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SOME RELATIONS FOR THE DEGENERATE ALTERNATING HARMONIC NUMBERS AND RELATED NUMBERS

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ABSTRACT. In last ten years, many mathematicians ([4]-[14], [15]) studied and investigated for harmonic numbers, hyperharmonic numbers and degenerate hyperharmonic numbers. In this work, we study the degenerate alternating harmonic numbers and the degenerate alternating hyperharmonic numbers defined by Kim in [6]. We give some relations between degenerate alternating numbers and the degenerate alternating Changhee-Genocchi numbers. Also, we give some identities between the alternating hyperharmonic numbers and the degenerate the third kind Korobov numbers.

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1. INTRODUCTION

In recent years, many mathematicians in ([1], [5]-[19], [23]) studied the various degenerate versions of many special numbers and polynomials. They have obtained some identities, recurrence relations and excellent results. T. Kim, D. S. Kim and coworker friends ([1]-[19], [23]) introduced and investigated harmonic numbers, hyperharmonic numbers, degenerate harmonic numbers and degenerate hyperharmonic numbers. D. S. Kim et al. ([8], [12]) and S.-H. Rim et al. [23] gave some identities explicit relations for the harmonic numbers and hyperharmonic numbers. T. Kim and D. S. Kim considered the relations between harmonic numbers and the type 2 polynomials.

Using umbral calculus, D. S. Kim et al. in ([11], [12]) gave combinatorial relations. T. Kim and D. S. Kim ([7], [18]) gave some relations. They proved combinatorial identities between degenerate harmonic numbers and degenerate Fubini polynomials. T. Kim and D. S. Kim ([5]-[19]) generalized the degenerate harmonic numbers. They gave degenerate hyperharmonic numbers and gave explicit relations and identities. H. K. Kim ([6], [10]) considered the degenerate hyperharmonic numbers from degenerate Sheffer

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sequences degenerate alternating numbers and degenerate alternating harmonic numbers. He gave an interesting relations and results.

The outline of this paper is as follows;

In section 1, we recall the degenerate exponentials, the degenerate logarithm, the degenerate harmonic numbers, the degenerate hyperharmonic numbers, the degenerate Frobenius-Euler numbers and the two variable degenerate Fubini polynomials. Also, we recall the Changhee-Genocchi polynomials and Korobov polynomials of the third kind.

In section 2, we obtain an expression between the degenerate harmonic numbers in Theorem 2.1 and also we give a relations between the degenerate hyperharmonic numbers and the ordered Bell numbers in Theorem 2.3.

In section 3, we give explicit relations between the degenerate Changhee-Genocchi numbers and the degenerate alternating numbers. Also, we give a relation the degenerate Korobov numbers of the third kind and degenerate harmonic number.

Throughout this paper, we always make use of the following notation; $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

For nonzero $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$e_{\lambda}^{x}(t) = (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \tag{1}$$

([1], [5]-[16]), where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $n \geq 1$.

Let $\log_{\lambda} t$ be an the compositional inverse function of $e_{\lambda}(t)$ such that

$$\log_{\lambda}(e_{\lambda}(t)) = e_{\lambda}(\log_{\lambda}(t)) = t.$$

The degenerate logarithm is given by

$$\log_{\lambda}(t+1) = \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(1)_{k,1/\lambda}}{k!} t^k = \frac{\left((1+t)^{\lambda}-1\right)}{\lambda}, \tag{2}$$

([1], [5]-[16]).

The generating function of Harmonic numbers given by

$$-\frac{1}{1-t}\log(1-t) = \sum_{n=1}^{\infty} H_n \, t^n, \tag{3}$$

([1], [5]-[16]).

The generating function of hyperharmonic numbers given by

$$-\frac{\log(1-t)}{(1-t)^r} = \sum_{n=1}^{\infty} H_n^{(r)} \, t^n, \tag{4}$$

([1], [5]-[16]).

From (3) and (4), the generating function of the degenerate harmonic numbers and the generating function of the degenerate hyperharmonic numbers are given by

$$-\frac{1}{(1-t)}\log_{\lambda}(1-t)=\sum_{n=1}^{\infty}H_{n,\lambda}\,t^{n} \tag{5}$$

and

$$-\frac{\log_{\lambda}(1-t)}{(1-t)^{r}}=\sum_{n=1}^{\infty}H_{n,\lambda}^{(r)}\,t^{n} \tag{6}$$

([1]-[12], [17]-[19]) respectively.

The degenerate Frobenius-Euler polynomials of order r are defined by

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,\lambda}^{(r)}(u\mid x)\,\frac{t^{n}}{n!}=\left(\frac{1-u}{e_{\lambda}(t)-u}\right)^{r}e_{\lambda}^{x}(t), \tag{7}$$

[22], when $x=0$, we get degenerate Frobenius-Euler numbers of order r are defined by

$$\sum_{n=0}^{\infty}\mathcal{H}_{n,\lambda}^{(r)}(u\mid 0)\,\frac{t^{n}}{n!}=\left(\frac{1-u}{e_{\lambda}(t)-u}\right)^{r}. \tag{8}$$

On the other hand, two variable higher order degenerate Fubini polynomials are defined by

$$\begin{aligned}\sum_{n=0}^{\infty}F_{n,\lambda}^{(r)}(x,y)\,\frac{t^{n}}{n!} &= \left(\frac{1}{1-y\left(e_{\lambda}(t)-1\right)}\right)^{r}e_{\lambda}^{x}(t)\\ &= \left(\frac{1-\frac{1+y}{y}}{e_{\lambda}(t)-\frac{1+y}{y}}\right)^{r}e_{\lambda}^{x}(t).\end{aligned} \tag{9}$$

From (8) and (9), we get

$$F_{n,\lambda}^{(r)}(0,y)=\mathcal{H}_{n,\lambda}^{(r)}\left(\frac{1+y}{y}\mid 0\right). \tag{10}$$

For $x=0$ and $y=1$,

$$F_{n,\lambda}^{(r)}(0,1)=\mathcal{H}_{n,\lambda}^{(r)}(2\mid 0).$$

The generating function of the Changhee-Genocchi polynomials given by

$$\sum_{n=0}^{\infty}CG_{n}(x)\,\frac{t^{n}}{n!}=\frac{2\log(1+t)}{2+t}(1+t)^{x},\ [14]. \tag{11}$$

When $x=0$, we get the Changhee-Genocchi number $CG_{n}(0):=CG_{n}$.

Korobov polynomials of the third kind defined as

$$\sum_{n=0}^{\infty}K_{n,3}(x\mid\mu)\,\frac{t^{n}}{n!}=\frac{\log(1+\mu t)}{\mu\log(1+t)}(1+t)^{x},\ ([4],\ [22]), \tag{12}$$

where $\mu\neq 0,\,1;\ \mu\in\mathbb{N}$.

When $x=0$, we get the third kind of Korobov numbers $K_{n,3}(0\mid\mu)=K_{n,3}(0)$.

The generating functions of the degenerate Stirling numbers of the first kind and the generating functions of the degenerate Stirling numbers of the second kind are defined as, respectively,

$$\frac{\left(\log_{\lambda}\left(1+t\right)\right)^{k}}{k!}=\sum_{n=k}^{\infty}S_{1,\lambda}\left(n,k\right)\frac{t^{n}}{n!},\ ([1]\text{-}[19]) \tag{13}$$

and

$$\frac{\left(e_{\lambda}\left(t\right)-1\right)^{k}}{k!}=\sum_{n=0}^{\infty}S_{2,\lambda}\left(n,k\right)\frac{t^{n}}{n!},\ ([1]\text{-}[19]). \tag{14}$$

When $\lambda\rightarrow 0$, $S_1\left(n,k\right)$ and $S_2\left(n,k\right)$ are the Stirling numbers of the first kind and the Stirling numbers of the second kind, respectively.

2. Some Relations for the Degenerate Harmonic Numbers

In this section, we give some relations between the degenerate Harmonic numbers, the degenerate Fubini numbers, the ordered Bell numbers and the degenerate Frobenius-Euler numbers.

Theorem 2.1. *There are the following relation between the degenerate harmonic numbers and the degenerate two variable Fubini numbers,*

$$\sum_{n=0}^{k-1}n!H_{n,\lambda}\ \left(-1\right)^{n}\ S_{2,\lambda}\left(k,n\right)=-\frac{1}{\lambda}\left(F_{k,\lambda}^{(1-\lambda)}\left(1\right)-F_{k,\lambda}\left(1\right)\right),\ k\geq 1. \tag{15}$$

Proof. Replacing t by $-1+e_{\lambda}\left(t\right)$ in (5) and using (2),

$$\begin{aligned}\sum_{n=1}^{\infty}H_{n,\lambda}\left(-1+e_{\lambda}\left(t\right)\right)^{n} &= \frac{1}{\lambda}\left\{\left(\frac{1}{2-e_{\lambda}\left(t\right)}\right)^{1-\lambda}-\frac{1}{2-e_{\lambda}\left(t\right)}\right\}\\ &= -\frac{1}{\lambda}\left\{\left(\frac{1}{1-\left(e_{\lambda}\left(t\right)-1\right)}\right)^{1-\lambda}-\frac{1}{1-\left(e_{\lambda}\left(t\right)-1\right)}\right\}.\end{aligned}$$

Using (9), for $x=0$ and $y=1$ and from the definition of the degenerate Stirling numbers of the second kind, we have

$$\sum_{k=1}^{\infty}\sum_{n=0}^{k-1}n!H_{n,\lambda}\ \left(-1\right)^{n}\ S_{2,\lambda}\left(k,n\right)\frac{t^{k}}{k!}$$

$$=-\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(F_{k,\lambda}^{(1-\lambda)}\left(0,1\right)-F_{k,\lambda}\left(0,1\right)\right)\frac{t^{k}}{k!},\ k\geq 1.$$

Comparing the coefficients of both sides, we have (15). □

Corollary 2.2. *There is the following relation between the degenerate harmonic numbers and the Frobenius-Euler number as*

$$\sum_{n=0}^{k-1}n!\ \left(-1\right)^{n}\ S_{2,\lambda}\left(k,n\right)=-\frac{1}{\lambda}\left(\mathcal{H}_{k,\lambda}^{(1-\lambda)}\left(2,0\right)-\mathcal{H}_{k,\lambda}\left(2,0\right)\right). \tag{16}$$

From (10) and (14), we get (16).

The generating function of the ordered Bell polynomials are defined by

$$\sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} = \frac{1}{2-e^t}e^{xt},\ ([2],\ [3]).$$

We define degenerate ordered Bell numbers of order r as

$$\sum_{n=0}^{\infty} b_{n,\lambda}^{(r)}\frac{t^n}{n!} = \left(\frac{1}{2-e_\lambda(t)}\right)^r. \tag{17}$$

Theorem 2.3. *There is the following relation between the degenerate hyperharmonic numbers and the ordered Bell numbers*

$$\sum_{k=0}^{n-1} n!H_{n,\lambda}^{(r)}\ S_{2,\lambda}(n,k) = -\frac{1}{\lambda}\left(b_{k,\lambda}^{(r-\lambda)} - b_{k,\lambda}^{(r)}\right). \tag{18}$$

Proof. Replacing t by $-1+e_\lambda(t)$ in (6) and from (17), we write as

$$\sum_{n=1}^{\infty} H_{n,\lambda}^{(r)}\ (-1+e_\lambda(t))^n = -\frac{1}{\lambda}\left\{\frac{1}{(2-e_\lambda(t))^{r-\lambda}} - \frac{1}{(2-e_\lambda(t))^r}\right\}.$$

Using (17) and (14), we get

$$\sum_{k=1}^{\infty}\sum_{k=0}^{n-1} n!H_{n,\lambda}^{(r)}\ (-1)^r\ S_{2,\lambda}(n,k)\frac{t^k}{k!} = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(b_{k,\lambda}^{(r-\lambda)} - b_{k,\lambda}^{(r)}\right)\frac{t^k}{k!}.$$

For $k \geq 1$, comparing the coefficients of both sides, we get (18). □

3. Explicit Relations for the Degenerate Alternating Harmonic Numbers

In this section, we introduce and investigate to alternating degenerate harmonic numbers. We give some identities, recurrance relations, explicit expressions.

The generating functions of the degenerate alternating harmonic numbers are defined by H. K. Kim in [6] as

$$\frac{\log_\lambda(1+t)}{1-t} = \sum_{n=1}^{\infty}\overline{H}_{n,\lambda}\ t^n,\ \overline{H}_{0,\lambda} = 1 \tag{19}$$

and also same author in [6] defined the generating functions of the degenerate alternating hyperharmonic numbers by

$$\frac{\log_\lambda(1+t)}{(1-t)^r} = \sum_{n=1}^{\infty}\overline{H}_{n,\lambda}^{(r)}\ t^n,\ r\in\mathbb{N}. \tag{20}$$

We define the degenerate Changhee-Genocchi numbers as

$$\sum_{n=0}^{\infty} CG_{n,\lambda}\frac{t^n}{n!} = \frac{2\log_\lambda(1+t)}{2+t}. \tag{21}$$

Theorem 3.1. *There are the following relation between the degenerate Changhee-Genocchi numbers and the degenerate alternating harmonic numbers as*

$$CG_{n,\lambda} = \sum_{k=0}^{n-1} n!\ \overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{n-k} + 2\sum_{k=0}^{n-2} n!\ \overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{n-k}. \tag{22}$$

Proof. From (20) and (21), we write as

$$\begin{aligned}\sum_{n=0}^{\infty} CG_{n,\lambda}\frac{t^n}{n!} &= \frac{2\log_\lambda(1+t)}{2+t} = \frac{\log_\lambda(1+t)}{1+\frac{t}{2}} \\ &= \frac{\log_\lambda(1+t)}{1-t}\frac{1-t}{1+\frac{t}{2}} \\ &= \sum_{k=1}^{\infty}\overline{H}_{k,\lambda}\ t^k\ (1-t)\sum_{l=0}^{\infty}\left(-\frac{t}{2}\right)^l \\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} n!\ \overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{n-k}\frac{t^n}{n!} + 2\sum_{n=0}^{\infty}\sum_{k=0}^{n-2} n!\ \overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{n-k}\frac{t^n}{n!} \\ &= 2\sum_{k=0}^{-2}\overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{(-k)} + \sum_{n=1}^{\infty}\left\{\sum_{k=0}^{n-1} n!\ \overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{n-k} + 2\sum_{k=0}^{n-2} n!\ \overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{n-k}\right\}\frac{t^n}{n!}.\end{aligned}$$

For $n \geq 1$, we have (22). □

Theorem 3.2. *The following relation holds*

$$CG_{n,\lambda} = \sum_{j=0}^{n-1} n!\ \overline{H}^{(r)}_{n-j,\lambda}\sum_{i=0}^{j}\binom{r}{i}2^i\left(-\frac{1}{2}\right)^j. \tag{23}$$

Proof. From (20) and (21), we write as

$$\begin{aligned}\sum_{n=0}^{\infty} CG_{n,\lambda}\frac{t^n}{n!} &= \frac{\log_\lambda(1+t)}{(1-t)^r}\frac{(1-t)^r}{1+\frac{t}{2}} = \sum_{k=1}^{\infty}\overline{H}^{(r)}_{k,\lambda}t^k\sum_{i=0}^{\infty}\binom{r}{i}(-t)^i\sum_{l=0}^{\infty}\left(-\frac{t}{2}\right)^l \\ &= \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} n!\overline{H}^{(r)}_{n-j,\lambda}\sum_{i=0}^{j}\frac{r!\,(-1)^j}{i!\,(r-i)!2^{j-1}}\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1} n!\ \overline{H}^{(r)}_{n-j,\lambda}\sum_{i=0}^{j}\binom{r}{i}2^i\left(-\frac{1}{2}\right)^j\right)\frac{t^n}{n!}.\end{aligned}$$

For $n \geq 1$, comparing the coefficient of $\frac{t^n}{n!}$, we get (23). □

Corollary 3.3. *For $n \geq 1$, we get*

$$\sum_{k=0}^{n-1} n!\overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{n-k} + 2\sum_{k=0}^{n-2} n!\ \overline{H}_{k,\lambda}\ \left(-\frac{1}{2}\right)^{n-k} = \sum_{j=0}^{n-1} n!\ \overline{H}^{(r)}_{k-j,\lambda}\sum_{i=0}^{j}\binom{r}{i}2^i\left(-\frac{1}{2}\right)^j.$$

From (22) and (23), we get this result.

Theorem 3.4. *The following relation holds*

$$CG_{n,\lambda} + \frac{1}{2}n\ CG_{n-1,\lambda} = \sum_{l=0}^{n-1}\frac{r!}{(r-l)!}(-1)^l\,(n-l)!\ \overline{H}_{n-l,\lambda}. \tag{24}$$

Proof. From (20) and (21), we write

$$(1-t)^r \sum_{k=1}^{\infty} \overline{H}_{k,\lambda}^{(r)}\ t^k = \left(1+\frac{t}{2}\right) \sum_{n=0}^{\infty} CG_{n,\lambda} \frac{t^n}{n!}.$$

L.H.S.

$$(1-t)^r \sum_{k=1}^{\infty} \overline{H}_{k,\lambda}^{(r)}\ t^k = \sum_{l=0}^{\infty} \binom{r}{l} (-t)^l \ \sum_{k=1}^{\infty} \overline{H}_{k,\lambda}^{(r)} t^k$$

$$= \sum_{n=1}^{\infty} \left(\sum_{l=0}^{n-1} \frac{r!}{(r-l)!}\ (n-l)!\,(-1)^l\ \overline{H}_{n-l,\lambda} \right) \frac{t^n}{n!}. \tag{25}$$

R.H.S.

$$\left(1+\frac{t}{2}\right) \sum_{n=0}^{\infty} CG_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} CG_{n,\lambda} \frac{t^n}{n!} + \frac{1}{2} \sum_{n=1}^{\infty} n\ CG_{n-1,\lambda} \frac{t^n}{n!}$$

$$= CG_{0,\lambda} + \sum_{n=1}^{\infty} \left(CG_{n,\lambda} + \frac{1}{2} n\ CG_{n-1,\lambda} \right) \frac{t^n}{n!}. \tag{26}$$

From (25) and (26), for $n \geq 1$, we get (24). □

From (12), we define degenerate Korobov numbers of third kind as follows,

$$\sum_{n=0}^{\infty} K_{n,\mu,3,\lambda} \frac{t^n}{n!} = \frac{\log_\lambda (1+\mu t)}{\mu \log_\lambda (1+t)},\ \mu \in \mathbb{N},\ \mu \neq 0,\ 1,\ \lambda \neq 0 \text{ and } \lambda \in \mathbb{R}. \tag{27}$$

Theorem 3.5. *There are the following relation between the degenerate Korobov numbers of the third kind and degenerate Harmonic numbers as*

$$\mu \left\{ \sum_{k=0}^{n-2} \frac{n!}{k!} K_{k,\mu,3,\lambda}\ \overline{H}_{n-1-k,\lambda} - \sum_{k=0}^{n-1} \frac{n!}{k!} K_{k,\mu,3,\lambda}\ \overline{H}_{n-k,\lambda} \right\} = n! \left(\overline{H}_{n,\lambda} - \mu^n\ \overline{H}_{n-1,\lambda} \right). \tag{28}$$

Proof. From (27) and (19), we write

$$\sum_{n=0}^{\infty} K_{n,\mu,3,\lambda} \frac{t^n}{n!} = \frac{\log_\lambda (1+\mu t)}{\mu \log_\lambda (1+t)} = \frac{(1-\mu t) \sum_{n=1}^{\infty}\ \overline{H}_{n,\lambda}\ t^n \mu^n}{\mu (1-t) \sum_{n=1}^{\infty}\ \overline{H}_{n,\lambda}\ t^n}$$

$$\mu \left(\sum_{n=1}^{\infty}\ \overline{H}_{n,\lambda}\ t^n - \sum_{n=1}^{\infty}\ \overline{H}_{n,\lambda}\ t^{n+1} \right) \sum_{k=0}^{\infty} K_{k,\mu,3,\lambda} \frac{t^k}{k!} = \sum_{n=1}^{\infty}\ \overline{H}_{n,\lambda}\ t^n - \mu \sum_{n=1}^{\infty}\ \overline{H}_{n,\lambda}\ t^{n+1} \mu^n. \tag{29}$$

L.H.S. of (29), we get

$$\mu \left\{ \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{n!}{k!} K_{k,\mu,3,\lambda} \overline{H}_{n-k,\lambda} \frac{t^n}{n!} - \sum_{n=0}^{\infty} \sum_{k=0}^{n-2} \frac{n!}{k!} K_{k,\mu,3,\lambda} \overline{H}_{n-1-k,\lambda} \frac{t^n}{n!} \right\}. \tag{30}$$

Also, R.H.S. of (29), we get

$$= \left(\sum_{n=1}^{\infty} \mu^n\ \overline{H}_{n,\lambda} - \sum_{n=0}^{\infty} \mu^n\ \overline{H}_{n-1,\lambda} \right) t^n \tag{31}$$

$$= \left(\sum_{n=1}^{\infty} n!\mu^n \overline{H}_{n,\lambda} - \sum_{n=0}^{\infty} n!\ \overline{H}_{n-1,\lambda}\ \mu^n \right) \frac{t^n}{n!}.$$

From (30) and (31), $n \geq 1$,

$$\mu \left\{ \sum_{k=0}^{n-1} \frac{n!}{k!}\ \overline{H}_{n-k,\lambda} - \sum_{k=0}^{n-2} \frac{n!}{k!}\ \overline{H}_{n-1-k,\lambda} \right\} K_{k,\mu,3,\lambda} = n!\mu^n \left(\overline{H}_{n-1,\lambda} - \overline{H}_{n,\lambda} \right).$$

We have (28). □

4. Conclusion

In this study, we give some identities, recurrence relation for the alternating degenerate harmonic numbers and the alternating degenerate hyperharmonic numbers. We consider the degenerate Changhee-Genocchi numbers, the degenerate Korobov polynomials of the third kind. We give some relations for these numbers.

References

[1] D. V. Dolgy, D. S. Kim, H. Y. Kim and T. Kim, *Degenerate harmonic and hyperharmonic numbers*, Proceed of the Jangjeon Math. Soc. 26(3) (2023), 259-268, doi: 10.17777/pjms2023.26.3.259.

[2] G.-W. Jang and T. Kim, *Some identities of ordered Bell numbers arising from differential equation*, Adv. Stud. Contemp. Math. 27(3) (2017), 385-397.

[3] T. Kim, D. S. Kim, G.-W. Jang and L.-C. Jang, *Degenerate ordered Bell numbers and polynomials associated with umbral calculus*, J. Nonlinear Sci. Appl. 10 (2017), 5142-5155.

[4] T. Kim and D. S. Kim, *Korobov polynomials of the third kind and of the fourth kind*, Springer plus (2015), 4.608, doi:10.1186/s40064-015-1405-9.

[5] T. Kim and D. S. Kim, *Some identities on degenerate harmonic and degenerate higher-order harmonic numbers*, Appl. Math. Comp. 486 (2025), 129045.

[6] H. K. Kim, *Degenerate alternating harmonic numbers and degenerate alternating hyperharmonic numbers*, Researchgate (2023) doi:10.13140/RG.22.26759.37287.

[7] T. Kim and D. S. Kim, *Combinatorial identities involving degenerate harmonic and hyperharmonic numbers*, Adv. in Appl. Math. 148 (2023), 102535.

[8] D. S. Kim, H. Kim and T. Kim, *Some identities on generalized harmonic numbers and generalized harmonic function*, Demonstratio Math. 56(1) (2023), 20220225, doi:10.1515/dema-2022-0229.

[9] T. Kim and D. S. Kim, *Higher-order degenerate harmonic numbers related to degenerate Riemann zeta function*, ArXiv:2308.00901vI, (2023).

[10] H. K. Kim, *The degenerate hyperharmonic numbers and polynomials derived from degenerate Sheffer sequences*, Researchgate (2023), doi:10.13140/RG.22/2290.66264.

[11] D. S. Kim, T. Kim and S.-H. Lee, *Combinatorial identities involving harmonic and hyperharmonic numbers*, Adv. Stud. Contempt. Math. 23(3) (2013), 393-413.

[12] D. S. Kim and T. Kim, *Identities involving harmonic and hyperharmonic numbers*, Adv. in Differ. Equa. (2013) 2013.235.

[13] T. Kim and D. S. Kim, *Some relations of two type 2 polynomials and discrete harmonic numbers and polynomials*, Symmetry 12(6) (2020) 12.905, doi: 10.3290/sym.12060905.

[14] D. S. Kim and T. Kim, *Some identities of Korobov-type polynomials associated with p-adic integrals on Z_p*, Adv. Differ. Equa. 282 (2015), 2015.282.

[15] T. Kim and D. S. Kim, *Some identities on degenerate hyperharmonic numbers*, Georgian Math. 30 (2023), 255-262, doi:10.1515/gmj-2022-2203.

[16] D. S. Kim, T. Kim, H.-I. Kwon and J.-W. Park, *Two variable higher-order Fubini polynomials*, J. Korean Math. Soc. 55(4) (2018) 975-986.

[17] H. K. Kim, D. S. Kim and T. Kim, *Identities involving degenerate harmonic and degenerate hyperharmonic numbers*, Arxiv:2301.02861VI, 2023.

[18] T. Kim and D. S. Kim, *Some results on degenerate harmonic numbers and degenerate Fubini numbers*, Arxiv:2208.12998VI, 2022.

[19] D. S. Kim and T. Kim, *Some results and numbers related to degenerate harmonic and degenerate hyperharmonic numbers*, Arxiv:2306.12009VI, 2023.

[20] B. Kurt, *Unified Apostol-Korobov type polynomials and related polynomials*, Bull. Korean Math. Soc. 58(2) (2021) 315-326.

[21] B. Kurt, *Some relations on the degenerate Korobov polynomials and poly-Korobov polynomials*, J. Pure Appl. Math. 55(2) (2024) 628-638.

[22] B. Kurt, *A note on the Apostol-type q-Frobenius-Euler polynomials and Generalizations of the Srivastava-Pintér addition theorems*, Filomat 30(1) (2016) 65-72.

[23] S.-H. Rim, T. Kim and S.-S. Pyo, *Identities between harmonic, hyperharmonic and Daehee numbers*, J. of Inequal. and Appl. 168 (2018) 2018.168.

[24] J. J. Seo and T. Kim, *Degenerate Korobov polynomials*, App. Math. Sci. 10(4) (2016) 167-173.

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