Plot 3D surface
figure;
surf(X1, X2, U);
xlabel('x_1');
ylabel('x_2');
zlabel('u(x_1, x_2)');
title('3D Surface Plot of u (x_1, x_2) with \ lambda = 0.5, E_1 = 0.7 ');
colormap jet;
shading interp;
colorbar;
grid on;
```

2. The MATLAB program to draw graph 1 of the Special case 2

```
clc; clear; close all;
% Define parameters
lambda = 0.5; % Given lambda value
E1 = 0.7;     Given E1 value
% Define function
[X1, X2] = meshgrid(-10:0.5:10, -10:0.5:10);
factor = (1 − ( pi * lambda) / (2 * (lambda * pi - 1)) );
U = factor * (X1 + X2).^2 + E1 * (X1 + X2);
% Plot 3D surface
figure;
surf(X1, X2, U);
xlabel('x_1');
ylabel('x_2');
zlabel('u(x_1, x_2)');
title('3D Surface Plot of u (x_1, x_2) with \lambda = 0.5, E_1 = 0.7 ');
colormap jet;
shading interp;
```

colorbar;
grid on;

Centre for Mathematical and Statistical Sciences, Peechi Campus, Peechi, Kerala, India-680653
E-mail address: mapathan@gmail.com

Department of Mathematics, D. A-V. Postgraduate College, Kanpur, Uttar Pradesh 208 001, India;
E-mail address: palhemant2007@rediffmail.com, hemantkumar_kn03@csjmu.ac.in

Department of Mathematics, College of Natural Sciences, Kwangwoon University, Seoul 139−701, Republic of Korea;
E-mail address: tkkim@kw.ac.kr

ESIME−Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista, CP 07738, CDMX, México,
E-mail address: jlopezb@ipn.mx