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Inclusion of two Summability Methods for Improper Integral

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Abstract. Introducing the concept of $|R,p|_k,\ (k \geq 1)$summability of improper integrals, Orhan established a result on the inclusion of two summability methods for improper integrals by extending his own result which was on infinite series. however in this paper we establish an relation between two index summability methods $|R,p;\delta|_k$ and $|R,q;\delta|_k,\ (k \geq 1)$ for improper integrals.

1. Introduction

Throughout this paper we assume that f is a real valued function which is continuous on $[0,\infty]$ and $s(x) = \int_0^\infty f(t)\ dt$. Let $\sigma(x)$ be the Cesáro mean of $s(x)$. Let $\nu(x) = \frac{1}{x}\int_0^\infty tf(t)\ dt$. As defined by Flett[2], the integral $\int_0^\infty f(t)\ dt$ is said to be integrable $|C,1|_k,\ k \geq 1$, if

$$\int_0^\infty x^{k-1}|\sigma'(x)|^k\ dx \quad = \quad \int_0^\infty \frac{|\nu(x)|^k}{x}\ dx \tag{1.1}$$

is convergent. In the present case, we call $\nu(x) = \frac{1}{x}\int_0^\infty tf(t)\ dt$ as a generator of the integral $\int_0^\infty f(t)\ dt$. Let p be a real valued, non-decreasing function defined on $[0,\infty)$ such that

$$P(x) \quad = \quad \int_0^x p(t)\ dt,\ p(x) \neq 0,\ p(0) = 0\ .$$

The Riesz mean of $s(x)$ is defined by

$$\sigma_p(x) \quad = \quad \frac{1}{P(x)}\int_0^x P(t)s(t)\ dt\ .$$

We say that the integral $\int_0^\infty f(t)\ dt$ is integrable $|R,p|_k,\ k \geq 1$, if

$$\int_0^\infty x^{k-1}|\sigma_p'(x)|^k\ dx \tag{1.2}$$

is convergent. As a special case if we take $P(x) = 1$ for all values of x, then $|R,p|_k,\ k \geq 1$ integrability reduces to $|C,1|_k$ integrability of improper integrals. For any functions f and g , it is customary to write $g(x) = O\Big(f(x)\Big)$, if there exists η and N, such that for every $x > N$, $\left|\frac{g(x)}{f(x)}\right| \leq \eta$. The difference between $s(x)$ and its nth weighted mean $\sigma_p(x)$, which is called the weighted Kronecker identity, is given by the identity

$$s(x) - \sigma_p(x) = \nu_p(x) \tag{1.3}$$

where

$$\nu_p(x) = \frac{1}{P(x)} \int_0^\infty p(u) f(u)\ du\ .$$

In particularly, by taking $p(x) = 1$, for all values of x, the identity (1.3) reduces to (See[1]) $s(x) - \sigma_p(x) = \nu(x)$. Since $\sigma_p'(x) = \frac{p(x)}{P(x)} \nu_p(x)$, condition (1.3) can be written as

$$s(x) \ = \ \nu_p(x) \ + \ \int_0^x \frac{p(u)}{P(u)} \nu_p(u)\ du\ . \tag{1.4}$$

In view of the identity (1.4), the function $v_p\left(x\right)$ is called the generator function of $s(x)$. Condition (1.1) can also be written as

$$\int_0^\infty x^{k-1} \left(\frac{p\left(x\right)}{P\left(x\right)} \right)^k \left| v_p\left(x\right) \right|^k dx \tag{1.5}$$

and is convergent. The improper integral $\int_0^\infty f(t)\ dt$ is integrable $\left| R, p; \delta \right|_k$, if

$$\int_0^\infty x^{\delta k + k - 1} \left(\frac{p\left(x\right)}{P\left(x\right)} \right)^k \left| v_p\left(x\right) \right|^k dx < \infty\ . \tag{1.6}$$

2. Known Result

It is noted that for infinite series, an analogous definition was introduced by Orhan[3] . Using this definition , Orhan [3] proved the following theorem dealing with $|R, p_n|_k$ and $|R, q_n|_k$ summability methods.

Theorem 1

The $|R, p_n|_k$, $(k \geq 1)$ summability implies the $|R, q_n|_k$, $(k \geq 1)$ summability provided that

$$nq_n = O\left(Q_n\right)\ , \tag{2.1}$$

$$P_n = O\left(np_n\right) \tag{2.2}$$

and

$$Q_n = O\left(nq_n\right)\ . \tag{2.3}$$

Dealing with integrability of improper integrals Orhan established the following theorem.

Theorem 2

Let p and q be real valued, non-decreasing functions on $[0,\infty)$ such that as $x \to \infty$

$$xq\left(x\right) = O\left(Q\left(x\right)\right) \ , \tag{2.4}$$

$$P(x) = O(xp)(x)) \tag{2.5}$$

and

$$Q\left(x\right) = O\left(xq\left(x\right)\right) \ . \tag{2.6}$$

If $\int_0^\infty f\left(t\right)dt$ is integrable $|R,p|_k$, then it is also integrable $\left|R,q\right|_k$, $(k \geq 1)$.

3. Main Result

Extending the result of Ohran, in the present paper we establish the following theorem.

Theorem 3

Let $p(x)$ and $q(x)$ be two real valued, non-decreasing functions on $[0,\infty)$ satisfying (2.4),(2.5),(2.6) together with

$$\int_t^m \frac{x^{k\delta}q(x)}{Q^2(x)}dx \quad = \quad O\bigg(\frac{t^{k\delta}}{Q(t)}\bigg) \tag{3.1}$$

and

$$\int_0^m t^{k\delta-1}\bigg|\nu_p(t)\bigg|^k dt \;\; = \;\; O(1). \tag{3.2}$$

If $\int_0^\infty f\left(t\right)dt$ is summable $|R,p;\delta|_k$, then it is also summable $\left|R,q;\delta\right|_k$, $(k \geq 1)$.

Proof

Let $\sigma_p\left(x\right)$ and $\sigma_q\left(x\right)$ be the functions of (R,p) and (R,q) means of the integral $\int_0^\infty f\left(t\right)dt$. If $\int_0^\infty f\left(t\right)dt$ is summable $|R,p:\delta|_k$, then

$$\int_0^\infty x^{k\delta+k-1}\left(\frac{p\left(x\right)}{P\left(x\right)}\right)^k\left|v_p\left(x\right)\right|^k dx$$

is convergent. Differentiating the equation (1.4) , we have

$$f(x) = v_p'(x) + \frac{p(x)}{P(x)} v_p(x) \ .$$

By definition, we obtain

$$\sigma_q(x) = \frac{1}{Q(x)} \int_0^x q(t) s(t)\, dt = \frac{1}{Q(x)} \int_0^x (Q(x) - Q(t)) f(t)\, dt$$

and

$$\sigma_q'(x) = \frac{q(x)}{Q^2(x)} \int_0^x Q(t) f(t)\, dt = \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \left[v'_p(t) + \frac{p(t)}{P(t)} v_p(t) \right] dt \ .$$

Integrating by parts of the first statement, we have

$$\begin{aligned}
\sigma_q'(x) &= \frac{q(x)}{Q^2(x)} \left[Q(x)\, v_p(x) - \int_0^x q(t)\, v_p(t)\, dt \right] + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t)\, dt \\
&= \frac{q(x)}{Q^2(x)} v_p(x) + \frac{q(x)}{Q^2(x)} \int_0^x Q(t) \frac{p(t)}{P(t)} v_p(t)\, dt - \frac{q(x)}{Q^2(x)} \int_0^x q(t)\, v_p(t)\, dt \\
&= \sigma_{q,1}(x) + \sigma_{q,2}(x) + \sigma_{q,3}(x) \, , say \ .
\end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$\int_0^m x^{\delta k + k - 1} |\sigma_{q,r}(x)|^k dx = O(1) \ \ as \ m \to \infty, for \ \ r = 1, 2, 3$$

Using conditions (3.1) and (3.2), we have

$$\begin{aligned}
\int_0^m x^{\delta k + k - 1} |\sigma_{q,1}(x)|^k dx &= \int_0^m x^{\delta k + k - 1} \left| \frac{q(x)}{Q(x)} v_p(x) \right|^k dx \\
&= \int_0^m x^{\delta k + k - 1} \left(\frac{q(x)}{Q(x)} \right)^k |v_p(x)|^k dx \\
&= O(1) \int_0^m x^{\delta k + k - 1} \left(\frac{p(x)}{P(x)} \right)^k |v_p(x)|^k dx \\
&= O(1) \int_0^m x^{\delta k + k - 1} \left| \sigma'_p(x) \right|^k dx \\
&= O(1) \, as \ \ m \to \infty
\end{aligned}$$

by virtue of the hypotheses of theorem 3. Applying Hölder's inequality with $k > 1$, we get

$$\begin{aligned}
\int_0^m x^{\delta k+k-1}|\sigma_{q,2}(x)|^k dx &= O(1)\int_0^m x^{\delta k+k-1}\left(\frac{q(x)}{Q^2(x)}\right)^k\left(\int_0^x \frac{Q(t)\,p(t)}{P(t)}|v_p(t)|\,dt\right)^k dx\\
&= O(1)\int_0^m \frac{q(x)}{Q^{k+1}(x)}\left(\int_0^x \frac{Q(t)\,p(t)}{P(t)}|v_p(t)|\,dt\right)^k dx\\
&= O(1)\int_0^m \frac{q(x)}{Q^2(x)}\left(\int_0^x \left(\frac{Q(t)}{q(t)}\right)^k q(t)\left(\frac{p(t)}{P(t)}\right)^k|v_p(t)|^k dt\right)\left(\frac{1}{Q(x)}\int_0^x q(t)\,dt\right)^{k-1} d\\
&= O(1)\int_0^m t^k q(t)\left(\frac{p(t)}{P(t)}\right)^k|v_p(t)|^k dt\int_t^m \frac{q(x)}{Q^2(x)}dx\\
&= O(1)\int_0^m t^k\frac{q(t)}{Q(t)}\left(\frac{p(t)}{P(t)}\right)^k|v_p(t)|^k dt\\
&= O(1)\int_0^m t^{k-1}\left(\frac{p(t)}{P(t)}\right)^k|v_p(t)|^k dt\\
&= O(1)\int_0^m t^{k-1}\left|{\sigma'}_p(t)\right|^k dt\\
&= O(1)\, as\ \ m\to\infty
\end{aligned}$$

by virtue of the hypotheses of theorem 3. Finally, again by Hölder's inequality with $k > 1$, we have

$$\begin{aligned}
\int_0^m x^{\delta k+k-1}|\sigma_{q,3}(x)|^k dx &= O(1)\int_0^m x^{\delta k+k-1}\left(\frac{q(x)}{Q^2(x)}\right)^k\left(\int_0^x q(t)\,|v_p(t)|^k dt\right)^k dx\\
&= O(1)\int_0^m \frac{q(x)}{Q^2(x)}\left(\int_0^x q(t)\,|v_p(t)|^k dt\right)x\left(\frac{1}{Q(x)}\int_0^x q(t)\,dt\right)^{k-1} dx\\
&= O(1)\int_0^m q(t)|v_p(t)|^k dt\int_t^m \frac{q(x)}{Q^2(x)}dx\\
&= O(1)\int_0^m \frac{q(x)}{Q(t)}|v_p(t)|^k dt\\
&= O(1)\quad as\quad m\to\infty
\end{aligned}$$

by virtue of the hypotheses of theorem 3. This completes the proof of theorem 3.

4. Conclusion

In the field of summability, there are many inclusion theorems of two summability methods for infinite series. In the present paper, we establish a result on inclusion of two summability methods for improper integrals. This result generalizes many results.Further study may be proceeded for other summability methods for improper integrals.

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