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IDENTITIES INVOLVING THE COMPONENTS OF AN ARBITRARY LORENTZ MATRIX

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Abstract. We exhibit identities satisfied by the components of a Lorentz matrix, which allows to deduce the Macfarlanes formula for the matrix S that governs the transformation of the Dirac 4−spinor under a Lorentz mapping; furthermore, we write S in terms of the gamma matrices.

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1. Introduction

The arbitrary complex quantities α, β, γ, δ verifying the condition $\alpha\delta-\beta\gamma=1$, generate a Lorentz transformation $L=(L^{\mu}{}_{\nu})$ via the expressions [1-10]:

$$L^0{}_0=\frac{1}{2}\Big(\alpha\bar{\alpha}+\beta\bar{\beta}+\gamma\bar{\gamma}+\delta\bar{\delta}\Big),\quad L^1{}_0=\frac{1}{2}\Big(\bar{\alpha}\gamma+\bar{\beta}\delta\Big)+cc,\quad L^2{}_0=-\frac{i}{2}\Big(\alpha\bar{\gamma}-\bar{\beta}\delta\Big)+cc,$$
$$L^0{}_1=\frac{1}{2}\Big(\bar{\alpha}\beta+\bar{\gamma}\delta\Big)+cc,\qquad L^1{}_1=\frac{1}{2}\Big(\bar{\alpha}\delta+\beta\bar{\gamma}\Big)+cc,\qquad L^2{}_1=-\frac{i}{2}\Big(\alpha\bar{\delta}+\beta\bar{\gamma}\Big)+cc,$$
$$L^0{}_2=-\frac{i}{2}\Big(\bar{\alpha}\beta+\bar{\gamma}\delta\Big)+cc,\qquad L^1{}_2=-\frac{i}{2}\Big(\bar{\alpha}\delta+\beta\bar{\gamma}\Big)+cc,\qquad L^2{}_2=\frac{1}{2}\Big(\bar{\alpha}\delta-\bar{\beta}\gamma\Big)+cc,$$
(1)
$$L^0{}_3=\frac{1}{2}\Big(\alpha\bar{\alpha}-\beta\bar{\beta}+\gamma\bar{\gamma}-\delta\bar{\delta}\Big),\qquad L^1{}_3=\frac{1}{2}\Big(\bar{\alpha}\gamma-\bar{\beta}\delta\Big)+cc,\qquad L^2{}_3=-\frac{i}{2}\Big(\alpha\bar{\gamma}+\bar{\beta}\delta\Big)+cc,$$
$$L^3{}_0=\frac{1}{2}\Big(\alpha\bar{\alpha}+\beta\bar{\beta}-\gamma\bar{\gamma}-\delta\bar{\delta}\Big),\quad L^3{}_1=\frac{1}{2}\Big(\bar{\alpha}\beta-\bar{\gamma}\delta\Big)+cc,\qquad L^3{}_2=-\frac{i}{2}\Big(\bar{\alpha}\beta-\bar{\gamma}\delta\Big)+cc,$$
$$L^3{}_3=\frac{1}{2}\Big(\alpha\bar{\alpha}-\beta\bar{\beta}-\gamma\bar{\gamma}+\delta\bar{\delta}\Big),$$

where cc means the complex conjugate of all the previous terms.

On the other hand, the Dirac 4−spinor obeys the transformation law [11, 12]:

(2) $$\tilde{\psi}(\tilde{x}^{\mu})=S(L)\psi(x^{\mu}).$$

for a non-singular matrix S such that:

(3) $$L^{\mu}{}_{\nu}\gamma^{\nu}=S^{-1}\gamma^{\mu}S,$$

in terms of the gamma matrices γ^{μ} verifying the anticommutators:

(4) $$\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}I_{4\times 4},\qquad (g^{\mu\nu})=Diag(1,-1,-1,-1).$$

In the Dirac-Pauli (or standard) representation [10-12]:

(5) $$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3,$$

with the Pauli matrices:

(6) $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In Sec. 2 we exhibit that the components L^μ_ν satisfy certain identities in terms of the quantities:

(7) $$b_0 = \frac{1}{4}(\bar{D}_+ + D_+), \quad b_1 = \frac{1}{4}(\bar{T}_+ - T_+), \quad b_2 = \frac{i}{4}(\bar{T}_- + T_-), \quad b_3 = \frac{1}{4}(\bar{D}_- - D_-),$$
$$d_0 = -\frac{i}{4}(\bar{D}_+ - D_+), \; d_1 = -\frac{i}{4}(\bar{T}_+ + T_+), \; d_2 = \frac{1}{4}(\bar{T}_- - T_-), \; d_3 = -\frac{i}{4}(\bar{D}_- + D_-),$$

$$D_\pm = \alpha \pm \delta, \qquad T_\pm = \beta \pm \gamma, \qquad \alpha\delta - \beta\gamma = 1,$$

which allow obtain a solution of (3), that is, to write S as a linear combination of the gamma matrices and also deduce the Macfarlane's formula [13] for it.

2. Identities involving components of a Lorentz matrix

In fact, the relations (1) generate the identities:

(8) $$L_{j\tau}L^{\tau}{}_{0} - L_{0\tau}L^{\tau}{}_{j} + (2 + \bar{D}_+ D_+)(L_{0j} - L_{j0}) = 4i(\bar{D}_+ + D_+)d_j, \quad j = 1, 2, 3,$$

$$L_{l\tau}L^{\tau}{}_{k} - L_{k\tau}L^{\tau}_{l} + (2 + \bar{D}_+ D_+)(L_{kl} - L_{lk}) = 4i(\bar{D}_+ + D_+)b_j,$$
$$(jkl) = (123), (231), (312),$$

$$tr\ L = L^\mu_\mu = \bar{D}_+ D_+, \qquad \epsilon_{\mu\nu\tau\varphi}L^{\mu\nu}L^{\tau\varphi} = 2i(\bar{D}^2_+ - D^2_+) = -32 b_0 d_0,$$

then S can be written in the following form using the quantities (7):

(9) $$S = b_0 I + i d_0 \gamma^5 + b_1 \sigma^{23} + b_2 \sigma^{31} + b_3 \sigma^{12} + \Sigma^3_{j=1} d_j \sigma^{0j},$$
$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \; \sigma^{0j} = i \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \; \sigma^{jk} = \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}, \; (jkl) = (123), (231), (312),$$

hence S acquires the structure:

(10) $$S = \begin{pmatrix} A & E \\ E & A \end{pmatrix}, \; A = \frac{1}{2}\begin{pmatrix} \bar{\alpha} + \delta & \bar{\beta} - \gamma \\ \bar{\gamma} - \beta & \alpha + \bar{\delta} \end{pmatrix}, \; E = \frac{1}{2}\begin{pmatrix} \bar{\alpha} - \delta & \bar{\beta} + \gamma \\ \bar{\gamma} + \beta & \bar{\delta} - \alpha \end{pmatrix},$$

for a given Lorentz matrix; furthermore, the Hermitian matrices $A^\dagger$ and $E^\dagger$ give the inverse of S:

$$S^{-1} = \begin{pmatrix} A^\dagger & -E^\dagger \\ -E^\dagger & A^\dagger \end{pmatrix}, \tag{11}$$

Finally, if the expressions (8) are applied in (9) we deduce the formula obtained by Macfarlane [13]:

$$S = \frac{1}{4\sqrt{G}} \Big[GI + \frac{i}{2}\epsilon_{\mu\nu\alpha\beta} L^{\mu\nu} L^{\alpha\beta}\gamma^5 + i\Gamma(L^2) - i\big(2 + tr\ L\big)\Gamma(L)\Big], \tag{12}$$

such that:

$$G = 2\big(1 + tr\ L\big) + \frac{1}{2}\Big[(tr\ L)^2 - tr\ L^2\Big], \quad tr\ L = \sum_{\mu=0}^{3} L^\mu{}_\mu, \quad tr\ L^2 = \sum_{\nu,\alpha=0}^{3} L^\nu{}_\alpha L^\alpha{}_\nu,$$

$$\Gamma(L) = \sum_{\mu,\nu=0}^{3} L_{\mu\nu}\sigma^{\mu\nu}, \qquad \Gamma(L^2) = \sum_{\alpha,\mu,\nu=0}^{3} L_{\mu\alpha} L^\alpha{}_\nu \sigma^{\mu\nu}. \tag{13}$$

References

[1] J. Rumer, *Spinorial analysis*, Moscow (1936).
[2] J. Aharoni, *The special theory of relativity*, Clarendon Press, Oxford (1959).
[3] J. L. Synge, *Relativity: the special theory*, North-Holland, Amsterdam (1965).
[4] R. Penrose, W. Rindler, *Spinors and space-time. I*, Cambridge University Press (1984).
[5] J. López-Bonilla, J. Morales, G. Ovando, *On the homogeneous Lorentz transformation*, Bull. Allahabad Math. Soc. 17, (2002) 53-58.
[6] M. Acevedo, J. López-Bonilla, M. Sánchez, *Quaternions, Maxwell equations and Lorentz transformations*, Apeiron 12, No. 4 (2005) 371-384.
[7] Z. Ahsan, J. López-Bonilla, B. M. Tuladhar, *Lorentz transformations via Pauli matrices*, J. of Advances in Natural Sci. 2, No. 1 (2014) 49-51.
[8] J. López-Bonilla, M. Morales-García, *Factorization of the Lorentz matrix*, Comput. Appl. Math. Sci. 5, No. 2 (2020) 32-33.
[9] R. Cruz-Santiago, J. López-Bonilla, N. Mondragón-Medina, *Unimodular matrix for a given Lorentz transformation*, Studies in Nonlinear Sci. 6, No. 1 (2021) 4-6.
[10] I. Miranda-Sńchez, J. López-Bonilla, D. Vázquez-Álvarez, *Dirac, Weyl and Majorana representations of the gamma matrices*, Studies in Nonlinear Sci. 6, No. 2 (2021) 24-28.
[11] J. Leite-Lopes, *Introduction to quantum electrodynamics*, Trillas, Mexico (1977).
[12] T. Ohlsson, *Relativistic quantum physics*, Cambridge University Press (2011).
[13] A. J. Macfarlane, *Dirac matrices and the Dirac matrix description of Lorentz transformations*, Commun. Math. Phys. 2, (1966) 133-146.

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