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α-Fixed Point Results for Fuzzy Enriched ϕ-ψ Contraction in $\mathcal{F}$-Metric Spaces with Application to Fuzzy Integrodifferential Equations

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Abstract: The paper aims to introduce the notion of $\mathcal{F}$-metric spaces and establish α-fixed point results for fuzzy enriched $\phi - \psi$ contraction in complete $\mathcal{F}$-metric spaces. These contributions extend the existing literature on fuzzy mappings and fixed point theory. Through illustrative examples, we showcase the practical applicability of our proposed results. Also, we explore as an application, the solution for fuzzy integrodifferential equations in the context of generalized Hukuhara derivative.

Keywords: α-fixed point, $\mathcal{F}$-metric space, ϕ-φ contraction mapping, Hukuhara derivative
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1 Introduction

Fixed point theory is one of the most fascinating fields of research in the growth on nonlinear analysis. One of the pioneer results of fixed point theory is the Banach fixed point theorem [27] which play a significant role in solving the existence and uniqueness of solution to different problems in mathematics. Numerous problems encountered in everyday life stem from incomplete information that is not well-expressed in conventional mathematics. In 1965, Zadeh[2] introduced the notion of fuzzy sets, that proffer efficient means to handle imprecise information, laying the foundation for subsequent research in fuzzy mathematics. Building upon Zadeh's work, Goguen [3] extended this concept of fuzzy set to L-fuzzy set thereby replacing the interval $[0, 1]$ by L that is completely distributive lattice. Heilpern [4] further extended the notion of fuzzy mappings and derived fixed point results in the metric linear space. For more, we refer [1, 7, 5, 9, 10, 23, 28]. Rashid et al. [6] introduced the conception of $\beta_{\mathcal{F}L}$ -admissible for two L-fuzzy mappings and derived numerous results for these mappings. Moreover, Jleli and Samet [15] introduced a contemporary metric space, which is referred to as $\mathcal{F}$-metric space, to extend the classical metric space. Samet et al. [25] introduced a new category of contractive type mappings referred to as β- ψ contractive type mapping and β- admissible mappings in metric spaces and obtained the existence of fixed point results. Further, Raji [21] generalized the concept of β-ψ contractive type mappings and obtained various common fixed point results for this generalized class of contractive mappings. Further results can be found in [8, 17, 19].
Recently, Lateef [22] introduced the notion of $\mathcal{F}$-metric space as a generalization of traditional metric space and proved Banach contraction principle in the setting of this generalized metric space and establish some common fixed point theorems for (β, ψ)-contractions.
Based on the above insight, we introduce the concepts of $\mathcal{F}$-metric spaces and subsequently establish α-fixed point results for fuzzy enriched ϕ-φ contraction within the framework of complete $\mathcal{F}$-metric spaces. To bolster our findings, we offer illustrative examples demonstrating the practical application of the presented results. Also, we explored as an application, the solution for fuzzy integrodifferential equations in the context of generalized Hukuhara derivative.

1.1 Preliminaries

We begin this section by presenting the concept of ϕ-φ contractive and $\mathcal{F}$-metric space.

Definition 1.1. [14] A mapping $\phi : [0, \infty) \to [0, \infty)$ is said to be a comparison function if it satisfies the following:

(i) ϕ is monotonic increasing,

(ii) $\lim_{n\to\infty} \phi^n(t) = 0$, for each $t > 0$.

Definition 1.2. [16] A mapping $\phi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if it satisfies the following:

(i) ϕ is monotonic increasing,

(ii) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$.

Clearly, every (c)-comparison function is a comparison function.
Remark 2.3 Let ϕ be a (c)-comparison function. Then:

(i) $\phi(0) = 0$,

(ii) $\phi(t) < t$, for each $t > 0$,

(iii) ϕ is right continuous at 0.

Definition 1.3. Let Φ be the set of all functions $\varphi : [0, \infty)^5 \to [0, \infty)$ satisfying the following:

(i) φ is continuous,

(ii) $\varphi(t_1, t_2, t_3, t_4, t_5) = 0$ if and only if $t_1t_2t_3t_4t_5 = 0$.

Example 1.4. The following functions $\varphi : [0, \infty)^5 \to [0, \infty)$ belong to Φ:

(i) $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1t_2t_3t_4t_5$,

(ii) $\varphi(t_1, t_2, t_3, t_4, t_5) = e^{t_1t_2t_3t_4t_5} - 1$,

(iii) $\varphi(t_1, t_2, t_3, t_4, t_5) = \ln(1 + t_1t_2t_3t_4t_5)$.

Definition 1.5. [13] A function T from a metric space (X, d) into itself is said to be a ϕ-contraction if there exists a function $\phi : [0, \infty) \to [0, \infty)$ satisfying

$$d(Tx, Ty) \le \phi(d(x, y)), \quad \forall x, y \in X. \tag{2.1}$$

Definition 1.6. A function T from a metric space (X, d) into itself is said to be a (ϕ, φ)-contraction if there exist functions $\phi : [0, \infty) \to [0, \infty)$ and $\varphi : [0, \infty)^5 \to [0, \infty)$ satisfying

$$d(Tx, Ty) \le \phi(d(x, y)) - \varphi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \quad \forall x, y \in X. \tag{2.2}$$

Definition 1.7. [8] Let X be a nonempty set. A fuzzy set in X is a function $\Omega : X \to [0, 1]$. If $x \in \Omega$, then $\Omega(x)$ is said to be the grade of membership of $x \in \Omega$. An α-level set of Ω denoted by $[\Omega]_\alpha$ is defined by

$$[\Omega]_\alpha = \{x : \Omega(x) \ge \alpha\} \text{ if } \alpha \in (0, 1],$$

$$\Omega_0 = \overline{\{x : \Omega(x) > 0\}}.$$

If X is a metric space, then I^X is the collection of all fuzzy sets in X.

Definition 1.8. An α-fuzzy fixed point of a fuzzy mapping $T : X \to I^X$ is defined as a point $x^* \in X$ where $\alpha \in (0, 1]$ and $x^* \in [Tx^*]_{\alpha(x^*)}$.

Definition 1.9. A common α-fuzzy fixed point of fuzzy mappings $T, f : X \to I^X$ is defined as a point $x^* \in X$ where $\alpha \in (0, 1]$ and $x^* \in [Tx^*]_{\alpha(x^*)} \cap [fx^*]_{\alpha(x^*)}$.

We now introduce $\mathcal{F}$-metric space as follows:
Let $g : (0, +\infty) \to \mathbb{R}$ and $\mathcal{F}$ refer to the set of functions g satisfying:

- $(\mathcal{F}_1)$ $0 < x < t \Rightarrow g(x) \le g(t)$,
- $(\mathcal{F}_2)$ for the sequence $\{x_n\} \subset \mathbb{R}^+$, $\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} g(x_n) = -\infty$.

Definition 1.10. [15] Let X be a nonempty set and $d_{\mathcal{F}} : X \times X \to [0, +\infty)$. Suppose there exists $(g, h) \in \mathcal{F} \times [0, +\infty)$ such that

(i) $(x, y) \in X \times X, d_{\mathcal{F}}(x, y) = 0 \iff x = y$,

(ii) $d_{\mathcal{F}}(x, y) = d_{\mathcal{F}}(y, x)$, for all $(x, y) \in X \times X$,

(iii) for every $(x, y) \in X \times X$, for every $N \in \mathbb{N}, N \geq 2$, and for each $\{u_i\}_{i=1}^{N} \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$d_{\mathcal{F}}(x, y) > 0 \Rightarrow g(d_{\mathcal{F}}(x, y)) \leq g\left(\sum_{i=1}^{N-1} d_{\mathcal{F}}(x_i, x_{i+1})\right) + h.$$

Then, $d_{\mathcal{F}}$ is referred to as an $\mathcal{F}$-metric on X and $(X, d_{\mathcal{F}})$ is called an $\mathcal{F}$-metric space.

Example 1.11. [15] Let $d_{\mathcal{F}} : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be a function defined by

$$d_{\mathcal{F}}(x, y) = \begin{cases} (x-y)^2, & \text{if } (x, y) \in [0, 3] \times [0, 3], \\ |x - y|, & \text{if } (x, y) \notin [0, 3] \times [0, 3], \end{cases}$$

with $g(t) = \ln(t)$ and $h = \ln(3)$, is an $\mathcal{F}$-metric.

Definition 1.12. [15] Let $(X, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space.

(i) Let $\{x_n\} \subset X$. The sequence $\{x_n\}$ is referred to as $\mathcal{F}$-convergent to $x \in X$ if $\{x_n\}$ is convergent to x in $\mathcal{F}$-metric $d_{\mathcal{F}}$.

(ii) The sequence $\{x_n\}$ is referred to as $\mathcal{F}$-Cauchy if

$$\lim_{n,m\to\infty} d_{\mathcal{F}}(x_n, x_m) = 0.$$

(iii) If every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to $x \in X$, then $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete.

Lemma 1.13. *[16] Assume X_1 and X_2 are nonempty compact subsets of $\mathcal{F}$-metric space $(X, d_{\mathcal{F}})$ that is closed. If $x \in X_1$, then*

$$d_{\mathcal{F}}(x, X_2) \leq H_{\mathcal{F}}(X_1, X_2).$$

Definition 1.14. [23] Let $(X, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space and $\beta : (X, d_{\mathcal{F}}) \times (X, d_{\mathcal{F}}) \to [0, +\infty)$. Let T, f be a pair of fuzzy mappings from X into $\mathcal{F}_L(X)$. Then, the pair (T, f) is referred to as an $\alpha_{\mathcal{F}}$-admissible if:

(i) for a point $x \in X$ and $y \in [Tx]_{\alpha_T(x)}$, where $\alpha_T(x) \in (0, 1]$ with $\beta(x, y) \geq 1$, then we have $\beta(y, z) \geq 1$, for all $z \in [fy]_{\alpha_f(y)} \neq \emptyset$ where $\alpha_f(y) \in (0, 1]$,

(ii) for a point $x \in X$ and $y \in [fx]_{\alpha_T(x)}$, where $\alpha_f(x) \in (0, 1]$ with $\beta(x, y) \geq 1$, then we have $\beta(y, z) \geq 1$, for all $z \in [Ty]_{\alpha_T(y)} \neq \emptyset$ where $\alpha_T(y) \in (0, 1]$.

2 Main Results

In this section, we present with the following theorem.

Theorem 2.1. *Let $(X, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space and T be a fuzzy mapping from X into I^X. Suppose for each $x \in X$, there exists $\alpha_T(x) \in (0, 1]$ such that $[Tx]_{\alpha_T(x)} \in C(2^X)$ satisfy the following conditions:*

(i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) for a point $x_0 \in X$, there exists $\alpha_T(x_0) \in (0, 1]$ such that $x_1 \in [Tx_0]_{\alpha_T(x_0)}$,

(iii) for all $x, y \in X$, there exist $\phi \in \Psi$ and $\varphi \in \Phi$ such that

$$\begin{aligned} &H_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [Ty]_{\alpha_T(y)}) \leq \phi(d_{\mathcal{F}}(x, y)) - \\ &\varphi(d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, [Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y, [Ty]_{\alpha_T(y)}), d_{\mathcal{F}}(x, [Ty]_{\alpha_T(y)}), d_{\mathcal{F}}(y, [Tx]_{\alpha_T(x)})). \end{aligned} \tag{3.1}$$

Then, T has an α-fixed point $x^* \in [Tx^*]_{\alpha_T(x^*)}$.

Proof. By condition (ii), we can choose a point $x_0 \in X$, there exists $\alpha_{\mathbf{T}}(x_0) \in (0, 1]$ such that

$$[Tx_0]_{\alpha_{\mathbf{T}}(x_0)} \in C(2^X)$$

is a nonempty compact subset of X, and there exists a point $x_1 \in [Tx_0]_{\alpha_{\mathbf{T}}(x_0)}$ such that

$$d_{\mathcal{F}}(x_1, x_2) = d_{\mathcal{F}}(x_1, [Tx_0]_{\alpha_T(x_0)}).$$

Again, for x_1, there exists $\alpha_{\mathbf{T}}(x_1) \in (0,1]$ such that

$$[Tx_1]_{\alpha_{\mathbf{T}}(x_1)} \in C(2^X).$$

Since $[Tx_1]_{\alpha_{\mathbf{T}}(x_1)}$ is a nonempty compact subset of X, there exists a point $x_2 \in [Tx_1]_{\alpha_{\mathbf{T}}(x_1)}$ such that

$$d_{\mathcal{F}}(x_1, x_2) = d_{\mathcal{F}}(x_1, [Tx_1]_{\alpha_{\mathbf{T}}(x_1)}).$$

With (3.1) and Lemma 2.1, we have

$$\begin{aligned} d_{\mathcal{F}}(x_1, x_2) &= d_{\mathcal{F}}(x_1, [Tx_1]_{\alpha_{\mathbf{T}}(x_1)}) \\ &\leq H_{\mathcal{F}}([Tx_0]_{\alpha_{\mathbf{T}}(x_0)}, [Tx_1]_{\alpha_{\mathbf{T}}(x_1)}) \\ &\leq \phi(d_{\mathcal{F}}(x_0, x_1)) \\ &\quad - \varphi(d_{\mathcal{F}}(x_0, x_1), d_{\mathcal{F}}(x_0, [Tx_0]_{\alpha_{\mathbf{T}}(x_0)}), d_{\mathcal{F}}(x_1, [Tx_1]_{\alpha_{\mathbf{T}}(x_1)}), \\ &\quad d_{\mathcal{F}}(x_0, [Tx_1]_{\alpha_{\mathbf{T}}(x_1)}), d_{\mathcal{F}}(x_1, [Tx_0]_{\alpha_{\mathbf{T}}(x_0)})) \end{aligned}$$

$$\leq \phi(d_{\mathcal{F}}(x_0, x_1)) - \varphi(d_{\mathcal{F}}(x_0, x_1), d_{\mathcal{F}}(x_0, x_1), d_{\mathcal{F}}(x_1, x_2), d_{\mathcal{F}}(x_0, x_2), d_{\mathcal{F}}(x_1, x_1)).$$

By applying definition 2.3 in the above equation, we have

$$d_{\mathcal{F}}(x_1, x_2) \leq \varphi(d_{\mathcal{F}}(x_0, x_1)).$$

Now, for $x_2 \in X$, there exists $\alpha_{\mathbf{T}}(x_2) \in (0,1]$ such that

$$[Tx_2]_{\alpha_{\mathbf{T}}(x_2)} \in C(2^X).$$

Since $[Tx_2]_{\alpha_{\mathbf{T}}(x_2)}$ is a nonempty compact subset of X, there exists a point $x_3 \in [Tx_2]_{\alpha_{\mathbf{T}}(x_2)}$ such that

$$d_{\mathcal{F}}(x_2, x_3) = d_{\mathcal{F}}(x_2, [Tx_2]_{\alpha_{\mathbf{T}}(x_2)}).$$

Again, by (3.1) and Lemma 2.1, we have

$$\begin{aligned} d_{\mathcal{F}}(x_2, x_3) &= d_{\mathcal{F}}(x_2, [Tx_2]_{\alpha_{\mathbf{T}}(x_2)}) \\ &\leq H_{\mathcal{F}}([Tx_1]_{\alpha_{\mathbf{T}}(x_1)}, [Tx_2]_{\alpha_{\mathbf{T}}(x_2)}) \\ &\leq \phi(d_{\mathcal{F}}(x_2, x_1)) \\ &\quad - \varphi(d_{\mathcal{F}}(x_2, x_1), d_{\mathcal{F}}(x_2, [Tx_2]_{\alpha_{\mathbf{T}}(x_2)}), d_{\mathcal{F}}(x_1, [Tx_1]_{\alpha_{\mathbf{T}}(x_1)}), \\ &\quad d_{\mathcal{F}}(x_2, [Tx_1]_{\alpha_{\mathbf{T}}(x_1)}), d_{\mathcal{F}}(x_1, [Tx_2]_{\alpha_{\mathbf{T}}(x_2)})) \end{aligned}$$

$$\leq \phi(d_{\mathcal{F}}(x_2, x_1)) - \varphi(d_{\mathcal{F}}(x_2, x_1), d_{\mathcal{F}}(x_2, x_3), d_{\mathcal{F}}(x_1, x_2), d_{\mathcal{F}}(x_2, x_2), d_{\mathcal{F}}(x_1, x_3)).$$

By applying definition 2.3 in the above equation, we have

$$d_{\mathcal{F}}(x_2, x_3) \leq \phi(d_{\mathcal{F}}(x_1, x_2)).$$

Continuing this process having chosen $x_1, x_2, x_3, x_4, \ldots$, we establish a sequence $\{x_n\}$ in X such that $x_{2n+1} \in [Tx_{2n}]_{\alpha_{\mathbf{T}}(x_{2n})}$, $x_{2n+2} \in [Tx_{2n+1}]_{\alpha_{\mathbf{T}}(x_{2n+1})}$, then for all n, we have

$$d_{\mathcal{F}}(x_{2n+1}, x_{2n+2}) \leq \phi(d_{\mathcal{F}}(x_{2n}, x_{2n+1})),$$

and

$$d_{\mathcal{F}}(x_{2n+2}, x_{2n+3}) \leq \phi(d_{\mathcal{F}}(x_{2n+1}, x_{2n+2})).$$

From these inequalities, we get

$$d_{\mathcal{F}}(x_n, x_{n+1}) \leq \phi(d_{\mathcal{F}}(x_{n-1}, x_n)) \leq \cdots \leq \phi^n(d_{\mathcal{F}}(x_0, x_1)).$$

Let $\varepsilon > 0$ and $n(\varepsilon) \in \mathbb{N}$ and $(g, h) \in \mathcal{F} \times [0, +\infty)$ be such that (iii) of definition 2.9 is satisfied. Again, let $\varepsilon > 0$ be fixed. By $\mathcal{F}_1$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow g(t) < g(\delta) - h.$$

Let $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\varepsilon)} \varphi^n(d_{\mathcal{F}}(x_0, x_1)) < \delta.$$

By the above inequality and $\mathcal{F}_1$, we have

$$g\left(\sum_{n \geq n(\varepsilon)} \phi^n(d_{\mathcal{F}}(x_0, x_1))\right) \leq g\left(\sum_{n \geq n(\varepsilon)} \phi^n(d_{\mathcal{F}}(x_0, x_1))\right) < g(\varepsilon) - h.$$

We have for all $m > n \geq \mathbb{N}$,

$$g(d_{\mathcal{F}}(x_n, x_m)) \leq g\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(x_i, x_{i+1})\right) + h \leq g\left(\sum_{i=n}^{m-1} \phi_n(d_{\mathcal{F}}(x_0, x_1))\right) + h$$

$$\leq g\left(\sum_{i=n}^{m-1} \phi_n(d_{\mathcal{F}}(x_0, x_1))\right) + h \leq g\left(\sum_{n\geq n(\varepsilon)}^{\infty} \phi_n(d_{\mathcal{F}}(x_0, x_1))\right) + h \leq g(\epsilon). \tag{3.2}$$

By $\mathcal{F}_1$, we have $d_{\mathcal{F}}(x_n, x_m) < \epsilon$, $m > n \geq \mathbb{N}$. It follows that the sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy. Since $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, there exists $x^* \in X$ such that the sequence $\{x_n\}$ is $\mathcal{F}$-convergent to x^*, that is,

$$\lim_{n\to\infty} d_{\mathcal{F}}(x_n, x^*) = 0 \tag{3.3}$$

To show that $x^* \in [Tx^*]_{\alpha_{\mathbf{T}}(x*)}$, we let $d_{\mathcal{F}}(x^*, [Tx^*]_{\alpha_{\mathbf{T}}(x*)} > 0$. By the definition of g and (iii) of definition 2.9, we have

$$\begin{aligned}
g(d_{\mathcal{F}}(x^*, [Tx^*]_{\alpha_{\mathbf{T}}(x*)}) &\leq g\left(d_{\mathcal{F}}(x^*, x_{2n}) + d_{\mathcal{F}}(x_{2n}, [Tx^*]_{\alpha_{\mathbf{T}}(x*)})\right) + h \\
&\leq g\left(d_{\mathcal{F}}(x^*, x_{2n}) + H_{\mathcal{F}}([Tx_{2n-1}]_{\alpha_{\mathbf{f}}(x_{2n-1})}, [Tx^*]_{\alpha_{\mathbf{T}}(x^*)})\right) + h \\
&\leq g\,(d_{\mathcal{F}}(x^*, x_{2n}) + \phi(d_{\mathcal{F}}(x^*, x_{2n-1})) \\
&\quad - \varphi\left(d_{\mathcal{F}}(x^*, x_{2n-1}), d_{\mathcal{F}}(x^*, [Tx^*]_{\alpha_{\mathbf{T}}(x^*)}),\right. \\
&\quad d_{\mathcal{F}}(x_{2n-1}, [Tx_{2n-1}]_{\alpha_{\mathbf{T}}(x_{2n-1})}), d_{\mathcal{F}}(x^*, [Tx_{2n-1}]_{\alpha_{\mathbf{T}}(x_{2n-1})}), \\
&\quad \left. d_{\mathcal{F}}(x_{2n-1}, [Tx^*]_{\alpha_{\mathbf{T}}(x^*)})\right) \\
&\quad + h \\
&\leq g\,(d_{\mathcal{F}}(x^*, x_{2n}) + \phi(d_{\mathcal{F}}(x^*, x_{2n-1})) \\
&\quad - \varphi\left(d_{\mathcal{F}}(x^*, x_{2n-1}), d_{\mathcal{F}}(x^*, [Tx^*]_{\alpha_{\mathbf{T}}(x^*)}, d_{\mathcal{F}}(x_{2n-1}, x_{2n}),\right. \\
&\quad \left. d_{\mathcal{F}}(x^*, x_{2n}), d_{\mathcal{F}}(x_{2n-1}, [Tx^*]_{\alpha_{\mathbf{T}}(x^*)}\right) \\
&\quad + h
\end{aligned}$$

Since $\{x_n\}$ is $\mathcal{F}$-convergent to x^*, by the properties of $\varphi \in \Psi$, $\varphi \in \Phi$, (3.3) with $\mathcal{F}_2$ and taking the limit in the above inequality, we have

$$\lim_{n\to\infty} g(d_{\mathcal{F}}(x^*, [Tx^*]_{\alpha_{\mathbf{T}}(x^*)}) = \lim_{n\to\infty} g(d_{\mathcal{F}}(x^*, x_{2n}) + d_{\mathcal{F}}(x^*, x_{2n-1})) + h = -\infty,$$

a contradiction. Therefore, we get $d_{\mathcal{F}}(x^*, [Tx^*]_{\alpha_{\mathbf{T}}(x^*)}) = 0$, which implies $x^* \in [Tx^*]_{\alpha_{\mathbf{T}}(x^*)}$. Thus, the T is α-fixed point $x^* \in [Tx^*]_{\alpha_{\mathbf{T}}(x^*)}$.

Example 2.2. Consider $X = [0, +\infty)$, for all $x, y \in X$, the $\mathcal{F}$-metric $d_{\mathcal{F}} : X \times X \to \mathbb{R}_+^0$ is defined as

$$d_{\mathcal{F}}(x, y) = \begin{cases} (x-y)^2 & \text{if } (x, y) \in [0, 4] \times [0, 4], \\ |x - y| & \text{if } (x, y) \notin [0, 4] \times [0, 4. \end{cases}$$

and for $t > 0$ and $h = \ln(4)$, $g(t) = \ln(t)$. Then, $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-complete $\mathcal{F}$-metric space but not a metric space because $d_{\mathcal{F}}$ does not satisfy the triangle inequality, as

$$d_{\mathcal{F}}(1, 4) = 9 > 5 = d_{\mathcal{F}}(1, 3) + d_{\mathcal{F}}(3, 4).$$

Moreover, let $\alpha(x) \in (0, 1]$ and define $T : X \to I^X$ as

$$(i) \text{ If } x = 0, \quad T(x)(t) = \begin{cases} 1 & \text{if } t \in 0, \\ 0 & \text{if } t \notin 0, \end{cases}$$

$$(ii) \text{ If } 0 < x < \infty, \quad T(x)(t) = \begin{cases} \alpha & \text{if } 0 \leq t < \frac{x^2}{60}, \\ \frac{\alpha}{3} & \text{if } \frac{x^2}{60} \leq t < \frac{x^2}{30}, \\ \frac{\alpha}{6} & \text{if } \frac{x^2}{30} \leq t < x^2, \\ 0 & \text{if } x^2 \leq t < \infty, \end{cases}$$

and

$$T(y)(t) = \begin{cases} \alpha & \text{if } 0 \le t < \frac{y^2}{40}, \\ \frac{\alpha}{3} & \text{if } \frac{y^2}{40} \le t < \frac{y^2}{30}, \\ \frac{\alpha}{24} & \text{if } \frac{y^2}{30} \le t < y^2, \\ 0 & \text{if } y^2 \le t < \infty. \end{cases}$$

We now define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{1}{2}t$ for $t > 0$ with $\phi \in \Psi$ and $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{1}{3}t$ with $\varphi \in \Phi$.

For all $x \in X$, there exists $\alpha_T(x) = \frac{\alpha}{3} \in (0, 1]$, such that $[Tx]_{\alpha_T(x)} \in C(2^X)$.

Case I: If $x = y = 0$, then

$$\begin{aligned} &[Tx]_{(\frac{\alpha}{3})} = 0, \\ &\text{then} \quad \mathcal{H}_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [Ty]_{\alpha_T(y)}) = 0 \\ &\le \phi(d_{\mathcal{F}}(x, y)) - \varphi\big(d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, [Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y, [Ty]_{\alpha_T(y)}), \\ &\quad d_{\mathcal{F}}(x, [Ty]_{\alpha_T(y)}), d_{\mathcal{F}}(y, [Tx]_{\alpha_T(x)})\big) \end{aligned}$$

Case II: If $x, y \in (0, \infty)$, then

$$[Tx]_{(\frac{\alpha}{3})} = \{t \in X : T(x)(t) \ge \frac{\alpha}{3}\} = \left[0, \frac{x}{6}\right]$$

and

$$[Ty]_{(\frac{\alpha}{3})} = \{t \in X : T(y)(t) \ge \frac{\alpha}{3}\} = \left[0, \frac{y}{6}\right].$$

For $x \neq y$ and by definition of $d_{\mathcal{F}}$, we have

$$\begin{aligned} &\mathcal{H}_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [Ty]_{\alpha_T(y)}) = \left(\frac{x}{30} - \frac{y}{30}\right)^2 \\ &\le \frac{|x + y|}{30}(|x - y|)^2 \\ &\le \frac{1}{12}|x - y|^2 = \frac{1}{12} d_{\mathcal{F}}(x, y) \\ &\le \phi(d_{\mathcal{F}}(x, y)) - \varphi\big(d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, [Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y, [Ty]_{\alpha_T(y)}), \\ &\quad d_{\mathcal{F}}(x, [Ty]_{\alpha_T(y)}), d_{\mathcal{F}}(y, [Tx]_{\alpha_T(x)})\big) \end{aligned}$$

Hence, all the conditions of Theorem 3.1 are satisfied. Thus, there exists $0 \in [0, +\infty)$, that is,

$$0 \in [T0]_{(\frac{\alpha}{3})}.$$

We consider the α-fixed points result for two fuzzy mappings as follows.

Corollary 2.3. *Let $(X, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space, and let T, f be fuzzy mappings from X into I^X. Suppose that for each $x \in X$, there exist $\alpha_T(x), \alpha_f(x) \in (0, 1]$ such that $[Tx]_{\alpha_T(x)}, [fx]_{\alpha_f(x)} \in C(2^X)$ satisfy the following conditions:*

(i) *$(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,*

(ii) *For a point $x_0 \in X$, there exists $\alpha_T(x_0)$ or $\alpha_f(x_0) \in (0, 1]$ such that $x_1 \in [Tx0]_{\alpha_T(x_0)}$ or $x_1 \in [fx0]_{\alpha_f(x_0)}$,*

(iii) *For all $x, y \in X$, there exist $\phi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\begin{aligned} &\mathcal{H}_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(y)}) \le \phi(d_{\mathcal{F}}(x, y)) - \\ &\varphi(d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, [Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y, [fy]_{\alpha_f(y)}), \\ &d_{\mathcal{F}}(x, [fy]_{\alpha_f(y)}), d_{\mathcal{F}}(y, [Tx]_{\alpha_T(x)})\big). \end{aligned}$$

Then, T and f have a common α-fixed point $x^* \in [Tx^*]_{\alpha_T(x^*)} \cap [fx^*]_{\alpha_f(x^*)}$.

Proof. The proof follows from Theorem 3.1 by taking fuzzy mappings T, f from X into I^X.

We now consider the α-fixed points results for multivalued mappings. ∎

Corollary 2.4. *Let $(X, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space. Let R be a fuzzy mapping from $(X, d_{\mathcal{F}})$ into $CB(X)$ satisfying the following:*

(i) *$(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,*

(ii) *For a point $x_0 \in X$, there exists $x_1 \in Rx_0$,*

(iii) *For all $x, y \in X$, there exist $\phi \in \Psi$ and $\varphi \in \Phi$ such that*
$\mathcal{H}_{\mathcal{F}}(Rx, Ry) \leq \phi(d_{\mathcal{F}}(x, y)) -$
$\varphi\left(d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, Rx), d_{\mathcal{F}}(y, Ry),\right.$
$\left.d_{\mathcal{F}}(x, Ry), d_{\mathcal{F}}(y, Rx)\right).$

Then, R has a fixed point $x^* \in Rx^*$.

Proof. Define α-fuzzy mappings $T : X \to I^X$, for some $\alpha_T \in (0, 1]$ by

$$T(x)(t) = \begin{cases} \alpha_T, & \text{if } t \in R_1 x, \\ 0, & \text{if } t \notin R_1 x. \end{cases}$$

Then,

$$[Tx]_{\alpha_T(x)} = \{t \in X : T(x)(t) \geq \alpha_T(x)\} = R_1 x.$$

Implies for all $x, y \in X$,

$$H_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(y)}) = H_{\mathcal{F}}(R_1 x, R_2 y).$$

The remaining proof follows from Corollary 3.3. Thus $x^* \in X$,

$$x^* \in [Tx^*]_{\alpha_T(x^*)} = Rx^*.$$

□

Corollary 2.5. *Let $(X, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space and R_1, R_2 be fuzzy mappings from $(X, d_{\mathcal{F}})$into $CB(X)$ satisfying the following:*

(i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) for a point $x_0 \in X$, there exists $x_1 \in R_1 x_0$ or $x_1 \in R_2 x_0$,

(iii) for all $x, y \in X$, there exist $\varphi \in \Psi$ and $\varphi \in \Phi$ such that

$$H_{\mathcal{F}}(R_1 x, R_2 y) \leq \phi(d_{\mathcal{F}}(x, y)) - \varphi(d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, R_1 x), d_{\mathcal{F}}(y, R_2 y), d_{\mathcal{F}}(x, R_2 y), d_{\mathcal{F}}(y, R_1 x)),$$

Then, R_1 and R_2 have a common fixed point $x^* \in R_1 x^* \cap R_2 x^*$.

Proof. Define α-fuzzy mappings $T, f : X \to I^X$, for some $\alpha_T, \alpha_f \in (0, 1]$ by

$$T(x)(t) = \begin{cases} \alpha_T, & \text{if } t \in R_1 x, \\ 0, & \text{if } t \notin R_1 x, \end{cases}$$

and

$$f(y)(t) = \begin{cases} \alpha_f, & \text{if } t \in R_2 y, \\ 0, & \text{if } t \notin R_2 y. \end{cases}$$

Then,

$$[Tx]_{\alpha_T(x)} = \{t \in X : T(x)(t) \geq \alpha_T(x)\} = R_1 x,$$

and

$$[fx]_{\alpha_f(x)} = \{t \in X : f(x)(t) \geq \alpha_f(x)\} = R_2 y.$$

Implies for all $x, y \in X$,

$$H_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(y)}) = H_{\mathcal{F}}(R_1 x, R_2 y).$$

The remaining proof follows from Theorem 3.1. Thus $x^* \in X$,

$$x^* \in [Tx^*]_{\alpha_T(x^*)} \cap [fx^*]_{\alpha_f(x^*)} = R_1 x^* \cap R_2 x^*.$$

□

3 Application

Here, we demonstrate applicability of the results developed in the previous sections to investigate the solution of fuzzy initial value problem by using generalized Hukuhara differentiability. For more details on this, we refer to [18, 20, 24, 26].
As a starting point, we introduce the symbols that will be used in this section. Let $\mathcal{H}_c^n$ denote the space of nonempty, compact, and convex subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. If $A, B \in \mathcal{H}_c^n$ and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$, then, the Hausdorff metric d on $\mathcal{H}_c^n$ is defined as

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

We now introduce the following definitions.

Definition 3.1. Let $u : \mathbb{R}^n \to [0,1]$ be a fuzzy mapping.

(a) u is said to be normal, if there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$.

(b) u is said to be fuzzy convex, if for all $x, y \in \mathbb{R}^n$ and $0 \le \mu \le 1$, we have

$$u(\mu x + (1-\mu)y) \ge \min\{u(x), u(y)\}.$$

(c) u is said to be upper semicontinuous, if for all $\alpha \in [0,1]$, $[u]^\alpha$ is closed.

(d) $[u]^0$ is compact.

Definition 3.2. [24] Suppose $u, v, w \in \mathcal{F}^n$. An element w is referred to as the Hukuhara difference of u and v, if it satisfies the equation $u = v + w$. Now, $u \ominus_H v$ denotes the Hukuhara difference points of u and v. Clearly, $u \ominus_H u = \{0\}$, and if $u \ominus_H v$ exists, then this is unique.

Definition 3.3. [24] Assume $g : (a,b) \to \mathcal{F}^n$ and $t_0 \in (a,b)$. g is referred to as strongly generalized differentiable or GH-differentiable at t_0, if there exists $g'_G(t_0) \in \mathcal{F}^n$ such that

$$g(t_0+h) \ominus_H g(t_0), \quad g(t_0) \ominus_H g(t_0+h)$$

and

$$\lim_{h\to 0^+} \frac{g(t_0+h) \ominus_H g(t_0)}{h} = \lim_{h\to 0^+} \frac{g(t_0) \ominus_H g(t_0+h)}{h} = g'_G(t_0).$$

Example 3.4. [24] Consider the fuzzy mapping $g : \mathbb{R} \to \mathcal{F}'$ defined by $g(t) = C \cdot t$, where C is a fuzzy number defined with its α-levels by $[C]^\alpha = [1+\alpha, 2(3-\alpha)]$. Then,

$$[g(t)]^\alpha = \begin{cases} [1+\alpha, 2(3-\alpha)t], & t \ge 0, \\ [2(3-\alpha)t, 1+\alpha], & t < 0. \end{cases}$$

Obviously, the functions g_l^α and g_r^α are not differentiable at $t = 0$. However, g is GH-differentiable on $\mathbb{R}$ and $g'_G(t) = C$, meaning that g is GH-differentiable at $t = 0$.

We now consider the following fuzzy initial value problem (FIVP) as follows:

$$\begin{cases} x'(t) = g(t, x(t)), & t \in J = [0,T], \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where x' derivative is considered in the sense of GH-differentiable, where at the endpoints of J. The fuzzy the fuzzy one-sided derivative is considered and the fuzzy function $g : J \times \mathcal{F}' \to \mathcal{F}'$ is continuous, and the initial data $x_0 \in \mathcal{F}'$. We denote $C^1(J, \mathcal{F}')$ as the collection of all continuous fuzzy functions $g : J \to \mathcal{F}'$ with continuous derivatives.

Lemma 3.5. *[22] A fuzzy function $x \in \mathcal{C}^1(J, \mathcal{F}')$ is a solution of (4.1) if and only if it verifies the integral equation*

$$x(t) = x_0 \ominus_H (-1)\int_0^t g(s, x(s))\, ds, \quad t \in J = [0,T].$$

Theorem 3.6. *Suppose $g : J \times \mathcal{F}' \to \mathcal{F}'$ is continuous such that*

(i) $g(t,x) < g(t,y)$ for $x < y$,

(ii) there exist some constants $\tau > 0$ large enough such that $\lambda \in \left(0, \frac{1}{2(\rho-\delta)}\right)$ and the metric for $x, y \in \mathcal{F}'$, with $x < y$ and $t \in J$ such that

$$\|g(t,x(t)) - g(t,y(t))\|_{\mathbb{R}} \le \tau \max_{t\in J}\left\{d_\infty(x,y)e^{-\tau(t-\delta)}\right\}$$

Then, (4.1) has a solution in $C^1(J,\mathcal{F}')$.

Proof. Let $C^1(J,\mathcal{F}') been endowed with$

$$d_\tau(x,y)=\sup_{t\in J}\max_{t\in J}\left\{d_\infty(x(t),y(t))e^{-\tau(t-\delta)}\right\},$$

$for x,y\in C^1(J,\mathcal{F}') and \tau>0. Then, with g(x)=\ln(x), x>0 and h=0, (C^1(J,\mathcal{F}'),d_\tau) is a complete metric space.$

Let $A,B:C^1(J,\mathcal{F}')\to(0,1]. For x\in C^1(J,\mathcal{F}')$,

$$L_x(t)=x_0\ominus_H(-1)\int_0^t g(s,x(s))\,ds.s$$

$Let x<y. Then, it follows from the assumption of definition 4.1(a) that$

$$L_x(t)=x_0\ominus_H(-1)\int_\delta^t g(s,x(s))\,ds<x_0\ominus_H(-1)\int_\delta^t g(s,y(s))\,ds=R_y(t).$$

$If L_x(t)\neq R_y(t)$ and $T:C^1(J,\mathcal{F}')\to\mathcal{F}^{C^1(J,\mathcal{F}')} as$

$$\beta_{Tx}(r)=\begin{cases}A(x), & \text{if } r(t)=L_x(t)\\ 0, & \text{otherwise.}\end{cases}$$

$$\beta_{Ty}(r)=\begin{cases}B(y), & \text{if } r(t)=L_y(t)\\ 0, & \text{otherwise.}\end{cases}$$

$Again, if \alpha_{T(x)}=A(x) and \alpha_{T(y)}=B(y), we get$

$$[Tx]_{\alpha_{T(x)}}=\{r\in X:(Tx)(t)\geq A(x)\}=\{L_x(t)\}.$$

Similarly,

$[\text{Ty}]_{\alpha_{T(y)}}=\{R_y(t)\}.$

Then,

$$H([Tx]_{\alpha_{T(x)}},[Ty]_{\alpha_{T(y)}})$$

$$=\max\left\{\sup_{x\in[Tx]_{\alpha_{T(x)}}}\inf\|x-y\|_{\mathbb{R}},\ \sup_{y\in[Ty]_{\alpha_{T(y)}}}\inf\|x-y\|_{\mathbb{R}}\right\}\leq\max\left\{\sup_{t\in J}\|L_x(t)-R_y(t)\|_{\mathbb{R}}\right\}$$

$=\sup_{t\in J}\|L_x(t)-R_y(t)\|_{\mathbb{R}}$

$=\sup_{t\in J}\left\|\int g(s,x(s))\,ds\,t_\delta-\int g(s,y(s))\,ds\,t_0\right\|_{\mathbb{R}}$

$\leq\sup_{t\in J}\left\{\int\|g(s,x(s))-g(s,y(s))\|\,ds\,t_\delta\right\}$

$\leq\sup_{t\in J}\left\{\int du\lambda\max\left\{d_\infty(x,y)e^{-\tau(t-\delta)}\right\}\,ds\,t_\delta\right\}$

$\leq\lambda\sup_{t\in J}\left\{(t-\delta)\max\left\{d_\infty(x,y)e^{-\tau(t-\delta)}\right\}\right\}$

$\leq\lambda(t-\delta)d_{\mathcal{F}}(x,y)$

$\leq\frac{1}{2}d_{\mathcal{F}}(x,y)=\phi(d_{\mathcal{F}}(x,y))$

$\leq\phi(d_{\mathcal{F}}(x,y))-\varphi(d_{\mathcal{F}}(x,[Tx]_{\alpha T(x)}),d_{\mathcal{F}}(y,[Ty]_{\alpha T(y)}),d_{\mathcal{F}}(x,[Ty]_{\alpha T(y)}),d_{\mathcal{F}}(y,[Tx]_{\alpha T(x)}))$.

Hence, all the conditions of Theorem 3.1 are satisfied with $\phi(t)=\frac{1}{2}t$and$\varphi(t)=\frac{1}{3}t$, for$t>0. Thus,$$x^*$is a solution of (4.1). □

4 Conclusion

The main findings of this study demonstrate applicability of $\mathcal{F}$-metric spaces in establishing α-fixed point results for fuzzy enriched φ-ψ contraction in a complete $\mathcal{F}$-metric spaces. This study provides significant advancements in the understanding of $\mathcal{F}$-metric space, through illustrative examples, we showcased the practical applicability of the results and explored as an application, the solution for fuzzy integrodifferential equations in the context of generalized Hukuhara derivative. Future work could also explore the extension of this results to other types of fuzzy mappings.

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Conflicting interests

The author declared that there are no competing interests.

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