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A Hybridized One-Step Direct Block Method for Third-Order Ordinary Differential Equations

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Abstract

Direct solution of Higher of ordinary differential equations have yet to gain more attention and be explored extensively despite it's vast presence in sciences and engineering. This study employs hybrid block methods within a step interval with higher order and good precision to directly solve third-order problems of ordinary differential equations without resulting in a reduction approach. The procedure used in constructing the method is by interpolating the trial function at $[x_n]$, collocating every other point on the grids within the interval $[0, 1]$. Block mode is employed in implementing the new method, which simultaneously evaluates the continuous schemes of the approximations at all points within the integration interval. The off-grid points introduced aim to improve the method's accuracy while circumventing the Dahlquist barrier within a small step number. We established the proposed method's characteristics, which include order, zero stability, and convergence. Application of the methods to third-order problems in sciences and engineering is given to assess its significance.

2000 Mathematics Subject Classification: 65L05, 65L10

Keywords: Third-order problems, Convergence, Hybrid block methods, Boundary value problems, Initial value problems

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1 Introduction

In this paper, consideration was given to the direct numerical approximation of third order Initial value and boundary value problems of ordinary differential equations. The direct methods for the solution of

$$y'''(t) = (t, y,\ y', y'') \tag{1}$$

with initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y'(0),\ y''(t_0) = y''(0) \tag{2}$$

or boundary conditions

$$g_1(y(t_0),\ \ y'(t_0),\ \ y''(t_0)) = y_a, \quad g_2(y(t_n),\ \ y'(t_n),\ \ y''(t_n)) = y_b \tag{3}$$

has been extensively studied by several authors as appeared in severally literatures [1-5]. The direct method or approach has shown advantages accuracy and speed over the reduction approach to a system of first order, which has a human computational burden [6-9]. All the above authors have either limited their research to applying their proposed methods to solve either third order initial or boundary value problems but not both.
While scholars like Abd El-Sala *et. al.* [1], Akram [4] and Al-Said & Rehman [6],consider different spline methods for the solution of third order boundary value problems of ordinary differential equations,it was observed that the accuracy of the method was small compare to the research work of Adoghe & Omole [3], Awoyemi *et. al.* [7], Reem & Fudziah [18] who worked on the linear multistep methods for the solution of third order intial value problems of ordianry differential equations.

The hybrid block method has been seen to be one of the numerical methods that perform favourably well because it combines the block method's advantage and overcomes the zero stability barrier in the linear multistep method according to Adesanya *et. al.* [2] and Adoghe & Omole [3]. Third-order ordinary differential equations are found in science and engineering, such as the sandwich boundary layer and laminar flow beam, thin-film flow, the motion of rockets, the study of stellar interiors, draining and coating flow, fluid dynamics and mechanics. The applications of these problems have critical applications in engineering and sciences. The introduction of numerical methods became necessary since it is not all ordinary differential equations that can be solved analytically. According [16], their are three primary approaches for solving Boundary value problems of ordinary differential equations which are the finite-difference methods [10],shooting method and methods that are based on approximating the solution by combination of trial functions [19]. Other approaches include but are not limited to variation iteration method [8], Adonian decomposition, Spline techniques [1, 4, 6], homotopy perturbation method[5], multistep methods[16, 17,20] , and modified Adomian decomposition techniques. Special attention were given to differential integral equations [10, 11, 15] where the precision and efficiency of the methods are of great importance.

2 Construction of Hybridized Block Method

Our interest is obtaining the approximate solution of $y(t)$ at the grid point $t_0 < t_1 < ... < t_N$ within the interval of integration $[t_0, t_n]$. In deriving the formula, assume $y(t)$ to be approximated within an interval $[t_n, t_{n+1}]$ by the given polynomial $q(t)$

$$y(t) = q(t) = \sum_{r=0}^{c+i-1} \varsigma_r t^r \tag{4}$$

where $\varsigma_r \in R$ are coefficients of the continuous scheme which will determined by collocating at some selected grid points and t is continuously differentiable. The collocation points is represented c while i is for the interpolation points. Since the polynomial must pass through the points $(x_n, y_n), (x_{n+\frac{1}{5}}, y_{n+\frac{1}{5}}),$

$(x_{n+\frac{2}{5}}, y_{n+\frac{2}{5}}), (x_{n+\frac{3}{5}}, y_{n+\frac{3}{5}}), (x_{n+\frac{4}{5}}, y_{n+\frac{34}{5}}), (x_{n+1}, y_{n+1})$, we demand the following $i+c$ equations must be satisfied.

$$y(t_n) = y_n, \quad y'(t_n) = y'_n, \quad y''(t_n) = y''_n, \quad f_{n+r} = y'''_{n+r}, \quad r = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$$

$$f(t, y, y', y'') = q(t) = \sum_{r=0}^{c+i-1} (r-1)(r-2)\varsigma_r t^r \tag{5}$$

Equation (4) and (5) is expressed in matrix form

$$TA = B$$

given as

$$\begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 & t_n^8 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 & 8t_n^7 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 & 56t_n^6 \\ 0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 & 336t_n^5 \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{1}{5}} & 60t_{n+\frac{1}{5}}^2 & 120t_{n+\frac{1}{5}}^3 & 210t_{n+\frac{1}{5}}^4 & 336t_{n+\frac{1}{5}}^5 \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{2}{5}} & 60t_{n+\frac{2}{5}}^2 & 120t_{n+\frac{2}{5}}^3 & 210t_{n+\frac{2}{5}}^4 & 336t_{n+\frac{2}{5}}^5 \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{3}{5}} & 60t_{n+\frac{3}{5}}^2 & 120t_{n+\frac{3}{5}}^3 & 210t_{n+\frac{3}{5}}^4 & 336t_{n+\frac{3}{5}}^5 \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{4}{5}} & 60t_{n+\frac{4}{5}}^2 & 120t_{n+\frac{4}{5}}^3 & 210t_{n+\frac{4}{5}}^4 & 336t_{n+\frac{4}{5}}^5 \\ 0 & 0 & 0 & 6 & 24t_{n+1} & 60t_{n+1}^2 & 120t_{n+1}^3 & 210t_{n+1}^4 & 336t_{n+1}^5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_n \\ y'_n \\ y''_n \\ f_n \\ f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix}$$

After solving for the values of a_n, where $n(\frac{u}{v})1$, $\frac{u}{v}$ stands for $\frac{1}{5}$ for the method. Substituting $x = x_n + rh$, We now have the polynomial in (4) now written as

$$\begin{aligned} q(t_n + rh) \;\; = \;\; & \alpha_0(r)y_n + h\alpha_1(r)y'_n + h^2\alpha_2(r)y'' + h^3\big(\beta_0(r)f_n + \beta_{\frac{1}{5}}(r)f_{n+\frac{1}{5}} + \beta_{\frac{2}{5}}(r)f_{n+\frac{2}{5}} \\ & + \beta_{\frac{3}{5}}(r)f_{n+\frac{3}{5}} + \beta_{\frac{4}{5}}(r)f_{n\frac{4}{5}} + \beta_1(r)f_{n+1}\big) \end{aligned} \tag{6}$$

where h is the step size and $\alpha_0(r), \alpha_1(r), \alpha_2(r), \beta_0(r), \beta_{\frac{1}{5}}(r), \beta_{\frac{2}{5}}(r), \beta_{\frac{3}{5}}(r), \beta_{\frac{4}{5}}(r), \beta_1$ are continuous coefficients of (6). Solving for the unknown coefficient and substituting their values into the continuous scheme (6), and evaluating at all points except y_n, we obtain the proposed hybridized one step methods as follows:

$$y_{n+\frac{1}{5}} = y_n + h\frac{1}{5}y' + h^2\frac{1}{50}y'' + h^3\left(\frac{3929f_n}{5040000} + \frac{199f_{n+\frac{1}{5}}}{201600} + \frac{173f_{n+\frac{3}{5}}}{360000} - \frac{1931f_{n+\frac{2}{5}}}{2520000} - \frac{883f_{n+\frac{4}{5}}}{5040000} + \frac{139f_{n+1}}{5040000}\right) \quad (7)$$

$$y_{n+\frac{2}{5}} = y_n + h\frac{2}{5}y' + h^2\frac{2}{25}y'' + h^3\left(\frac{317f_n}{78750} + \frac{f_{n+1}}{5625} + \frac{367f_{n+\frac{1}{5}}}{39375} + \frac{122f_{n+\frac{3}{5}}}{39375} - \frac{38f_{n+\frac{2}{5}}}{7875} - \frac{89f_{n+\frac{4}{5}}}{78750} + \frac{1f_{n+1}}{5625}\right) \quad (8)$$

$$y_{n+\frac{3}{5}} = y_n + h\frac{3}{5}y' + h^2\frac{9}{50}y'' + h^3\left(\frac{783f_n}{80000} + \frac{16119f_{n+\frac{1}{5}}}{560000} + \frac{423f_{n+\frac{3}{5}}}{56000} - \frac{2187f_{n+\frac{2}{5}}}{280000} - \frac{1539f_{n+\frac{4}{5}}}{560000} + \frac{243f_{n+1}}{560000}\right) \quad (9)$$

$$y_{n+\frac{4}{5}} = y_n + h\frac{4}{5}y' + h^2\frac{8}{25}y'' + h^3\left(\frac{712f_n}{39375} + \frac{2336f_{n+\frac{1}{5}}}{39375} + \frac{704f_{n+\frac{3}{5}}}{39375} - \frac{8f_{n+\frac{4}{5}}}{1575} - \frac{32f_{n+\frac{2}{5}}}{5625} + \frac{32f_{n+1}}{39375}\right) \quad (10)$$

$$y_{n+1} = y_n + hy' + h^2\frac{1}{2}y'' + h^3\left(\frac{233f_n}{8064} + \frac{815f_{n+\frac{1}{5}}}{8064} + \frac{5f_{n+\frac{2}{5}}}{4032} + \frac{155f_{n+\frac{3}{5}}}{4032} - \frac{5f_{n+\frac{4}{5}}}{1152} + \frac{11f_{n+1}}{8064}\right) \quad (11)$$

The Hybridized method above need to be completed by evaluating $q'(t)$ and $q''(t)$ at all points except at y'_n and y''_n. The resulting formulas is given by

$$y'_{n+\frac{1}{5}} = y' + h\frac{1}{5}y'' + h^2\left(\frac{1231f_n}{126000} + \frac{863f_{n+\frac{1}{5}}}{50400} + \frac{941f_{n+\frac{3}{5}}}{126000} - \frac{761f_{n+\frac{2}{5}}}{63000} - \frac{341f_{n+\frac{4}{5}}}{126000} + \frac{107f_{n+1}}{252000}\right) \quad (12)$$

$$y'_{n+\frac{2}{5}} = y' + h\frac{2}{5}y'' + h^2\left(\frac{71f_n}{3150} + \frac{544f_{n+\frac{1}{5}}}{7875} + \frac{136f_{n+\frac{3}{5}}}{7875} - \frac{37f_{n+\frac{2}{5}}}{1575} - \frac{101f_{n+\frac{4}{5}}}{15750} + \frac{8f_{n+1}}{7875}\right) \quad (13)$$

$$y'_{n+\frac{3}{5}} = y' + h\frac{3}{5}y'' + h^2\left(\frac{123f_n}{3500} + \frac{3501f_{n+\frac{1}{5}}}{28000} + \frac{87f_{n+\frac{3}{5}}}{2800} - \frac{9f_{n+\frac{4}{5}}}{875} - \frac{9f_{n+\frac{2}{5}}}{3500} + \frac{9f_{n+1}}{5600}\right) \quad (14)$$

$$y'_{n+\frac{4}{5}} = y' + h\frac{4}{5}y'' + h^2\left(\frac{376f_n}{7875} + \frac{1424f_{n+\frac{1}{5}}}{7875} + \frac{176f_{n+\frac{2}{5}}}{7875} + \frac{608f_{n+\frac{3}{5}}}{7875} - \frac{16f_{n+\frac{4}{5}}}{1575} + \frac{16f_{n+1}}{7875}\right) \quad (15)$$

$$y'_{n+1} = y' + hy'' + h^2\left(\frac{61f_n}{1008} + \frac{475f_{n+\frac{1}{5}}}{2016} + \frac{25}{504}f_{n+\frac{2}{5}} + \frac{125f_{n+\frac{3}{5}}}{1008} + \frac{25f_{n+\frac{4}{5}}}{1008} + \frac{11f_{n+1}}{2016}\right) \quad (16)$$

$$y''_{n+\frac{1}{5}} = y'' + h\left(\frac{19f_n}{288} + \frac{1427f_{n+\frac{1}{5}}}{7200} + \frac{241f_{n+\frac{3}{5}}}{3600} - \frac{133f_{n+\frac{2}{5}}}{1200} - \frac{173f_{n+\frac{4}{5}}}{7200} + \frac{3f_{n+1}}{800}\right) \quad (17)$$

$$y''_{n+\frac{2}{5}} = y'' + h\left(\frac{14f_n}{225} + \frac{43f_{n+\frac{1}{5}}}{150} + \frac{7f_{n+\frac{2}{5}}}{225} + \frac{7f_{n+\frac{3}{5}}}{225} - \frac{1f_{n+\frac{4}{5}}}{75} + \frac{f_{n+1}}{450}\right) \quad (18)$$

$$y''_{n+\frac{3}{5}} = y'' + h\left(\frac{51f_n}{800} + \frac{219f_{n+\frac{1}{5}}}{800} + \frac{57f_{n+\frac{2}{5}}}{400} + \frac{57f_{n+\frac{3}{5}}}{400} - \frac{21f_{n+\frac{4}{5}}}{800} + \frac{3f_{n+1}}{800}\right) \quad (19)$$

$$y''_{n+\frac{4}{5}} = y'' + h\left(\frac{14f_n}{225} + \frac{64f_{n+\frac{1}{5}}}{225} + \frac{8f_{n+\frac{2}{5}}}{75} + \frac{64f_{n+\frac{3}{5}}}{225} + \frac{14f_{n+\frac{4}{5}}}{225}\right) \quad (20)$$

$$y''_{n+1} = y'' + h\left(\frac{19f_n}{288} + \frac{25f_{n+\frac{1}{5}}}{96} + \frac{25f_{n+\frac{2}{5}}}{144} + \frac{25f_{n+\frac{3}{5}}}{144} + \frac{25f_{n+\frac{4}{5}}}{96} + \frac{19f_{n+1}}{288}\right) \quad (21)$$

3 Analysis of basic properties of the method

Examination of the basic properties of the method in terms of the order of accuracy and error constant, consistency, zero stability and convergence analysis is carried out in this section

3.1 Local truncation error

The local truncation error (LTE) is difference between the exact solution $y\left(x_{n+i}\right)$ at $x_n = x_{n+i}$ and the numerical approximation

3.1.1 Order

3.1.2 Lemma

The order one-step hybridized method is 6

3.1.3 Proof

The block method is of the form

$$y(t) = \sum_{i=0}^{2} \alpha_i y_n^i(t_n) h^i + h^3 \sum_{i=0}^{1} \beta_i(t_n) f_{n+i} \tag{22}$$

Assuming,

$$y_{n+v} \approx y(t_n + vh), f_{n+j} \equiv (t_n + jh, y(t_n + jh))$$

and $y(x_n)$ differentiable on the interval $[a, b]$ continuously.
The local truncation error (LTE) is represented as the linear operator $L[y(x); h]$ for the developed method such that

$$L[y(t); h] = y(t_n + ih) - \left(\alpha_1 h y'(t) - \alpha_2 h^2 y''(t) - h^3 \sum_{j=0}^{1} \beta_j(t) f_{n+j} \right) \tag{23}$$

By expansion using Taylor's series approach about point t of the right hand side (RHS), the order of the method is 6 i.e $p = 6$

The Local Truncation Error (L.T.E) is

$$L.T.E.(t_n) \quad = \quad -y(t) + y_n\alpha_1 hy'(t) + \alpha_2 h^2 y''(t) + \alpha_5 h^3 y'''(t) + h^3 \sum_{j=0}^{1} \beta_j(t) f_{n+j} (24)$$

For each of the member of the block formula (7) - (21), the local truncation error was obtained using taylor's series expansion,

$$L[y(t_{n+\frac{1}{5}}); h] = -\frac{9809 h^9 y^{(9)}(0)}{7087500000000} + O\left(h^{10}\right), \quad L[y(t_{n+\frac{2}{5}}); h] = -\frac{491 h^9 y^{(9)}(0)}{55371093750} + O\left(h^{10}\right)$$

$$L[y(t_{n+\frac{3}{5}}); h] = -\frac{1917 h^9 y^{(9)}(0)}{87500000000} + O\left(h^{10}\right), \quad L[y(t_{n+\frac{14}{5}}); h] = -\frac{1136 h^9 y^{(9)}(0)}{27685546875} + O\left(h^{10}\right)$$

$$L[y(t_{n+1}); h] = -\frac{149 h^9 y^{(9)}(0)}{2268000000} + O\left(h^{10}\right), \quad L[y'(t_{n+\frac{1}{5}}); h] = -\frac{199 h^8 y^{(9)}(0)}{9450000000} + O\left(h^{9}\right)$$

$$L[y'(t_{n+\frac{2}{5}}); h] = -\frac{19 h^8 y^{(9)}(0)}{369140625} + O\left(h^{9}\right), \quad L[y'(t_{n+\frac{3}{5}}); h] = -\frac{141 h^8 y^{(9)}(0)}{1750000000} + O\left(h^{9}\right)$$

$$L[y'(t_{n+\frac{4}{5}}); h] = -\frac{8 h^8 y^{(9)}(0)}{73828125} + O\left(h^{9}\right), \quad L[y'(t_{n+1}); h] = -\frac{11 h^8 y^{(9)}(0)}{75600000} + O\left(h^{9}\right)$$

$$L[y''(t_{n+\frac{1}{5}}); h] = -\frac{863 h^7 y^{(9)}(0)}{4725000000} + O\left(h^{8}\right), \quad L[y''(t_{n+\frac{2}{5}}); h] = -\frac{37 h^7 y^{(9)}(0)}{295312500} + O\left(h^{8}\right)$$

$$L[y''(t_{n+\frac{3}{5}}); h] = -\frac{29 h^7 y^{(9)}(0)}{175000000} + O\left(h^{8}\right), \quad L[y''(t_{n+\frac{4}{5}}); h] = -\frac{8 h^7 y^{(9)}(0)}{73828125} + O\left(h^{8}\right)$$

$$L[y''(t_{n+1}); h] = -\frac{11 h^7 y^{(9)}(0)}{37800000} + O\left(h^{8}\right)$$

The method is of uniform order are $[6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6]^T$

3.2 Convergence Analysis of the Hybridized One-Step Block Method

The convergence of the block method will be established by expressing formulas in matrix form adopting the following notations.

Theorem1

We assume $\bar{\text{Z}}$ as the approximation of the true solution of vector Z in vectorial form for the system obtained from OSHBM on the block interval $[0,1]$. If the error vector is denoted by $E = (e_0, e_{\frac{1}{5}}, e_{\frac{2}{5}}, e_{\frac{3}{5}}, e_{\frac{4}{5}}, e_1)$ where $e_j = y(t_j) - y_j$. Assume hat the solution is in closed form and also differentiable om $[t_0, t_N]$ and that $\| E \| = \| \bar{\text{Z}}$ - Z $\|$, then as $h \to 0$, the method is said to be a convergent method of order six. Proof: Suppose the $N \times N$-matrices of the coefficient of OSHBM method is defined as follows:

$$
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\ddots & & \ddots & & \ddots & & \ddots & & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

$$
V=\begin{pmatrix}
\eta_{0,1} & \eta_{\frac{1}{5},1} & \eta_{\frac{2}{5},1} & \eta_{\frac{3}{5},1} & \eta_{\frac{4}{5},1} & \eta_{1,1} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\eta_{0,2} & \eta_{\frac{1}{5},2} & \eta_{\frac{2}{5},} & \eta_{\frac{3}{5},1} & \eta_{\frac{4}{5},1} & \eta_{1,2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\eta_{0,3} & \eta_{\frac{1}{5},3} & \eta_{\frac{2}{5},3} & \eta_{\frac{3}{45},3} & \eta_{\frac{4}{5},3} & \eta_{1,3} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\eta_{0,4} & \eta_{\frac{1}{5},4} & \eta_{\frac{2}{5},4} & \eta_{\frac{3}{5},4} & \eta_{\frac{4}{5},4} & \eta_{1,4} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\eta_{0,5} & \eta_{\frac{1}{5},5} & \eta_{\frac{2}{5},5} & \eta_{\frac{3}{5},5} & \eta_{\frac{4}{5},5} & \eta_{1,5} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & \eta_{0,1} & \eta_{\frac{1}{5},1} & \eta_{\frac{2}{5},1} & \eta_{\frac{3}{5},1} & \eta_{\frac{4}{5},1} & \eta_{1,1} & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & \eta_{0,2} & \eta_{\frac{1}{5},2} & \eta_{\frac{2}{5},} & \eta_{\frac{3}{5},1} & \eta_{\frac{4}{5},2} & \eta_{1,2} & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & \eta_{0,3} & \eta_{\frac{1}{5},3} & \eta_{\frac{2}{5},3} & \eta_{\frac{3}{5},3} & \eta_{\frac{4}{5},3} & \eta_{1,3} & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & \eta_{0,4} & \eta_{\frac{1}{5},4} & \eta_{\frac{2}{5},4} & \eta_{\frac{3}{5},4} & \eta_{\frac{4}{5},4} & \eta_{1,4} & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & \eta_{0,5} & \eta_{\frac{1}{5},5} & \eta_{\frac{2}{5},5} & \eta_{\frac{3}{5},5} & \eta_{\frac{4}{5},5} & \eta_{1,5} & \cdots & 0\\
\ddots & & \ddots & & \ddots & & \ddots & & \ddots & \ddots & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_{0,1} & \eta_{\frac{1}{5},1} & \eta_{\frac{2}{5},1} & \eta_{\frac{3}{5},1} & \cdots & \eta_{\frac{1}{5},1}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_{0,2} & \eta_{\frac{1}{5},2} & \eta_{\frac{2}{5},2} & \eta_{\frac{3}{5},2} & \cdots & \eta_{\frac{2}{5},2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_{0,3} & \eta_{\frac{1}{5},3} & \eta_{\frac{2}{5},3} & \eta_{\frac{3}{5},2} & \cdots & \eta_{\frac{3}{5},3}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_{0,4} & \eta_{\frac{1}{5},4} & \eta_{\frac{2}{5},4} & \eta_{\frac{3}{5},4} & \cdots & \eta_{\frac{4}{5},4}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_{0,5} & \eta_{\frac{1}{5},5} & \eta_{\frac{2}{5},5} & \eta_{\frac{3}{5},5} & \cdots & \eta_{1,5}
\end{pmatrix}
$$

and the known values contained in the N-vector given by

$$C=\Big(-y_0-h\eta_0 f_0,-y_0-h\eta_{0,1}f_0,-y_0-h\eta_{0,2}f_0,-y_0-h\eta_{0,3}f_0,...,0\Big)$$

According to Rufai & Ramos [17], we define the following vectors corresponding to the exact values $y(x)$ and its derivatives

$$\bar{Z}=\big(y(x_1),y(x_2),\cdots,y(x_{N-1}),y'(x_1),\cdots,y'(x_{N-1}),y''(x_1),\cdots,y''(x_{N-1})\big)^T$$

$$\bar{F}=\big(f(x_0,y(x_0),y'(x_0),y''(x_0),f(x_1,y(x_1),y'(x_1),y''(x_1),f(x_N,y(x_N),y'(x_N),y''(x_N)\big)^T$$

the Exact form of the system formed is

$$UZ - VF + C = -L(h). \tag{25}$$

the approximate value can be written as

$$U\bar{Z} - V\bar{F} + C = 0. \tag{26}$$

The subtraction of (25) from (26) gives

$$U(\bar{Z} - Z) - V(\bar{F} - F) = L(h). \tag{27}$$

recalling that $E = \bar{Z} - Z = (e_0, e_{\frac{1}{5}}, e_{\frac{2}{5}}, e_{\frac{3}{5}}, e_{\frac{4}{5}}, e_1)^T$, therefore equation (27) becomes

$$UE - V(\bar{F} - F) = L(h). \tag{28}$$

To obtain $\bar{F} - F = JE$, we apply the Mean-Value Theorem. J is the jacobian matrix and the partial derivaties are applied at intermediate points, we have (28) as c having $\psi = -VJ$ as a matrix, we have

$$(U + \psi)E = L(h). \tag{29}$$

For a very small h, the matrix $U + \psi$, therefore

$$(U + \psi)^{-1} = \Omega. \tag{30}$$

Expanding the terms of Ω in taylor's series while considering the norm, $\| \Omega \| = O(h^{-3})$, we can observe that

$$\| E \| = \| \Omega L(h) \| = \| \Omega \| = \| L(h) \| \ = O(h^{-3})O(h^9) = O(h^6)$$

$$= O(h^{-3})O(h^9) = O(h^6)$$

The method is sixth-order convergent.

3.3 Zero stability of the Hybridized One-Step Block Method

The block method is generally represented as

$$A^{(0)}Y_\tau = A^{(r)}Y_{\tau-1} + h^\mu[B^{(i)}F_m + B^{(0)}F_{\tau-1} \tag{31}$$

The hybridized method is said to be zero stable, if the roots of

$$det[\lambda A^{(0)} - A^{(r)}] = 0, \tag{32}$$

that is, the first characteristic polynomial lie inside or on the unit circle, i.e $|\lambda| \leq 1$ and it's roots with $|\lambda| = 1$ as long as the multiplicity did not exceed the order of the differential equations, according to Lambert [11].

For h$\rightarrow$ 0, the method in (31) which is a system of equations can be written as;

$$A^{(0)}Y_\tau = A^{(r)}Y_{\tau-1} \tag{33}$$

where $A^{(0)}$ is identity matrix.

The roots of the method is

$$(\lambda - 1)^3\lambda^{12} = 0 \tag{34}$$

4 Numerical Application

In this section, the performance and efficiency of the newly developed method is tested on some third order initial and boundary value problems of ordinary differential equations. The tables [1 -5] shows the comparison between the Maximum Absolute Error of the news method with selected problems in literatures.

- OSHBM: Maximum error in the One-Step Hybridized Block Method as developed in this article.
- TSHBM: Maximum error in Two-Step Hybrid Block Method referenced as [16]
- NPST: Maximum error in Non-Polynomial Spline Technique referenced as [1]

- CSM: Maximum error in Cubic Spline Method proposed referenced as [6]
- FSDM: Five-step direct method of order nine [7]
- QSM: Maximum error in Quartic Spline Method referenced as [4]
- FDM: Maximum error in Finite Difference Method of algebraic order-six referenced as [12]
- CBIHM: Maximum error in Continuous Block Implicit Hybrid Method referenced as [3]
- ITPBDM: Implicit Three-Point Block Direct Method referenced as [18]

Problem 1

A modeled singularly perturbed problem in fluid mechanics

$$-\varepsilon y''' + y(t) = 81\varepsilon^2 cos(3t) + \varepsilon sin(3t)$$

$$y(0) = 0, \quad y(1) = 3\varepsilon sin(3), \qquad y''(0) = 0, \quad 0 \leq t \leq 1.$$

Exact Solution

$$y(t) = 3\varepsilon sin(3t)$$

Problem 2

A Boundary Value Problem

$$y''' = ty + (t^3 - 2t^2 - 5t - 3)e^t, \qquad 0 \leq t \leq 1$$

$$y(0) = 0, \quad y'(0) = 1, \qquad y(1) = 0$$

Exact Solution

$$y(t) = t(1-t)e^t$$

Problem 3

Consider a third order boundary value problems with mixed boundary value conditions

$$y''' = -\frac{y''(t)}{t} + \frac{y'(t)}{t^2} + \frac{1}{t},$$

$$y(2) = 0, \quad y''(1) + 0.3y'(1) = 0, \quad y''(2) + 0.15y'(2) = 0, \quad 1 \leq t \leq 2.$$

whose exact solution is

$$y(t) = c_1 + c_2 log(t) + c_3 t^2 - \frac{t^2}{4} + \frac{1}{4} t^2 log(t)$$

where

$$c_1 = \frac{33}{26} + \frac{log(2)(7 + 26 log(2))}{21}, \quad c_2 = -\frac{26 log(2)}{21}, \quad c_3 = -\frac{7}{104} - \frac{log(2)}{3}.$$

Problem 4

Considering the third order linear initial value problem

$$y''' = 2y'' + 3y' - 10y + 34te^{-2t} - 16e^{-2t} - 10t^2 + 6t + 34 \qquad 0 \leq t \leq 1.$$

$$y(0) = 3, \quad y'(0) = y''(0) = 0.$$

whose exact solution is

$$y(t) = t^2 e^{-2t} - t^2 + 3$$

Problem 5

Considering the third order initial value problem

$$y''' = t - 4y$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1. \quad h = 0.01$$

Exact Solution

$$y(t) = \frac{-3}{16}Cos(2t) + \frac{3}{16} + \frac{t^2}{8}$$

Table 1: Maximum Absolute Errors for Problem 1

M	ε	$Method$	MAX-ABS Error	M	ε	$Method$	MAX-ABS Error
10	$\frac{1}{16}$	OSHBM	2.20100×10^{-11}	20	$\frac{1}{16}$	OSHBM	3.49032×10^{-13}
10	$\frac{1}{16}$	TSHBM	2.50472×10^{-9}	20	$\frac{1}{16}$	TSHBM	3.97180×10^{-11}
10	$\frac{1}{16}$	QSM	2.50000×10^{-3}	20	$\frac{1}{16}$	QSM	1.90000×10^{-4}
10	$\frac{1}{32}$	OSHBM	7.94394×10^{-12}	20	$\frac{1}{32}$	OSHBM	1.24820×10^{-13}
10	$\frac{1}{32}$	TSHBM	9.02593×10^{10}	20	$\frac{1}{32}$	TSHBM	1.43122×10^{-11}
10	$\frac{1}{32}$	QSM	6.80000×10^{-4}	20	$\frac{1}{32}$	QSM	5.70000×10^{-5}
10	$\frac{1}{64}$	OSHBM	2.55692×10^{-12}	20	$\frac{1}{64}$	OSHBM	4.02300×10^{-14}
10	$\frac{1}{64}$	TSHBM	3.00527×10^{-10}	20	$\frac{1}{64}$	TSHBM	4.67482×10^{-12}
10	$\frac{1}{64}$	QSM	1.20000×10^{-4}	20	$\frac{1}{64}$	QSM	1.30000×10^{-5}

Table 2: Maximum Absolute Errors for Problem 2

h	$Method$	MAX-ABS Error	h	$Method$	MAX-ABS Error	h	$Method$	MAX-ABS Error
$\frac{1}{16}$	OSHBM	4.67885×10^{-14}	$\frac{1}{32}$	OSHBM	7.31430×10^{-16}	$\frac{1}{64}$	OSHBM	1.14300×10^{-17}
$\frac{1}{16}$	NPST	5.29920×10^{-7}	$\frac{1}{32}$	NPST	2.61270×10^{-8}	$\frac{1}{64}$	NPST	1.49990×10^{-9}
$\frac{1}{16}$	CSM	1.68610×10^{-3}	$\frac{1}{32}$	CSM	4.45100×10^{-4}	$\frac{1}{64}$	CSM	1.12930×10^{-4}

Table 3: Maximum Absolute Errors for Problem 3

h	$Method$	MAX-ABS Error	h	$Method$	MAX-ABS Error
$\frac{1}{8}$	OSHBM	8.56097×10^{-9}	$\frac{1}{32}$	OSHBM	2.15011×10^{-12}
$\frac{1}{8}$	TSHBM	8.17298×10^{-7}	$\frac{1}{32}$	TSHBM	2.37099×10^{-10}
$\frac{1}{8}$	FDM	1.40000×10^{-5}	$\frac{1}{32}$	FDM	5.00000×10^{-9}
$\frac{1}{16}$	OSHBM	1.36814×10^{-10}	$\frac{1}{40}$	OSHBM	5.64135×10^{-13}
$\frac{1}{16}$	TSHBM	1.44941×10^{-8}	$\frac{1}{40}$	TSHBM	6.26244×10^{-11}
$\frac{1}{16}$	FDM	2.91000×10^{-7}	$\frac{1}{40}$	FDM	1.33000×10^{-9}

Table 4: Error Comparison for Problem 4

h	$Exact$	$OSHBM$	$FSDM$	$ITPBDM$
0.1	2.99819	5.3919E-14	6.6218E-13	2.593481E-13
0.2	2.98681	2.2563E-12	6.2238E-11	4.361134E-11
0.3	2.95939	5.2986E-12	3.5134E-09	2.967204E-11
0.4	2.91189	9.8984E-11	6.1100E-07	9.981296E-11
0.5	2.84197	1.6376E-11	6.4183E-07	2.342377E-10
0.6	2.74843	2.5145E-11	1.8082E-06	4.550881E-10
0.7	2.63083	3.6721E-10	1.3511E-06	7.912180E-10
0.8	2.48921	5.1726E-10	1.3367E-06	1.275017E-09
0.9	2.32389	7.08953E-09	7.9041E-06	1.945292E-09
1.0	2.13534	9.50875E-09	3.7360E-05	2.849440E-08

Table 5: Errors Comparison for Problem 5

h	$Exact$	$OSHBM$	$CBIHM$
0.1	0.004987516655	1.6316E-14	2.1530E-13
0.2	0.019801063624	2.5492E-13	8.5054E-12
0.3	0.043999572204	1.2567E-13	6.8439E-11
0.4	0.076867491997	3.8568E-12	2.9415E-10
0.5	0.117443317650	9.1145E-11	8.9993E-10
0.6	0.164557921036	1.8234E-11	2.2216E-09
0.7	0.216881160706	3.2478E-12	4.7276E-09
0.8	0.003194884367	5.3069E-13	7.4418E-13
0.9	0.004041807602	8.1096E-13	1.1100E-12
1.0	0.004987516655	1.1741E-13	1.5931E-12

5 Conclusion

A uniform sixth-order OSHBM for the direct solution of third-order Ordinary Differential Equations using collocation and interpolation techniques has been derived in this article. Basic properties of the method such as the order, zero-stability, the convergence and stability analysis of the method have been well studied. The new approach performs better in comparison with the absolute and maximum error than some cited existing methods in literatures. The limitation of this approach is that it was not extended to the solution of systems of third-order differential.

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7 Conflict of Interest

No conflict of interests.

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