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THE INDEPENDENT DEGREE DOMINATION NUMBER OF A GRAPH

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Abstract:

A set D of vertices in a graph G is said to be dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set D is said to be an independent dominating set, if the induced subgraph $\langle D \rangle$ has no edges. An independent degree dominating (IDDF) is a function $f : V(G) \rightarrow \{0, 1, 2, ..., \triangle(G) + 1\}$ having the property that every vertex V of D is assigned with $deg(V)+1$ and all remaining vertices with zero. The weight of an independent degree dominating function f is designed by $w(f) = \sum_{v \in D} deg(v)+1$. The independent degree domination number, denoted by $\gamma_i deg$, is the minimum weight of all possible IDDF. In this paper, we study a new domination parameter called independent degree domination in graphs.

Keywords: Independent domination number, independent dominating set, independent degree domination, independent degree dominating function.

AMS subject classification: $05C69, 05C76, 68R10$

1 Introduction

Let $G = (V, E)$ be a simple graph with n vertices and m edges. All terminology not defined here can be found in [1, 2]. The Domination in graph theory is one of the most studied concepts that has attracted many researches to study on it due to its many applications in many fields, such as in facility location problems where one attempts to minimize the distance that a person needs to travel to reach to the nearest facility.

Dominating set is a set D of vertices in a graph G if every vertex in $V-D$ is adjacent to some vertex in D. A dominating set D is said to be an independent dominating set, if the induced subgraph $\langle D\rangle$ has no edges. The cordinality of the minimum independent dominating set is the independent domination number of a graph G, denoted by $\gamma_i(G)$. For a detailed treatment of these parameters, the reader is referred to [5]. There are many types of domination depending on the structures of dominating sets. One of these types, the weighted domination number γ_w of (G,w) is the minimum weight $w(D)=\sum_{v\in D} w(v)$ of a set $D\subseteq V$ with $N[D]=V$ i.e. a dominating set of G [3]. The Roman domination number, denoted by γ_R, is the minimum weight of all possible Roman dominating functions which is defined as a function $f: V\rightarrow\{0,1,2\}$ satisfying the condition that every vertex v for which $f(v)=0$ is adjacent to at least one vertex u for which $f(u)=2$ [7]. The concept of the degree dominating function in graph was introduced by Demirpolat and Kilie [4].

Definition 1.1. *The independent degree dominating function (IDDF) is a function* $f: V(G)\rightarrow\{0,1,2,...,\triangle(G)+1\}$ *having the property that every vertex* v *of the independent dominating set* D *is assigned with* $deg(v)+1$ *and all remaining vertices with zero. The weight of an independent degree dominating function* f *is defined by*
$w(f)=\sum_{v\in D} deg(v)+1$.

The independent degree domination number, denoted by γ_{ideg}, *is the minimum weight of all possible IDDFs.*

For illustration, we consider the following graph G.

In Figure 1, there are many independent dominating sets, but the set that gives the minimum vertices weight should be chosen. The set $D=\{v_4,v_7\}$ *is the minimum inde-*

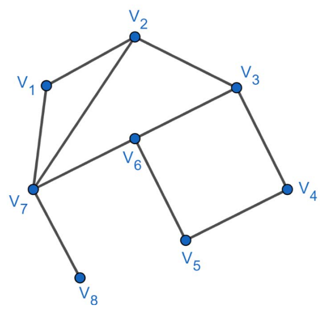

Figure 1: A graph G with eight vertices

pendent dominating set. The maximum degree of the graph G *is* $\triangle(G) = 4$.

By the definition of IDDF, $f : V(G) \to \{0, 1, 2, 3, 4, 5\}$ *and the IDDF must consist of vertices* $\{deg(v_4) + 1, deg(v_7) + 1\}$. *Hence, the independent degree domination number is* $\gamma_{ideg}(G) = \sum_{v \in D} f(v) = (2 + 1) + (4 + 1) = 8.$

2 RESULTS

Theorem 2.1. *For any, complete graph* $K_n, n \geq 1, \gamma_{ideg}(K_n) = n$

Proof: *The complete graph* $K_n, n \geq 1$ *is a* $(n-1)$*-regular graph and* $\gamma_i(K_n) = 1,$. *Also for* K_n, *each vertex dominates every other vertex. Let this vertex be* v. *By definition of IDDF,* $\gamma_{ideg}(K_n) = deg(v) + 1 = (n - 1) + 1 = n.$

Theorem 2.2. *For any wheel graph* W_n, *if* $n \geq 4$, *then,* $\gamma_{ideg}(W_n) = n + 1$

Proof: *Consider any wheel graph* W_n *with* n *vertices formed by sum of cycle graph with one vertex* v_1 *and cycle graph with* $n - 1$ *vertices are* $v_2, v_3, ..., v_{n-1}, v_n$, *that is the wheel* W_n *can be defined as the graph* $K_1 + C_{n-1}$. *Hence* v_1 *has degree* $n - 1$ *so it is internal vertex to all other vertices and* $deg(v_2) = deg(v_3) = ... = deg(v_n) = 3$.

It is clear that $\gamma_i(W_n) = 1$ *and* $D = v_1$ *is the minimum independent dominating set of* W_n. *By the definition of IDDF,* $\gamma_{ideg}(W_n) = deg(v) + 1 = n + 1$.

Helm Graph

The helm graph H_n is the graph obtained from wheel W_n by attaching a pendant edge to each of its rim vertices.

Theorem 2.3. *For the Helm graph $H_n (n \geq 4), \gamma_{ideg}(H_n) = 2n + 3$.*

Proof: *The Helm graph H_n has $2n + 1$ vertices and $3n$ edges. It contains Wheel W_n and $n - 1$ pendant vertices. In order to dominate the pendant vertices of H_n, at least $n - 1$ vertices of H_n are required. Moreover, it is also possible to take $n - 1$ pairwise non adjacent vertices which can dominate the pendant vertices as well as remaining vertices of the helm graph H_n. There fore, for any independent dominate set D of the helm graph H_n, has at least n, which independent $i(H_n) = n$. Let $D = X \cup Y$, be the partition of minimum independent set, where X is a set of pendant vertices, $v_1, v_2, ...v_{n-1}$ and Y is a set of vertex x of degree 4 which is not adjacent to any vertex of X.*

The maximum degree of the graph H_n is $\triangle(H_n) = n$. Then we define $f; V(H_n) \Rightarrow \{0, 1, ...n + 1\}$ and IDDF must consist of vertices $\{deg(v_1) + 1, deg(v_2) + 1, ...deg(v_{n-1}) + 1, deg(x) + 1$

Hence, the independent degree domination number is

$$\gamma_{ideg}(H_n) = \sum_{v \in D} f(v)$$
$$= (1 + 1) + (1 + 1) + ...(1 + 1) + (4 + 1)$$
$$= 2 + 2 + ... + 2 + 5$$
$$= 2(n - 1) + 5$$
$$= 2n + 3$$

Theorem 2.4. *Let $G : K_{r,t}$ be the complete bipartite graph, where $r \leq t$. Then $\gamma_{ideg}(K_{r,t}) = t(1 + 1)$*

Proof:*Let $K_{r,t}$ be the complete bipartite graph with bipartite sets v_1 and v_2 of order v*

and t, respectively. We know that $\gamma_i(K_{r,t}) = min\{r,t\}$ say r and $D = \{v_1, v_2, ..., v_r\}$ is the minimum independent dominating set of $K_{r,t}$. Then $deg(x) = t$, where $x \in V_1$ and $deg(y) = r$, where $y \in V_2$.

By definition of IDDF,

$\gamma_{ideg}(K_{r,t}) = (deg(x) + 1 + ... + deg(x) + 1)$

$=(t + 1) + ... + (t + 1)(r - times)$

$=r(t + 1)$

Theorem 2.5. *For the closed helm graph $CH_n, (n \geq 3)$,*

$$\gamma_{ideg}(CH_n) = \begin{cases} \frac{7n+3}{3} & if \quad n \equiv 0(mod3) \\ \frac{7n+7}{3} & if \quad n \equiv 2(mod3) \\ \frac{7n+11}{3} & if \quad n \equiv 1(mod3) \end{cases}$$

Proof: *Let CH_n be the closed helm graph $2n + 1$ vertices and $4n$ edges. Let X be the set of vertices of outer cycle C_n and $deg(v) = \triangle = n$. Then clearly $\gamma_i(CH_n) = [\frac{n}{3}] + 1$.*

Then we define $f : V(CH_n) = \{0, 1, 2, ...n + 1\}$

If $n \equiv 0(mod3)$, then $f(v_{3i-1}) = 4$ for $1 \leq i \leq \frac{n}{3}$ and $f(v) = n + 1$.

If $n \equiv 2(mod3)$, then $f(v_{3i-1}) = 4$ for $1 \leq i \leq \frac{n-2}{3}$ and $f(v_n) = 4$ and $f(v) = n + 1$.

If $n \equiv 1(mod3)$, then $f(v_{3i-1}) = 4$ for $1 \leq i \leq \frac{n-1}{3}$ and $f(v_n) = 4$ and $f(v) = n + 1$.

For all remaining vertices $f(v) = 0$. It is easy to generalize that f is IDDF of CH_n of weight

$$4.\frac{n}{3} + n + 1 = \frac{7n+3}{3}, \text{ if } n \equiv 0(mod3),$$

$$4.\frac{n-2}{3} + 4 + n + 1 = \frac{7n+7}{3}, \text{ if } n \equiv 2(mod3)$$

$$\text{and } 4.\frac{n-1}{3} + 4 + n + 1 = \frac{7n+11}{3}, \text{ if } n \equiv 1(mod3)$$

Theorem 2.6. *For any graph G with $n \geq 2$ vertices, $\gamma_i(G) \leq \gamma_{ideg}(G)$.*

Proof: *Suppose that X is an independent dominating set and D is an independent degree dominating set of G. Let $|X| = t$ with $t \geq 1$. It is clear by the definition of IDDF, D*

consists of $deg(v) + 1$.

Therefore, $|X| \leq |D|$ *and hence* $\gamma_i(G) \leq \gamma_{ideg}(G)$.

Observation 2.7. *For any connected graph* G *with* $n \geq 2$ *vertices,* $\gamma_{ideg}(G) \geq deg(v_i)$, *where* $1 \leq i \leq n$.

Lemma 2.8. *Let* G *be* r*-regular graph. then* $\gamma_{ideg}(G) = (r+1)\gamma_i(G)$.

Proof: *Suppose that* X *is a minimum independent dominating set of* G. *Let* $|X| = t$ *with* $t \geq 1$. *It is clear that the degree of all vertices of* X *is* r, *by the definition of IDDF,* $\gamma_{ideg}(G) = \sum_{i=1}^{t} r + 1$.

Therefore, $\gamma_{ideg}(G) = (r+1)t = (r+1)|X| = (r+1)\gamma_i(G)$.

The binomial tree B_0 *consists of a single vertex. The binomial tree* B_n *is an ordered tree defined recursively. The binomial tree* B_n *consists of two binomial trees* B_{n-1} *that are liked together, the root of one is the left most child of the root of the other. Note that there are* 2^n *vertices in the binomial tree* B_n.

Theorem 2.9. *: Let* B_n *be a binomial tree. Then* $\gamma_{ideg}(B_n) = 2^n$.

Proof: *Let* X *be an independent dominating set of* B_n. *Then by definition of* B_n *includes two* B_{n-1}. *With this logic, the recursive structure of the* B_n *obtained as* $B_n = 2^i.(B_{n-i})$ *for* $1 \leqslant i \leqslant n-2$. *Let* $n \leq 3$. *Then it is clear that when* $n = 0$ *there is no pendant vertex and there is only* v *isolated vertex. By definition of IDDF,* $\gamma_{ideg}(B_0) = deg(v) + 1 = 1$. *When* $n = 1$, *the number of pendant vertices of* B_1 *is* 1 *and* $\gamma_{ideg}(B_1) = 2\gamma_{ideg}(B_0) = 2$. *When* $n = 2$, $\gamma_{ideg}(B_2) = 2\gamma_{ideg}(B_1) = 4$. *For* $n \geqslant 3$, *if we continue the same process, we obtain that independent degree dominating set of* B_n *is* $\gamma_{ideg}(B_n) = 2\gamma_{ideg}(B_{n-1})$. *If we put this result in recursive formula, we have* $\gamma_{ideg}(B_n) = 2^i\gamma_{ideg}(B_{n-i})$, *for* $1 \leqslant i \leqslant n-1$.

We prove this formula, by induction on i, *for* $n \geqslant 3$.

Let $i = 0$. *Then* $\gamma_{ideg}(B_0) = 1$.

Let $i = 1$. *Then,* $\gamma_{ideg}(B_1) = 2\gamma_{ideg}(B_0) = 2$

Let $i = k$ *and this result is true. We assume that* $\gamma_{ideg}(B_n) = 2^k.\gamma_{ideg}(B_{n-k})$.

Now, we prove it for $i = k+1$. *Then we get* $\gamma_{ideg}(B_n) = 2^k.(2.\gamma_{ideg}(B_{n-k-1}))$

$= 2^{k+1}.\gamma_{ideg}(B_{n-(k+1)})$

Hence the formula is true for $i = k+1$ *and we have* $\gamma_{ideg}(B_n) = 2^i.\gamma_{ideg}(B_{n-i})$ *for* $1 \leqslant i \leqslant n-1$.

We obtain the following formula by putting $i = n-1$ *in* $\gamma_{ideg}(B_n) = 2^i.\gamma_{ideg}(B_{n-1})$.

Therefore,

$\gamma_{ideg}(B_n) = 2^i.\gamma_{ideg}(B_{n-i})$

$\gamma_{ideg}(B_n) = 2^{n-1}.\gamma_{ideg}(B_{n-(n-1)})$

$\gamma_{ideg}(B_n) = 2^{n-1}.\gamma_{ideg}(B_1)$

$= 2^{n-1}.2$

$= 2^n$.

Theorem 2.10. *For* $n \geq 3, \gamma_{ideg}(P_n) = \begin{cases} n, & if \quad n \equiv 0,2(mod3) \\ n+1, & if \quad n \equiv 1(mod3) \end{cases}$

Proof: *Let* $P_n = \{v_1, v_2, ..., v_4\}$ *be a path of order* n. *We know that the degree of all vertices except the pendant vertices is* 2, *and* $\triangle(f_n) = 2$. *Then, we define* $f : V(P_n) \rightarrow \{0,1,2,3\}$.

If $n \equiv 0(mod3)$, *then* $f(v_{3i-1}) = 3$ *for* $1 \leq i \leq \frac{n}{3}$.

If $n \equiv 2(mod3)$, *then* $f(v_{3i-1}) = 3$ *for* $1 \leq i \leq \frac{n-2}{3}$. *and* $f(v_n) = 2$.

If $n \equiv n(mod3)$, *then* $f(v_{3i-1}) = 3$ *for* $1 \leq i \leq \frac{n-1}{3}$ *and* $f(v_n) = 3$

For all remaining vertices $f(v) = 0$. *It is every generalize that* f *is IDDF of* P_n *of weight* $3.\frac{n}{3} = n$, *if* $n \equiv 0(mod3)$, $3.\frac{n-2}{3} + 2 = n$, *if* $n \equiv 2(mod3)$ *and* $3.\frac{n-1}{3} + 2 = \frac{3n-3+6}{3} = n+1$, *if* $n \equiv 1(mod3)$

Thus, $\gamma_{ideg}(P_n) = \begin{cases} n, & if \quad n \equiv 0,2(mod3) \\ n+1, & if \quad n \equiv 1(mod3) \end{cases}$

Theorem 2.11. *For* $n \geq 3, \gamma_{ideg}(C_n) = \begin{cases} n, & if \quad n \equiv 0(mod3) \\ n+1, & if \quad n \equiv 2(mod3) \\ n+2, & if \quad n \equiv 1(mod3) \end{cases}$

Proof: *Let* $C_n = \{v_1, v_2, ...v_n\}$ *be the set of* n *vertices of* C_n*. It is known that* C_n *is regular graph of edges 2. Then we define* $f : V(Cn) \rightarrow \{0, 1, 2, 3\}$

If $n \equiv 0(mod3)$*, then* $f(v_{3i-1}) = 3$ *for* $1 \leq i \leq \frac{n}{3}$.

If $n \equiv 2(mod3)$*, then* $f(v_{3i-1}) = 3$ *for* $1 \leq i \leq \frac{n-2}{3}$ *and* $f(v_1) = 3$.

If $n \equiv 1(mod3)$*, then* $f(v_{3i-1}) = 3$ *for* $1 \leq i \leq \frac{n-1}{3}$ *and* $f(v_n) = 3$

For all remaining vertices $f(v) = 0$*. It is easy generalize that* f *is IDDF of* C_n *of weight*

$3.\frac{n}{3} = n,$ *if* $n \equiv 0(mod3)$,

$3.\frac{n-2}{3} + 3 = \frac{3n-6+9}{3} = n + 1,$ *if* $n \equiv 2(mod3)$

and $3.\frac{n-1}{3} + 3 = \frac{3n-3+9}{3} = n + 2,$ *if* $n \equiv 1(mod3)$

Thus, $\gamma_{ideg}(C_n) = \begin{cases} n, & if \quad n \equiv 0(mod3) \\ n+1, & if \quad n \equiv 2(mod3) \\ n+2, & if \quad n \equiv 1(mod3) \end{cases}$

Crown Graph

The graph $CW_n = C_n \circ K_1$ *is called a crown graph [7].*

For example: *The graph* $CW_n = C_n \circ K_1$

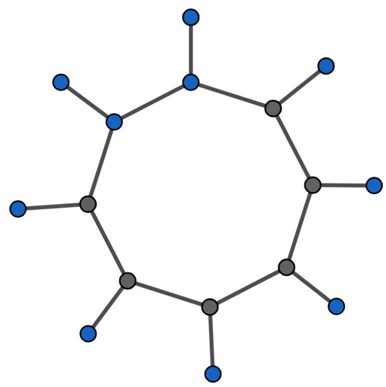

Figure 2: The crown graph CW_8

Theorem 2.12. *Let $G = CW_n$ be the crown graph. Then $\gamma_{ideg}(CW_n) = 2n$.*

Proof: *Let $CW_n = \{v_1, v_2, ..., v_n, u_1, u_2, ...u_n\}$ be set of vertices of CW_n of order $2n$. It is known that, there are two types of degrees: i) vertices of degree one and ii) vertices of degree three. Then we define $f : V(CW_n) \to \{0, 1, 2, 3, 4\}$. It is seen that set $D = \{v_1, v_2, ...v_n\}$ be the set of pendent vertices and $|D| = \gamma_i$, and the IDDF must consists of vertices $\{deg(v_1) + 1, deg(v_2) + 1, ...deg(v_n) + 1\}$.*

Hence, the independent degree domination number is

$$\gamma_{ideg}(CW_n) = \sum_{v \in D} f(v) = (1 + 1) + (1 + 1) + ... + (1 + 1)$$
$$= 2 + 2+, ..., +2 = 2n$$

CONCLUSION

In this paper the new domination parameter called independent degree domination number and denoted by γ_{ideg}. We find some results for γ_{ideg} of some well known graphs such as path, cycle, star, complete graph, complete bipartite graph,wheel, helm graph, closed helm graph, binomial tree and crown graph.

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