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ON THE 4×4 MATRICES ASSOCIATED WITH THE QUATERNIONIC UNITS

J. D. BULNES, T. KIM, AND J. LÓPEZ-BONILLA

Abstract. Here we consider real and unitary quaternions which generate 3-rotations, and we show the corresponding 4×4 matrices associated with the quaternionic units.

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1. Introduction

For an arbitrary matrix $M_{4\times 4}$ it is possible to obtain its expression in terms of the sixteen Dirac matrices in the standard representation [1, 2, 3, 4, 5]:

$$I, \qquad \gamma^\mu, \qquad \gamma^5, \qquad \gamma^\mu\gamma^5, \qquad \sigma^{\mu\nu}, \tag{1}$$

in fact:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = e_0\gamma^0\gamma^5 + b_0 I + c_0\gamma^0 + id_0\gamma^5 + b_1\sigma^{23} + b_2\sigma^{31} + b_3\sigma^{12} +$$

$$+\sum_{j=1}^{3}\Big(e_j\gamma^j + c_j\gamma^j\gamma^5 + d_j\sigma^{0j}\Big), \tag{2}$$

$$= \sum_{k=0}^{3}\left[e_k\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} + b_k\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} + c_k\begin{pmatrix} \sigma_k & 0 \\ 0_k & -\sigma_k \end{pmatrix} + id_k\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\right],$$

in terms of the Pauli matrices with $\sigma_0 = I_{2\times 2}$, therefore:

$$A = \sum_{k=0}^{3}\Big(b_k + c_k\Big)\sigma_k, \qquad B = \sum_{k=0}^{3}\Big(e_k + id_k\Big)\sigma_k,$$

$$C = \sum_{k=0}^{3}\Big(-e_k + id_k\Big)\sigma_k, \qquad D = \sum_{k=0}^{3}\Big(b_k - c_k\Big)\sigma_k,$$

whose solution is given by:

$$e_k = \frac{1}{4}\,tr\Big[\Big(B - C\Big)\sigma_k\Big], \qquad d_k = -\frac{i}{4}\,tr\Big[\Big(B + C\Big)\sigma_k\Big],$$

$$k = 0, 1, 2, 3,$$

$$c_k = \frac{1}{4}\, tr\Big[\Big(A - D\Big)\sigma_k\Big], \qquad b_k = \frac{1}{4}\, tr\Big[\Big(A + D\Big)\sigma_k\Big]. \tag{3}$$

That is, if we have M, therefore we know A, B, C, D and we can calculate $(A \pm D)$ and $(B \pm C)$ and their products with the Pauli matrices, finally we obtain the corresponding traces to determine the coefficients of the expansion (2) in according with (3). For example, M could be any Lorentz matrix [6, 7], or the matrix that transforms a Dirac spinor under Lorentz mappings [8, 9], the Frenet−Serret curvature matrix for the motion of point particles in Minkowski spacetime [10, 11, 12], or the Faraday complex matrix [13], etc.

Here we consider real and unitary quaternions [14, 15, 16, 17]:

$$\mathbf{A} = a_0 + a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}, \quad \bar{\mathbf{A}} = a_0 - a_1\mathbf{I} - a_2\mathbf{J} - a_3\mathbf{K}, \quad a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1, \tag{4}$$

with which we can associate the following 4×4 matrix [18, 19]:

$$M = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \tag{5}$$

in terms of the Euler−Olinde Rodrigues parameters [20, 21, 22, 23]. In Sec. 2 we employ the relations (2), ..., (5) to determine the corresponding 4×4 matrices associated with the quaternionic units.

2. Quaternionic units and its association with 4×4 matrices

If we use (5) in (3) we obtain that the only non-zero values are:

$$b_0 = a_0, \qquad c_2 = ia_1, \qquad d_2 = a_3, \qquad e_0 = a_2, \tag{6}$$

then (2) gives the expansion:

$$M = a_0 I + ia_1\gamma^2\gamma^5 + a_2\gamma^0\gamma^5 + a_3\sigma^{02}, \tag{7}$$

which implies the following matrix association for the quaternionic units:

$$1 \;\to\; I, \qquad \mathbf{I} \;\to\; i\gamma^2\gamma^5, \qquad \mathbf{J} \;\to\; \gamma^0\gamma^5, \qquad \mathbf{K} \;\to\; \sigma^{02}, \tag{8}$$

in harmony with the expressions:

$$\mathbf{IJ} = \mathbf{K}, \quad \mathbf{JK} = \mathbf{I}, \quad \mathbf{KI} = \mathbf{J}, \quad \mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -1. \tag{9}$$

On the other hand, the quaternion (4) allows generate arbitrary 3−rotations [24, 25] via the relation [19, 26, 27, 28, 29, 30, 31]:

$$\tilde{\mathbf{X}} = \mathbf{A}\mathbf{X}\bar{\mathbf{A}}, \qquad \mathbf{X} = ct - ix\mathbf{I} - iy\mathbf{J} - iz\mathbf{K}, \tag{10}$$

where all a_ν are real, then in (10) we can apply the matrix association (7) to obtain that $\tilde{t} = t$ and:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad RR^T = I, \tag{11}$$

such that the matrix R represents a passive rotation (in the terminology of Ryder [32]) [19, 30, 31, 33, 34, 35]:

$$R = \begin{pmatrix} 1-2(a_2^2+a_3^2) & 2(a_1a_2-a_0a_3) & 2(a_1a_3+a_0a_2) \\ 2(a_1a_2+a_0a_3) & 1-2(a_1^2+a_3^2) & 2(a_2a_3-a_0a_1) \\ 2(a_1a_3-a_0a_2) & 2(a_0a_1+a_2a_3) & 1-2(a_1^2+a_2^2) \end{pmatrix}. \tag{12}$$

Remark.- The expression (5) can be obtained from (7) and the relations [1, 2, 4, 5, 36, 37]:
(13)

$$\gamma^2\gamma^5 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \ \gamma^0\gamma^5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \ \sigma^{02} = i \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

References

[1] R. H. Good, *Properties of the Dirac matrices*, Rev. Mod. Phys. 27, No. 2 (1955) 187-211.
[2] J. D. Bjorken, S. D. Drell, *Relativistic quantum mechanics*, McGraw-Hill, New York (1964).
[3] N. Hamdan, A. Chamaa, J. López-Bonilla, *On the relativistic concept of the Diracs electron spin*, Lat. Am. J. Phys. Educ. 2, No. 1 (2008) 65-70.
[4] J. López-Bonilla, L. Rosales-Roldán, A. Zúñiga-Segundo, it Dirac matrices via quaternions, J. Sci. Res. (India) 53 (2009) 253-255.
[5] V. G. Bagrov, D. Gitman, *The Dirac equation and its solutions*, Walter de Gruyter GmbH, Berlin (2014).
[6] J. López-Bonilla, M. Morales-García, *Factorization of the Lorentz matrix*, Comput. Appl. Math. Sci. 5, No. 2 (2020) 32-33.
[7] J. López-Bonilla, D. Morales-Cruz, S. Vidal-Beltrán, *On the Lorentz matrix*, Studies in Nonlinear Sci. 6, No. 1 (2021) 1-3.
[8] J. López-Bonilla, J. Morales, G. Ovando, *Dirac spinor's transformations under boosts and 3-rotations*, Studies in Nonlinear Sci. 5, No. 3 (2020) 45-49.
[9] J. López-Bonilla, G. Sánchez-Meléndez, D. Vzquez-Álvarez, *2×2 complex matrices and 4-spinors*, Studies in Nonlinear Sci. 6, No. 1 (2021) 7-9.
[10] F. Gürsey, *Relativistic kinematics of a classical point particle in spinor form*, Nuovo Cim. 5, No. 4 (1957) 784-809.
[11] J. López-Bonilla, G. Ovando, J. M. Rivera-Rebolledo, *Intrinsic geometry of curves and the Bonnor's equation*, Proc. Indian Acad. Sci. (Math. Sci.) 107, No. 1 (1997) 43-55.
[12] J. D. Bulnes, S. Ray, R. Cruz-Santiago, J. López-Bonilla, *Lorentz transformation and the intrinsic geometry of a time-like curve in Minkowski spacetime*, Prespacetime J. 14, No. 3 (2023) 354-361.
[13] J. Morales, G. Ovando, J. López-Bonilla, *Faraday complex matrix in terms of Dirac matrices*, Sarcouncil J. Appl. Sci. 2, No. 2 (2022) 1-4.
[14] L. Byron Mc Allister, *A quick introduction to quaternions*, Pi Mu Epsilon J. 9, No. 1 (1989) 23-25.
[15] P. V. Arunachalam, W. R. *Hamilton and his quaternions*, Math. Ed. 6, No. 4 (1990) 261-266.

[16] A. J. Hanson, *Visualizing quaternions*, Elsevier, San Francisco CA, USA (2006).

[17] T. Bannon, *The origin of quaternions*, Coll. Math. J. 46, No. 1 (2015) 43-50.

[18] J. Pedro Morais, S. Georgiev, W. Sprössig, *Real quaternionic calculus handbook*, Birkhuser, Basel (2014).

[19] J. D. Bulnes, T. Kim, J. López-Bonilla, *Factorization of rotation and Lorentz matrices via Euler-Olinde Rodrigues parameters*, Appl. Math. Inf. Sci. 16, No. 4 (2022) 659-663.

[20] L. Euler, *Formulae generals pro translatione quacunque corporum rigidorum*, Novi Comm. Acad. Sci. Imp. Petrop. 20 (1775) 189-207.

[21] L. Euler, *Nova methodus motum corporum rigidorum determinandi*, Novi. Comm. Acad. Sci. Imp. Petrop. 20 (1775) 208-238.

[22] B. Olinde Rodrigues, *Des lois géométriques qui régissent les déplacements d'un systme solide*, J. de Math. (Liouville) 5 (1840) 380-440.

[23] R. E. Roberson, *Kinematic equations for bodies whose rotation is described by the Euler-Rodrigues parameters*, AIAA Journal 6 (1968) 5-71.

[24] H. Cheng, K. C. Gupta, *An historical note on finite rotations*, J. of Appl. Mechanics 56 (1989) 139-145.

[25] V. Chi, *Quaternions and rotations in 3-space: How it works*, Univ. of North Carolina at Chapel Hill Dept. of Computer Science (1998) 1-10.

[26] L. Silberstein, *Quaternionic form of relativity*, Phil. Mag. 23, No. 137 (1912) 790-809.

[27] C. Cailler, *Sur quelques formules de la théorie de la relativité*, Archs. Sci. Phys. Nat. Genéve Ser. IV 44 (1917) 237-255.

[28] C. Lanczos, *The variational principles of mechanics*, University of Toronto Press (1970).

[29] J. L. Synge, *Quaternions, Lorentz transformations, and the Conway-Dirac-Eddington matrices*, Comm. Dublin Inst. Adv. Stud. Series A, No. 21 (1972).

[30] J. López-Bonilla, J. Morales, G. Ovando, *Quaternions, 3-rotations and Lorentz transformations*, Indian J. Theor. Phys. 52, No. 2 (2004) 91-96.

[31] I. Guerrero-Moreno, J. López-Bonilla, L. Rosales-Roldán, *Rotations in three and four dimensions via 2×2 complex matrices and quaternions*, The Icfai Univ. J. Phys. 1, No. 2 (2008) 7-13.

[32] L. H. Ryder, *Quantum field theory*, Cambridge University Press (1996).

[33] J. Prajapati, J. López-Bonilla, R. López-Vázquez, *Rotations in three dimensions*, J. of Interdisciplinary Maths. 18, No. 1-2 (2015) 97-102.

[34] J. Morales, G. Ovando, J. López-Bonilla, R. López-Vázquez, *Rotations via quaternions*, World Eng. Appl. Sci. J. 10, No. 1 (2019) 8-12.

[35] J. López-Bonilla, M. Morales-García, S. Vidal-Beltrán, 3-Rotations, Studies in Nonlinear Sci. 5, No. 3 (2020) 38-40.

[36] J. Leite-Lopes, *Introduction to quantum electrodynamics*, Trillas, México (1977).

[37] J. López-Bonilla, I. Miranda-Sánchez, D. Vázquez-Álvarez, *Dirac, Weyl and Majorana representations of the gamma matrices*, Studies in Nonlinear Sci. 6, No. 2 (2021) 24-28.

Departamento de Ciências Exatas e Tecnologias, Universidade Federal do Amapá, Rod. J. Kubitschek, 68903-419, Macapá, AP, Brazil,

E-mail address: `bulnes@unifap.br`

Department of Mathematics, College of Natural Sciences, Kwangwoon University, Seoul 139−701, Republic of Korea,

E-mail address: `tkkim@kw.ac.kr`

ESIME−Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista, CP 07738, CDMX, México,

E-mail address: `jlopezb@ipn.mx`