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Absolute almost convergence of Hardy-Littlewood series

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Abstract. Extending a result of Mohanty and Mohapatra on absolute Cesáro summability of Hardy-Littlewood series, a result on absolute almost convergent of Hardy-Littlewood series has been established in this paper.

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1. Definitions

Let $\sum a_n$ be a given infinite series, which we denote by a, with the sequence of partial sums $\{s_n\}$. Let us consider the transformation

$$t_{m,n} = \frac{1}{m+1}\sum_{k=0}^{m} s_{n+k} \tag{1.1}$$

If $t_{m,n} \to s$, uniformly in n, then the series $\sum a_n$ is said to be almost convergent to s (see[11]). Let $\hat{c}$ denote the set of all almost convergent sequences.The series a (or the sequence $\{s_n\}$) is said to be absolutely almost convergent (see [6],[4],[5],[3]) , if

$$\sum_{m=0}^{\infty} |\phi_{m,n}| < \infty, \tag{1.2}$$

uniformly in n, where

$$\phi_{m,n} = \frac{1}{m(m+1)}\sum_{k=0}^{m} k\ a_{n+k}, (n \geq 1), \quad \phi_{0,n} = a_n \tag{1.3}$$

An infinite series $\sum a_n$ is said to be absolutely (C,α), $\alpha > 0$, if

$$\sum_{n=0}^{\infty} \frac{\tau_n^{\alpha}}{n} < \infty, \tag{1.4}$$

where τ_n^{α} is the (C,α) mean ([11]) of the sequence $\{na_n\}$, that is

$$\tau_n^{\alpha} = \frac{1}{A_n^{\alpha}}\sum_{k=0}^{n} A_{n-k}^{\alpha} k a_k \tag{1.5}$$

and A_n^{α}'s are given by

$$\frac{1}{(1-x)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^{\alpha} x^n, \quad |x| < 1. \tag{1.6}$$

Let l and $\hat{l}$ denote the set of absolutely convergent series and almost convergence series respectively.We have the following known results :

$$(i)\ \ l \ \subset\ \hat{l}\ \subset\ |C,1| \tag{1.7}$$

$$(ii)\ \ \hat{l}\ and\ \ |C,\alpha|,\ 0<\alpha<1,\ \text{are mutually exclusive.} \tag{1.8}$$

Let f be a 2π-periodic integrable function defined over $(-\pi,\pi)$. The Fourier series of f at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(\ a_n\ cos\ nx\ +\ b_n\ sin\ nx\ ) = \sum_{n=0}^{\infty}A_n(x) \tag{1.9}$$

The series conjugate to Fourier series (1.9) is given by

$$\sum_{n=1}^{\infty}(\ a_n\ sin\ nx\ -\ b_n\ cos\ nx\ ) = \sum_{n=1}^{\infty}B_n(x) \tag{1.10}$$

We write

$$\phi_x(t) = \frac{1}{2}[f(x+t) + f(x-t) - 2\ f(x)] \tag{1.11}$$

$$\psi_x(t) = \frac{1}{2}[f(x+t) - f(x-t)] \tag{1.12}$$

Let $S_n(f;x)$ and $S_n^*(f;x)$ denote respectively the nth. partial sum and modified nth. partial sum of the Fourier series (1.9). Then

$$S_n(f;x) = \sum_{k=0}^{n}A_k(x) \tag{1.13}$$

$$S_n^*(f;x) = \sum_{k=0}^{n-1}A_k(x) + \frac{1}{2}A_n(x) \tag{1.14}$$

It is known that ([16], page 50) that

$$S_n(f;x) - f(x) = \frac{2}{\pi}\int_0^{\pi}\phi_x(t)\frac{sin(n+\frac{1}{2})t}{2\ sin(t/2)}dt \tag{1.15}$$

$$S_n^*(f;x) - f(x) = \frac{2}{\pi}\int_0^{\pi}\phi_x(t)\frac{sin(n+\frac{1}{2})t}{2\ tan(t/2)}dt \tag{1.16}$$

2. Introduction

It is known [8] that the series

$$\sum_{n=1}^{\infty}\frac{S_n(f;x) - f(x)}{n} \tag{2.1}$$

is usually called Hardy-Littlewood series or simply HL-series.Hardy and Littlewood [8] have shown that (2.1) is $(C, 1)$ summable to the value

$$\frac{1}{\pi}\int_0^{\pi}\left[\left(\frac{\pi - u}{2}\right)cot\frac{1}{2}u - log(2\ sin\frac{u}{2})\right]\phi_x(u)\ du \tag{2.2}$$

whenever the integral

$$\int_0^{\infty}\phi_x(u)\frac{1}{2}cot\frac{1}{2}u\ du \tag{2.3}$$

exists. Further [8], if

$$\int_0^{\infty}|\phi_x(u)|\ du = O(t),\ as\ t \to 0^+, \tag{2.4}$$

then (2.1)converges if and only if (2.3) exists.As the interest of HL-series lies in the relation to the integral (2.3), there are relations very similar to those between the conjugate series $\sum B_n(x)$ and to the integral

$$\int_0^{\pi}\frac{\psi_x(u)}{u}\ du. \tag{2.5}$$

It is known([16], page 50), that if $f \in L(0, \pi)$, then (2.5) exists almost everywhere. On the otherhand there exists a continuous function for which the integral given in (2.3) diverges for almost all x[8]. At this stage we remark that above results on HL-series remain unaltered, if we replace the H-L series by

$$\sum_{n=1}^{\infty}\frac{S_n^*(f;x) - f(x)}{n}. \tag{2.6}$$

The series (2.6) is summable $(C, 1)$ to the value

$$\int_{0^+}^{\pi}\phi_x(u)\frac{u}{2}cot\frac{1}{2}u\ du, \tag{2.7}$$

if it exists. Further, if (2.4) holds, then a necessary and sufficient condition([16], page 125) for the convergence or summability $(C, 1)$ problems of (2.6) is the convergence or the integral (2.7).Thus the convergence or $(C, 1)$ summability problem of (2.6) is same as that of (2.1) though their sums are different and hence we may term (2.6) as an HL-series.

3. Known Result

Prössdorf[14] studied the degree of approximation in the Hölder metric and proved the following theorem:

Theorem A

Let $f \in H_\alpha, (0 < \alpha \leq 1)$ and $0 \leq \beta < \alpha \leq 1$. Then

$$||\sigma_n(f) - f||_{(\beta,\alpha)} = O(1) \begin{cases} n^{\beta-\alpha}, & \text{for } 0 < \alpha < 1 \\ n^{\beta-1}(1 + logn)^{1-\beta}, & \text{for } \alpha = 1 \end{cases} , \qquad (3.1)$$

where $\sigma_n(f)$ is the Fejer mean of the Fourier series of f.
The case $\beta = 0$ of Theorem A is due to Alexits. With regards to the approximation of functions of L_p norm, the following is due to Quade[15]

Theorem B

Let $f \in lip(\alpha, p, (0 < \alpha \leq 1)$. Then

$$||\sigma_n(f) - f||_{(0,p)} = O(1) \begin{cases} n^{-\alpha}, & \text{for } p > 1 \\ n^{-\alpha}, & \text{for } p = 1, 0 < \alpha < 1 \\ (logn)/n, & \text{for } p = 1, \alpha = 1 \end{cases} . \qquad (3.2)$$

In 1996 [2], the degree of approximation in the generalized Hölder metric has been introduced and the following result has been obtained.

Theorem C

Let $s_n(x)$ be the nth partial sum of (1.9). Suppose that $A \in \mathfrak{T}$ and there exists a positive non-decreasing sequence (μ_n) such that

$$\sum_{k=\mu_n}^{\infty} (k+1)|a_{n,k}| = O(\mu_n). \qquad (3.3)$$

Then for $p \geq 1$ and $f \in H(\alpha, p), 0 < \alpha \leq 1, 0 \leq \beta < \alpha$

$$||\sum_{k=0}^{\infty} a_{n,k}s_k - f||_{(\beta,p)} = O(1) \begin{cases} (1 + log(\mu_n/\lambda_n)^{\beta/\alpha}\lambda_n^{\beta-\alpha} + \psi(n)\lambda_n^{1-\alpha+\beta}, & \text{for } 0 < \alpha \leq 1 \\ \frac{(1+log(\mu_n/\lambda_n))^\beta}{\lambda_n^{1-\beta}} + \psi(n)\lambda_n^\beta(log\lambda_n)^{1-\beta}, & \text{for } \alpha = 1 \end{cases} , \qquad (3.4)$$

where λ_n is any positive non-decreasing sequence such that $\lambda_n \leq \mu_n$.

In the middle of 1998, Das et al [6] had determined the degree of approximation of the series

$$(1/2)c_0 + \sum_{n=1}^{\infty} \frac{S_n(f;x) \quad - \quad f(x)}{n} \tag{3.5}$$

by means of A-transform in the generalized Hölder metric in the following form :

Theorem D

Suppose that $A \in \Im$ and there exists a positive non-decreasing sequence (μ_n) such that

$$\sum_{k=\mu_n}^{\infty} (k+1)|a_{n,k}| = O(\mu_n). \tag{3.6}$$

Let $M_n(x)$ be the A-transform of the series (2.12).Then for $p \geq 1$ and $f \in H(\alpha, p), 0 < \alpha \leq 1, 0 \leq \beta < \alpha$

$$||M_n(x) - \chi_x(\pi/\mu_n)||_{(\beta,p)} = O(1) \begin{cases} (log\mu_n)^{\beta/\alpha}\Big[(1 + log(\mu_n/\lambda_n)^{\beta/\alpha}\lambda_n^{\beta-\alpha} + \psi(n)\lambda_n^{1-\alpha+\beta}\Big], & \text{for } 0 < \alpha < 1 \\ (log\mu_n)^{\beta}\Big[\frac{(1+log(\mu_n/\lambda_n))^{\beta}}{\lambda_n^{1-\beta}} + \psi(n)\lambda_n^{\beta}(log\lambda_n)^{1-\beta}\Big], & \text{for } \alpha = 1 \end{cases}, \tag{3.7}$$

where λ_n is any positive non-decreasing sequence such that $\lambda_n \leq \mu_n$ and $\psi(n) = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}|$.
Very recently Manish Kumar et al[12], dealing with Euler, Borel and (e, c) mean of H-L series for functions of Lipschitz class, have established the following theorems :

Theorem E

Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha,p}$. Then

$$||E_n^q(\hat{T}) - f||_{(\beta,p)} = O(1) \begin{cases} n^{\beta-\alpha}, & \text{for } \alpha - \beta \neq 1 \\ (logn)/n, & \text{for } p = 1, \alpha - \beta = 1 \end{cases}, \tag{3.8}$$

where $E_n^q(\hat{T})$ is the (E, q) mean of the H-L series.

Theorem F

Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha,p}$. Then

$$||B_p(\hat{T}) - f||_{(\beta,p)} = O(1)\begin{cases} p^{\beta-\alpha}, & \text{for } \alpha-\beta \neq 1 \\ (logp)/p, & \text{for } \alpha-\beta = 1 \end{cases}, \qquad (3.9)$$

where $B_p(\hat{T})$ is the Borel mean of the H-L series.

Theorem G

Let $0 \leq \beta < \alpha \leq 1$ and let $f \in H_{\alpha,p}$. Then

$$||e_n(\hat{T}) - f||_{(\beta,p)} = O(1)\begin{cases} n^{\beta-\alpha}, & \text{for } 0 < \alpha-\beta \neq 1/2 \\ 1/\sqrt{n}, & \text{for } 1/2 < \alpha-\beta \leq 1 \end{cases}, \qquad (3.10)$$

where $e_n(\hat{T})$ is the (e,c) mean of the H-L series.
However, the absolute convergence, absolute Riesz summability and absolute Cesàro summability problems of HL-series were first studied by Mohanty and Mohapatro[13]. Their result on absolute Cesàro summabilty is as follows :

Theorem H

If $\frac{\phi_x(t)}{t} \in L(0,\pi)$, then the HL-series is summable $|C,\alpha|$, $\alpha > 0$.

4. Main Result

Dealing with absolute almost convergence of H-L series, in the present work , we prove the following theorem :

Theorem

If $\frac{\phi_x(t)}{t} \in L(0,\pi)$, then the HL-series (2.6) is summable $\hat{l}$.

5. Lemma

In order to prove the theorem we require the following lemma :

Lemma

If

$$l_m(n,t) \quad = \quad \sum_{k=1}^{m} \frac{k}{n+k} \; sin(n \; + \; k) \; t \tag{5.1}$$

then

$$l_m(n,t) \quad = \begin{cases} O(m^2 \; t) \; , & 0 \; \leq t < \frac{\pi}{n+1}, \\ O(t^{-1}), & \frac{\pi}{n+1} < t \leq \pi \end{cases} \tag{5.2}$$

Proof of the lemma :

For $0 \leq t < \frac{\pi}{n+1}$, we have

$$\begin{aligned} |l_m(n,t)| &= \left| \sum_{k=1}^{m} \frac{k \; sin \; (n+k) \; t}{n+k} \right| \\ &\leq \sum_{k=1}^{k=m} kt \; , \text{ as } \; |sin(n+k) \; t| \leq (n+k) \; t \\ &= \mathcal{O}(m^2 t). \end{aligned}$$

Next for all t

$$\begin{aligned} |l_m(n,t)| &= \left| \sum_{k=1}^{m} \frac{k \; sin \; (n+k) \; t}{n+k} \right| \\ &\leq \frac{m}{n+m} \text{Max} \sum_{k=m_1}^{k=m_2} sin \; (n+k) t \; , \text{ where max is taken for } m_1 \leq k < m_2 \\ &\leq \text{Max} \left| \sum_{k=m_1}^{k=m_2} sin \; (n+k) \; t \right| \\ &= \mathcal{O}(t^{-1}). \end{aligned}$$

6.Proof of the theorem

By the definition, we have that the series $\sum_{n=1}^{\infty} \frac{S_n^*(f;x) \; - \; f(x)}{n} \in \; \hat{l}$, if and only if,

$$\sum_{m=1}^{\infty} \frac{1}{m(m+1)} \left| \sum_{k=1}^{m} \int_0^{\pi} \frac{\phi_x(t) \; k \; sin \; (n+k) \; t}{(n+k) \; 2 \; tan\frac{t}{2}} \; dt \right| \; < \; \infty,$$

uniformly in n. As by the hypothesis $\int_0^\pi \frac{|\phi_x(t)|}{t} < \infty$, in order to establish the theorem it is enough to show that

$$\sum = \sum_{m=1}^{\infty} \frac{1}{m(m+1)} \left| \sum_{k=1}^{m} \frac{k \ sin \ (n+k) \ t}{(n+k)} \ dt \right| = O(1), \tag{6.1}$$

uniformly in $0 < t \leq \pi$ and n .
We have

$$\begin{aligned} \sum &= \left(\sum_{m=1}^{T} + \sum_{m=T+1}^{\infty} \right) |l_m(n,t)| \ , \text{ where } T = [t_{-1}] \\ &= I_1 + I_2 \ , \text{ say.} \end{aligned} \tag{6.2}$$

Now

$$\begin{aligned} I_1 &= \sum_{m=1}^{T} \frac{1}{m(m+1)} |l_m(n,t)| \\ &= \sum_{m=1}^{T} \frac{1}{m(m+1)} O(m^2 t) \\ &= O(t)T \\ &= O(1) \ . \end{aligned} \tag{6.3}$$

Next

$$\begin{aligned} I_2 &= \sum_{m=T+1}^{\infty} \frac{1}{m(m+1)} |l_m(n,t)| \\ &= O(t^{-1}) \sum_{m=T+1}^{\infty} \frac{1}{m^2} \\ &= O\left(\frac{t^{-1}}{T}\right). \\ &= O(1) \ . \end{aligned} \tag{6.4}$$

Collecting the results (6.2) , (6.3) and (6.4), , we obtain (6.1) and this completes the proof of the theorem.

Conclusion

The result in this article is quite independent of the result of Mohanty and Mohapatra.One can find results for the convergence of operators associated with H-L series.

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