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PROPERTIES OF NORMAL SUBGROUPS RELATED TO PSEUDOCHARACTERS

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Abstract. We prove several properties of normal subgroups (the so-called center of a pseudocharacter and the so-called big center of a pseudocharacter) introduced in our recent papers. We correct some details concerning these normal subgroups and prove that, if the pseudocharacter in question vanishes on the center of the group. then the center of the group is contained in the center of the pseudocharacter.

§ 1. Introduction

In this note, we prove several properties of normal subgroups (the so-called center of a pseudocharacter and the so-called big center of a pseudocharacter) introduced in [1] and [2]. We correct some details in these papers concerning these normal subgroups and prove that, if the pseudocharacter in question vanishes on the center of the group. then the center of the group is contained in the center of the pseudocharacter. For the definition and properties of pseudocharacters on groups, see [3]–[6].

§ 2. Preliminaries

We begin with the corrected version of the theorem in [2], which was in fact proved in the proof of the theorems in [1] and [2].

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Theorem. *Let G be a group, let f be a pseudocharacter on G, and let $N = \ker f$, i.e.,*

$$N = \{g \in G : f(g) = 0\}.$$

Consider the set N_0 of all elements $n \in G$ such that

$$f(gn) = f(g) + f(n)$$

for all $g \in G$. Then

(1) $N_0^{-1} \subset N_0$;

(2)

$$f(ng) = f(n) + f(g)$$

for all $n \in N_0$ and all $g \in G$;

(3) N_0 contains the products of its elements, i.e.,

$$f(gn_1n_2) = f(g) + f(n_1n_2) = f(g) + f(n_1) + f(n_2)$$

for every $n_1, n_2 \in N_0$ and all $g \in G$;

(4) N_0 is invariant under the inner automorphisms of G;

(5) N_0 is a normal subgroup of G;

(6) N_0 contains the maximal normal subgroup N_1 in the kernel of f, which coincides with the set N_1 of all elements $n \in N$ such that

$$f(gn) = f(g)$$

for all $g \in G$; this set has the properties similar to (1)–(5).

Definition. *The normal subgroup N_1 of G is called the center of the pseudocharacter f, and the normal subgroup N_0 of G is called the big center of the pseudocharacter f.*

§ 3. Main Results

Recall that, for a pseudocharacter f on a group G, we have $f(g^n) = nf(g)$, $g \in G$, $n \in \mathbb{Z}$ (the integers), and hence

$$f(g) = \lim_{n\to\infty} n^{-1} f(g^n).$$

This fact immediately implies the following assertion.

Theorem 1. *Let G be a group, let f be a pseudocharacter on G, and let $n \in G$. The element $n \in G$ belongs to the big center N_0 of the pseudocharacter f if and only if*

$$f((gn)^m) - f(g^m) - f(n^m) = o(m) \quad \text{as} \quad m \to +\infty \tag{1}$$

for every $g \in G$. In particular, if $n \in N$ in addition, then n belongs to the center N_1 of the pseudocharacter f.

This has an obvious corollary.

Theorem 2. *If, under the assumptions of Theorem 1, n belongs to the center Z of the group G, then n belongs to the big center of f, and hence $Z \subset N_0$ and $Z \cap N \subset N_1$.*

Proof. The proof follows immediately from Theorem 1 due to formula (1), since, for $n \in Z$, we have

$$(gn)^m = g^m n^m,$$

and hence the left-hand side of (1) vanishes.

§ 4. Discussion

The Rademacher pseudocharacter [3] on $G = \mathrm{SL}(2, \mathbb{Z})$ vanishes on the center of the group G; it can readily be seen that the center of the Rademacher pseudocharacter coincides with the center of the group G.

The Guichardet–Wigner pseudocharacter [4] on the universal covering group of a simple Hermitian symmetric Lie group G is nontrivial on the center of G, and hence the center of G does not belong to the center of the Guichardet–Wigner pseudocharacter and belongs to the big center of the pseudocharacter,

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