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Analysis of Fourth-Order Geometric Properties in P^h-Generalized Recurrent Finsler Manifolds

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Abstract

In this paper, we delve into a comprehensive analysis of fourth-order geometric properties within the framework of $P-$generalized recurrent Finsler manifolds. By employing advanced geometric techniques, we investigate the intricate relationships between curvature tensors, torsion tensors, and other relevant geometric quantities. Our findings reveal novel insights into the structure and behavior of these manifolds, particularly concerning the implications of recurrence conditions on higher-order geometric invariants. Moreover, we establish several new theorems and propositions that enrich the existing body of knowledge in Finsler geometry. The results obtained in this study have potential applications in various areas of physics and engineering, including cosmology and robotics.

In this paper, we investigate the properties of fourth-order tensors in the context of tensor-generalized recurrent Finsler manifolds. We begin by introducing the concept of a tensor $P^i_{jkh}-$generalized recurrent Finsler manifold and establishing some fundamental results. We then proceed to derive the equations for the fourth-order curvature tensor and its covariant derivative. Finally, we apply our results to study the geometry of tensor-generalized recurrent Finsler manifolds with non-vanishing fourth-order curvature tensor.

Subject Classification: 53B40, 53C60 .
Keywords: Tensor P^i_{jkh}-generalized recurrent, Fourth-order curvature tensor, Geometry and Recurrent Finsler Manifolds..

1 Introduction and Preliminaries

Finsler geometry, as a generalization of Riemannian geometry, provides a powerful framework for studying the geometry of spaces with anisotropic metrics. In recent years, there has been a growing interest in investigating higher-order geometric properties of Finsler manifolds, motivated by their potential applications in physics and engineering. Recurrent Finsler manifolds, characterized by the parallel transport of curvature tensors along certain directions, form an important class of Finsler manifolds with rich geometric structures.

In this paper, we focus on P-generalized recurrent Finsler manifolds, which constitute a broader class of recurrent Finsler manifolds. Our objective is to conduct a thorough analysis of fourth-order geometric properties within this setting. By exploring the interplay between curvature and torsion tensors, we aim to uncover new geometric invariants and establish novel relationships between different geometric quantities.

Finsler manifolds are a generalization of Riemannian manifolds in which the distance between two points is not given by a Euclidean metric but by a more general function called a Finsler function. Tensor P^i_{jkh}-generalized recurrent Finsler manifolds are a special type of Finsler manifold in which a certain tensor field satisfies a particular equation. The study of tensor P^i_{jkh}-generalized recurrent Finsler manifolds has been an active area of research in recent years, and there is a rich body of literature on this topic.

Fourth-order curvature tensors are a generalization of the second-order curvature tensor that is used to study the geometry of higher-dimensional manifolds. The study of fourth-order curvature tensors in Finsler manifolds is a relatively new area of research, and there are still many open problems in this area.

Previous research on recurrent Finsler spaces,particularly their three- dimensional Riemannian counterparts, has been conducted by numerous scholars. Notable contributors include Rund [16], Mishra, Pande [23], Pandey, Saxena, Goswani [22], and Dikshit [24]. Several researchers, such as AL-Qashbari (see [1, 3, 4, 5, 6, 7, 8, 9]), have explored generalized curvature tensors within the framework of recurrent Finsler spaces, utilizing both Berwald and Cartan curvature tensors. Additionally, AL-Qashbari [2] investigated the properties of Weyl's curvature tensor in this context. Higher-order recurrent Finsler spaces, including birecurrent, trirecurrent, and n-dimensional recurrent spaces, have also been subjects of study (see [11, 13, 14, 15, 16, 17, 18, 19, 20]). Owing to the existence of multiple connections in Finsler spaces, the recurrence of various tensors has been a focus of research. Mishra and Pande [23], and Pandey [21] have contributed to this area, with Dikshit [10] specifically demonstrating the birecurrence of Cartan's third curvature tensor. Qasem [6] expanded on this by examining both generalized and special generalized birecurrence of the same tensor. Furthermore, Qasem and Saleem [12] studied the h-curvature tensor U^i_{jkh}.

A P^h-recurrent space is defined by the following equation:

$$P^i_{jkh|l} = \lambda_l P^i_{jkh} + \mu_l\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big), \qquad P^i_{jkh} \neq 0, \tag{1}$$

where P^i_{jkh} is a tensor, $|l$ denotes the covariant derivative, and λ_l is a non-zero covariant vector field referred to as the recurrence vector field.
Subsequently, we introduced the concept of P^h-birecurrent spaces, characterized by:

$$P^i_{jkh|l|m} = a_{lm} P^i_{jkh} + b_{lm}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big), \qquad P^i_{jkh} \neq 0. \tag{2}$$

Here, a_{lm} and b_{lm} are non-zero covariant tensor fields of second order, known as the birecurrence tensor fields.
The tensor g_{kh} and the associate tensor g^{kh} are covariant constant, i.e.

$$\Big\{ a)\ g_{kh|r} = 0 \qquad b)\ g^{kh}_{|r} = 0. \tag{3}$$

$$g_{kr} g^{rh} = \delta^h_k = \begin{cases} 1 & \text{if} \quad k = h, \\ 0 & \text{if} \quad k \neq h. \end{cases} \tag{4}$$

The covariant derivative of the vectors y^i and y_i, vanish identically, i.e.

$$\begin{cases} a)\ y^i_{|k} = 0 \\ b)\ y_{i|k} = 0. \end{cases} \tag{5}$$

$$\Big\{ a)\ y^i y_i = F^2 \qquad b)\ g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i. \tag{6}$$

The vectors y_i and δ^i_k are satisfy the following

$$\Big\{ a)\ \delta^i_k y^k = y^i \qquad b)\ \delta^i_k y_i = y_k. \tag{7}$$

$$\begin{cases} a)\ \delta^i_j g^{jk} = g^{ik} \qquad b)\ \delta^i_k \delta^k_h = \delta^i_h. \end{cases} \tag{8}$$

$$\begin{cases} a)\ \delta^i_k g_{ji} = g_{jk} \qquad b) g_{jh} y^j = y_h. \end{cases} \tag{9}$$

Using Euler's on homogeneous properties, this tensor satisfies the identities

$$\begin{cases} a)\ C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0 \\ b)\ C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0. \end{cases} \tag{10}$$

The hv-curvature tensor P^i_{jkh}, its associate curvature tensor P_{ijkh}, the $v(hv)$-torsion tensor P^i_{kh}, the P-Ricci tensor P_{jk}, tensor P^i_k and the scalar curvature P satisfy [11]

$$\begin{cases} a)\ P^i_{jkh} y^j = P^i_{kh} & b)\ g_{ir} P^r_{jkh} = P_{jikh} & c)\ P^i_{jkh} = P_{pjkh} g^{ip} \\ d)\ P^i_{jki} = P_{jk} & e)\ P^i_{ki} = P_k & f)\ P^i_{hk} y^h = P^i_k \\ g)\ P^i_{jkh} = \dot{\partial}_j P^i_{kh} & h)\ g_{ir} P^r_{kh} = P_{kih} & i)\ P_h y^h = P. \end{cases} \tag{11}$$

A brief introduction to the hv-curvature tensor P^i_{jkh}, also known as Cartan's second curvature tensor, is essential for understanding the intricate geometry of various mathematical and physical spaces. This tensor plays a crucial role in characterizing the curvature of a manifold equipped with a non-metric connection. By providing a measure of the failure of parallel transport along different curves, the hv-curvature tensor offers deep insights into the intrinsic properties of the underlying space.

2 Necessary and Sufficient Generalized Conditions for Identifying P^h-Recurrent

This paper investigates the necessary and sufficient conditions for identifying P-recurrent patterns in a generalized setting. By establishing these conditions, we aim to provide a rigorous framework for detecting and analyzing P-recurrent phenomena in various domains. Generalized four recurrent affinely connected spaces are a special type of manifold that has been studied by mathematicians for many years. These spaces have a number of interesting properties, and they are used in a variety of applications.

In this paper, we will discuss the necessary and sufficient conditions for a space to be a generalized four recurrent affinely connected space. We will also discuss some of the applications of these spaces.
Second Cartan's curvature tensor P^i_{jkh} satisfied the generalized recurrence condition

$$P^i_{jkh|l} = \lambda_l P^i_{jkh} + \mu_l \Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big), \qquad P^i_{jkh} \neq 0, \tag{12}$$

where $|l$ denotes is h-covariant derivative of order one with respect to x^l, and λ_l,μ_l are non-zero covariant vectors field and the space is called it a generalized P^h-recurrent space.
Also, curvature tensor P^i_{jkh} satisfied the generalized birecurrence condition

$$P^i_{jkh|l|m} = a_{lm} P^i_{jkh} + b_{lm} \Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big), \qquad P^i_{jkh} \neq 0. \tag{13}$$

where $|l|m$ is h-covariant derivative of order two with respect to x^l and x^m, successively, a_{lm} and b_{lm} are non-zero covariant vectors field and the space is called a generalized P^h-birecurrent space.
Differentiating (13) covariantly with respect to x^n and applying [(5)a] yields

$$P^i_{jkh|l|m|n} = c_{lmn} P^i_{jkh} + d_{lmn} \Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big). \tag{14}$$

where $|l|m|n$ is h-covariant derivative of order three with respect to x^l,x^m and x^n successfully, $c_{lmn} = a_{l|mn} + a_{lm}\lambda_n$ and $d_{lmn} = a_{lm}\mu_n + b_{l|m|n}$ are non-zero covariant tensors fields of order three.
Upon covariant differentiation of (14) with respect to x^s and using [(5)a], we obtain

$$P^i_{jkh|l|m|n|s} = c_{lmns} P^i_{jkh} + d_{lmns} \Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big). \tag{15}$$

where $|l|m|n|s$ denotes the fourth-order h-covariant derivative with respect to x^l,x^m,x^n and x^s respectively, $c_{lmns} = a_{l|mns} + a_{lms}\lambda_n + a_{lm}\lambda_{ns}$ and $d_{lmns} = a_{lms}\mu_{nm} + a_{lm}\mu_{nsm} + b_{l|m|n|s}$ are non-zero covariant tensors fields of rank four.

The space and the tensor satisfying (15) is called P^h-generalized four recurrent spaces. We shall denote them briefly by $P^h - G - FRF_n$.

Result 2.1 *Every generalized P^h-Fourecurrent space is generalized P^h-Trirecurrent space.*

Transvecting (15) by g_{ir}, using [(3)a], [(9)a] and [(11)b], we get

$$P_{jrkh|l|m|n|s} = c_{lmns} P_{jrkh} + d_{lmns}\Big(g_{kr}g_{jh} - g_{hr}g_{jk}\Big). \tag{16}$$

Conversely, transvecting (16) by g^{ir}, using [(3)a], (4) and [(11)c], yields condition (15).
Therefore, we can state the following theorem

Theorem 2.2 *In $P^h - G - FRF_n$,the h-covariant derivative of fourth order for the associate curvature tensor P_{ijkh} of the curvature tensor P^i_{jkh} is given by* (16).

Transvecting (15) by y^j, using [(5)a], [(11)a] and [(9)b], we get

$$P^i_{kh|l|m|n|s} = c_{lmns} P^i_{kh} + d_{lmns}\Big(\delta^i_k y_h - \delta^i_h y_k\Big). \tag{17}$$

Transvecting (17) by y^k, using [(5)a], [(11)f], [(7)a]and [(6)a], we get

$$P^i_{h|l|m|n|s} = c_{lmns} P^i_h + d_{lmns}\Big(y^i y_h - \delta^i_h F^2\Big). \tag{18}$$

Consequently, the following theorem holds

Theorem 2.3 *In $P^h - G - FRF_n$,the h-covariant derivative of fourth order for the $h(v)$-torsion tensor P^i_{kh} and the deviation tensor P^i_h given by* (17) *and* (18)*, respectively.*

Differentiating (17), with respect to y^j, using [(6)b] and [(11)g], we get

$$\dot\partial_j(P^i_{kh|l|m|n|s}) = (\dot\partial_j c_{lmns})P^i_{kh} + c_{lmns}P^i_{jkh} + (\dot\partial_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) \tag{19}$$
$$+d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big).$$

The hv-curvature tensor P^i_{jkh} (Cartan's second curvature tensor) is positively homogeneous of degree zero in the directional argument and is defined by

$$P^i_{jkh} = \dot\partial_h \Gamma^{*i}_{jk} + C^i_{jm}P^m_{kh} - C^i_{jh|k}.$$

Using the above formula for $(P^i_{kh|l|m|n})$ in (19), we get

$$\begin{aligned}&\dot{\partial}_j(P^i_{kh|l|m|n})_{|s} + P^r_{kh|l|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs}) - P^i_{rh|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks})\\&-P^i_{kr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs}) - P^i_{kh|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{ls}) - P^i_{kh|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms})\\&-P^i_{kh|l|m|r}(\dot{\partial}_j\Gamma^{*r}_{ns}) - \dot{\partial}_r(P^i_{kh|l|m|n})P^r_{js} = (\dot{\partial}_j c_{lmns})P^i_{kh}\\&+c_{lmns}P^i_{jkh} + (\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) + d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big).\end{aligned} \tag{20}$$

Also, applying the previous formula for $(P^i_{kh|l|m})$ in (20), we get

$$\begin{aligned}&\dot{\partial}_j(P^i_{kh|l|m|n})_{|n|s} + [P^r_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) - P^i_{rh|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn})\\&-P^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) - P^i_{kh|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - \dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s}\\&+P^i_{kh|l|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs}) - P^i_{rh|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) - P^i_{kr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs})\\&-P^i_{kh|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{ls}) - P^i_{kh|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) - P^i_{kh|l|m|r}(\dot{\partial}_j\Gamma^{*r}_{ns})\\&-[\dot{\partial}_r(P^i_{kh|l|m|n})_{ls} + P^q_{kh|l|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) - P^i_{qh|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks})\\&-P^i_{kq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs}) - P^i_{kh|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) - P^i_{kh|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms})\\&-P^i_{kh|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns}) - \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} = (\dot{\partial}_j c_{lmns})P^i_{kh}\\&+c_{lmns}P^i_{jkh} + (\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) + d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big).\end{aligned} \tag{21}$$

Again, applying the previous formula for $(P^i_{kh|l})$ in (21), we get

$$\begin{aligned}&(\dot{\partial}_j P^i_{kh|l})_{|m|n|s} + [P^r_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - P^i_{rh|l}(\dot{\partial}_j\Gamma^{*r}_{km})\\&-P^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - P^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \dot{\partial}_r(P^i_{kh|l})P^r_{jm}]_{|n|s}\\&+[P^i_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) - P^i_{rh|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - P^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn})\\&-P^i_{kh|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - \dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s} - P^r_{kh|l|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs})\\&-P^i_{rh|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) - P^i_{kr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs}) - P^i_{kh|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{ls})\\&-P^i_{kh|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) - P^i_{kh|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ns}) - [\dot{\partial}_r(P^i_{kh|l|m|n})_{ls}\\&+P^q_{kh|l|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) - P^q_{qh|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks}) - P^i_{kq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs})\\&-P^i_{kh|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) - P^i_{kh|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) - P^i_{kh|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns})\\&-\dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} = (\dot{\partial}_j c_{lmns})P^i_{kh} + c_{lmns}P^i_{jkh}\\&+(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) + d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big).\end{aligned}$$

Further, applying the previous formula for (P^i_{kh}) in above equation, we get

$$\begin{aligned}
&(\dot\partial_j P^i_{kh})_{|l|m|n|s} + [P^r_{kh}(\dot\partial_j \Gamma^{*i}_{rl}) - P^i_{rh}(\dot\partial_j \Gamma^{*r}_{kl}) \\
&-P^i_{kr}(\dot\partial_j \Gamma^{*r}_{hl}) - \dot\partial_r(P^i_{kh})P^r_{jl}]_{|m|n|s} + [P^i_{kh|l}(\dot\partial_j \Gamma^{*i}_{rm}) - P^i_{rh|l}(\dot\partial_j \Gamma^{*r}_{km}) \\
&-P^i_{kr|l}(\dot\partial_j \Gamma^{*r}_{hm}) - P^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{lm}) - \dot\partial_r(P^i_{kh|l})P^r_{jm}]_{|n|s} + [P^i_{kh|l|m}(\dot\partial_j \Gamma^{*i}_{rn}) \\
&-P^i_{rh|l|m}(\dot\partial_j \Gamma^{*r}_{kn}) - P^i_{kr|l|m}(\dot\partial_j \Gamma^{*r}_{hn}) - P^i_{kh|r|m}(\dot\partial_j \Gamma^{*r}_{ln}) - \dot\partial_r(P^i_{kh|l|m})P^r_{jn}]_{|s} \\
&+P^r_{kh|l|m|n}(\dot\partial_j \Gamma^{*i}_{rs}) - P^i_{rh|l|m|n}(\dot\partial_j \Gamma^{*r}_{ks}) - P^i_{kr|l|m|n}(\dot\partial_j \Gamma^{*r}_{hs}) - P^i_{kh|r|m|n}(\dot\partial_j \Gamma^{*r}_{ls}) \\
&-P^i_{kh|l|r|n}(\dot\partial_j \Gamma^{*r}_{ms}) - P^i_{kh|l|m|n}(\dot\partial_j \Gamma^{*r}_{ns}) - [\dot\partial_r(P^i_{kh|l|m|n})_{ls} + P^q_{kh|l|m|n}(\dot\partial_r \Gamma^{*i}_{qs}) \\
&-P^q_{qh|l|m|n}(\dot\partial_r \Gamma^{*q}_{ks}) - P^i_{kq|l|m|n}(\dot\partial_r \Gamma^{*q}_{hs}) - P^i_{kh|q|m|n}(\dot\partial_r \Gamma^{*q}_{ls}) - P^i_{kh|l|q|n}(\dot\partial_r \Gamma^{*q}_{ms}) \\
&-P^i_{kh|l|m|q}(\dot\partial_r \Gamma^{*q}_{ns}) - \dot\partial_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} = (\dot\partial_j c_{lmns})P^i_{kh} + c_{lmns}P^i_{jkh} \\
&+(\dot\partial_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) + d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big).
\end{aligned} \tag{22}$$

Using [(11)g] in (22), we get

$$\begin{aligned}
&P^i_{jkh|l|m|n|s} + [P^r_{kh}(\dot\partial_j \Gamma^{*i}_{rl}) - P^i_{rh}(\dot\partial_j \Gamma^{*r}_{kl}) \\
&-P^i_{kr}(\dot\partial_j \Gamma^{*r}_{hl}) - \dot\partial_r(P^i_{kh})P^r_{jl}]_{|m|n|s} + [P^i_{kh|l}(\dot\partial_j \Gamma^{*i}_{rm}) - P^i_{rh|l}(\dot\partial_j \Gamma^{*r}_{km}) \\
&-P^i_{kr|l}(\dot\partial_j \Gamma^{*r}_{hm}) - P^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{lm}) - \dot\partial_r(P^i_{kh|l})P^r_{jm}]_{|n|s} + [P^i_{kh|l|m}(\dot\partial_j \Gamma^{*i}_{rn}) \\
&-P^i_{rh|l|m}(\dot\partial_j \Gamma^{*r}_{kn}) - P^i_{kr|l|m}(\dot\partial_j \Gamma^{*r}_{hn}) - P^i_{kh|r|m}(\dot\partial_j \Gamma^{*r}_{ln}) - \dot\partial_r(P^i_{kh|l|m})P^r_{jn}]_{|s} \\
&+P^r_{kh|l|m|n}(\dot\partial_j \Gamma^{*i}_{rs}) - P^i_{rh|l|m|n}(\dot\partial_j \Gamma^{*r}_{ks}) - P^i_{kr|l|m|n}(\dot\partial_j \Gamma^{*r}_{hs}) - P^i_{kh|r|m|n}(\dot\partial_j \Gamma^{*r}_{ls}) \\
&-P^i_{kh|l|r|n}(\dot\partial_j \Gamma^{*r}_{ms}) - P^i_{kh|l|m|n}(\dot\partial_j \Gamma^{*r}_{ns}) - [\dot\partial_r(P^i_{kh|l|m|n})_{ls} + P^q_{kh|l|m|n}(\dot\partial_r \Gamma^{*i}_{qs}) \\
&-P^q_{qh|l|m|n}(\dot\partial_r \Gamma^{*q}_{ks}) - P^i_{kq|l|m|n}(\dot\partial_r \Gamma^{*q}_{hs}) - P^i_{kh|q|m|n}(\dot\partial_r \Gamma^{*q}_{ls}) - P^i_{kh|l|q|n}(\dot\partial_r \Gamma^{*q}_{ms}) \\
&-P^i_{kh|l|m|q}(\dot\partial_r \Gamma^{*q}_{ns}) - \dot\partial_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} = (\dot\partial_j c_{lmns})P^i_{kh} + c_{lmns}P^i_{jkh} \\
&+(\dot\partial_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) + d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big).
\end{aligned} \tag{23}$$

This shows that

$$P^i_{jkh|l|m|n|s} = c_{lmns}P^i_{jkh} + d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big). \tag{24}$$

If and only if

$$
\begin{aligned}
&[P^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - P^i_{rh}(\dot{\partial}_j\Gamma^{*r}_{kl}) - P^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{hl}) - \dot{\partial}_r(P^i_{kh})P^r_{jl}]_{|m|n|s} \\
&+[P^i_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - P^i_{rh|l}(\dot{\partial}_j\Gamma^{*r}_{km}) - P^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - P^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) \\
&-\dot{\partial}_r(P^i_{kh|l})P^r_{jm}]_{|n|s} + [P^i_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) - P^i_{rh|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - P^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) \\
&-P^i_{kh|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - \dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s} + P^r_{kh|l|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs}) - P^i_{rh|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) \\
&-P^i_{kr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs}) - P^i_{kh|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{ls}) - P^i_{kh|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) - P^i_{kh|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ns}) \\
&-[\dot{\partial}_r(P^i_{kh|l|m|n})_{ls} + P^q_{kh|l|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) - P^q_{qh|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks}) - P^i_{kq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs}) \\
&-P^i_{kh|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) - P^i_{kh|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) - P^i_{kh|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns}) \\
&-\dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} - (\dot{\partial}_j c_{lmns})P^i_{kh} - (\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) = 0.
\end{aligned} \tag{25}
$$

Hence, we have the following theorem

Theorem 2.4 *In $P^h - G - FRF_n$, at the differentiation of the tensor P^i_{jkh} is a generalized four-recurrent if and only if the condition* (25) *holds good.*

Transvecting (23) by g_{ip} , using [(3)a],[(11)b],[(11)h] and [(9)a], we get

$$
\begin{aligned}
&P_{jpkh|l|m|n|s} + [g_{ip}P^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - P_{rph}(\dot{\partial}_j\Gamma^{*r}_{kl}) \\
&-P_{kpr}(\dot{\partial}_j\Gamma^{*r}_{hl}) - g_{ip}\dot{\partial}_r(P^i_{kh})P^r_{jl}]_{|m|n|s} + [g_{ip}P^i_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - P_{rph|l}(\dot{\partial}_j\Gamma^{*r}_{km}) \\
&-P_{kpr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - P_{kph|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - g_{ip}\dot{\partial}_r(P^i_{kh|l})P^r_{jm}]_{|n|s} + [g_{ip}P^i_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) \\
&-P_{rph|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - P_{kpr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) - P_{kph|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - g_{ip}\dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s} \\
&+g_{ip}P^r_{kh|l|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs}) - P_{rphm|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) - P_{kpr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs}) - P_{kph|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{ls}) \\
&-P_{kph|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) - P_{kph|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ns}) - [g_{ip}\dot{\partial}_r(P^i_{kh|l|m|n})_{ls} + g_{ip}P^q_{kh|l|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) \\
&-P_{qph|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks}) - P_{kpq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs}) - P_{kph|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) - P_{kph|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) \\
&-P_{kph|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns}) - g_{ip}\dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} = (\dot{\partial}_j c_{lmns})P_{kph} + c_{lmns}P_{jpkh} \\
&+g_{ip}(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) + d_{lmns}\Big(g_{kp}g_{jh} - g_{hp}g_{jk}\Big).
\end{aligned} \tag{26}
$$

This shows that

$$
P_{jpkh|l|m|n|s} = c_{lmns}P_{jpkh} + d_{lmns}\Big(g_{kp}g_{jh} - g_{hp}g_{jk}\Big). \tag{27}
$$

If and only if

$$
\begin{aligned}
&[g_{ip}P^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - P_{rph}(\dot{\partial}_j\Gamma^{*r}_{kl}) \\
&-P_{kpr}(\dot{\partial}_j\Gamma^{*r}_{hl}) - g_{ip}\dot{\partial}_r(P^i_{kh})P^r_{jl}]_{|m|n|s} + [g_{ip}P^i_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - P_{rph|l}(\dot{\partial}_j\Gamma^{*r}_{km}) \\
&-P_{kpr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - P_{kph|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - g_{ip}\dot{\partial}_r(P^i_{kh|l})P^r_{jm}]_{|n|s} + [g_{ip}P^i_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) \\
&-P_{rph|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - P_{kpr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) - P_{kph|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - g_{ip}\dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s} \\
&+g_{ip}P^r_{kh|l|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs}) - P_{rphm|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) - P_{kpr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{hs}) - P_{kph|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{ls}) \\
&-P_{kph|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) - P_{kph|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ns}) - [g_{ip}\dot{\partial}_r(P^i_{kh|l|m|n})_{ls} + g_{ip}P^q_{kh|l|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) \\
&-P_{qph|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks}) - P_{kpq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{hs}) - P_{kph|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) - P_{kph|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) \\
&-P_{kph|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns}) - g_{ip}\dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} - (\dot{\partial}_j c_{lmns})P_{kph} \\
&-g_{ip}(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) = 0.
\end{aligned} \tag{28}
$$

Accordingly, the following theorem is established

Theorem 2.5 *In $P^h - G - FRF_n$, at the differentiation of the associative tensor p_{jpkh} of tensor P^i_{jkh} is a generalized four-recurrent if and only if the condition* (28) *holds good.*

Contracting the indices i and h in (23), using [(11)d],[(11)e], [(7)b] and (4), we get

$$
\begin{aligned}
&P_{jk|l|m|n|s} + [P^r_{ki}(\dot{\partial}_j\Gamma^{*i}_{rl}) - P_r(\dot{\partial}_j\Gamma^{*r}_{kl}) \\
&-P^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{il}) - \dot{\partial}_r(P_k)P^r_{jl}]_{|m|n|s} + [P^r_{ki|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - P_{r|l}(\dot{\partial}_j\Gamma^{*r}_{km}) \\
&-P^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{im}) - P_{k|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \dot{\partial}_r(P_{k|l})P^r_{jm}]_{|n|s} + [P^r_{ki|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) \\
&-P_{r|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - P^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{in}) - P_{k|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - \dot{\partial}_r(P_{k|l|m})P^r_{jn}]_{|s} \\
&+P^r_{ki|l|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs}) - P_{r|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) - P^i_{kr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{is}) - P_{k|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{ls}) \\
&-P^i_{k|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) - P_{k|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ns}) - [\dot{\partial}_r(P_{k|l|m|n})_{ls} + P^q_{ki|l|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) \\
&-P_{q|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks}) - P^i_{kq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{is}) - P_{k|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) - P_{k|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) \\
&-P_{k|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns}) - \dot{\partial}_q(P_{k|l|m})P^q_{rn}]P^r_{js} = (\dot{\partial}_j c_{lmns})P_k + c_{lmns}P_{jk} \\
&+(\dot{\partial}_j d_{lmns})\Big(y_k - y_h\Big) + (1-n)d_{lmns}g_{jk}.
\end{aligned} \tag{29}
$$

This shows that

$$
P_{jk|l|m|n|s} = c_{lmns}P_{jk} + (1-n)d_{lmns}g_{jk}. \tag{30}
$$

If and only if

$$
\begin{aligned}
&[P^r_{ki}(\dot{\partial}_j\Gamma^{*i}_{rl}) - P_r(\dot{\partial}_j\Gamma^{*r}_{kl}) \\
&-P^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{il}) - \dot{\partial}_r(P_k)P^r_{jl}]_{|m|n|s} + [P^r_{ki|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - P_{r|l}(\dot{\partial}_j\Gamma^{*r}_{km}) \\
&-P^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{im}) - P_{k|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \dot{\partial}_r(P_{k|l})P^r_{jm}]_{|n|s} + [P^r_{ki|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) \\
&-P_{r|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - P^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{in}) - P_{k|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - \dot{\partial}_r(P_{k|l|m})P^r_{jn}]_{|s} \\
&+P^r_{ki|l|m|n}(\dot{\partial}_j\Gamma^{*i}_{rs}) - P_{r|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ks}) - P^i_{kr|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{is}) - P_{k|r|m|n}(\dot{\partial}_j\Gamma^{*r}_{ls}) \\
&-P^i_{k|l|r|n}(\dot{\partial}_j\Gamma^{*r}_{ms}) - P_{k|l|m|n}(\dot{\partial}_j\Gamma^{*r}_{ns}) - [\dot{\partial}_r(P_{k|l|m|n})_{ls} + P^q_{ki|l|m|n}(\dot{\partial}_r\Gamma^{*i}_{qs}) \\
&-P_{q|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{ks}) - P^i_{kq|l|m|n}(\dot{\partial}_r\Gamma^{*q}_{is}) - P_{k|q|m|n}(\dot{\partial}_r\Gamma^{*q}_{ls}) - P_{k|l|q|n}(\dot{\partial}_r\Gamma^{*q}_{ms}) \\
&-P_{k|l|m|q}(\dot{\partial}_r\Gamma^{*q}_{ns}) - \dot{\partial}_q(P_{k|l|m})P^q_{rn}]P^r_{js} - (\dot{\partial}_j c_{lmns})P_k - (\dot{\partial}_j d_{lmns})\Big(y_k - y_h\Big) = 0.
\end{aligned}
\tag{31}
$$

Condition (30) demonstrates that the P-Ricci tensor P_{jk} cannot be zero, as its vanishing would imply $d_{lmns} = 0$, condition (31) if and only if it holds, a contradiction.
Therefore, we can state the following theorem

Theorem 2.6 *In $P^h - G - FRF_n$,at the differentiation of the P-Ricci tensor P_{jk}can't vanish if and only if the condition* (31) *holds good.*

3 On Generalized P^h- Four-Recurrent Affinely Connected Space

The study of recurrent spaces has been a cornerstone in differential geometry. In this paper, we introduce a novel generalization, focusing on four-recurrent-affinely connected spaces. Our research aims to contribute to the ongoing exploration of higher-order recurrent structures and their potential applications in physics and engineering.
In this section, we shall introduce new definition for $P^h - G - FRF_n$, whose also be in possession the properties of an affinely connected space.

Definition 3.1 *Finsler space F_n, whose coefficient of parameter, G^i_{jk} is independent of y^i is called affinely connected space and equivalent the equations*

$$\Big\{ a)\ G^i_{jkh} = 0 \qquad b) C_{ijk|h} = 0. \tag{32}$$

The coefficients parameters Γ^{*i}_{kh} and G^i_{kh} are independent of directional argument [16], i.e.

$$\begin{cases} a)\ \dot{\partial}_j G^i_{kh} = 0 \qquad b)\ \dot{\partial}_j(\Gamma^{*i}_{kh}) = 0. \end{cases} \tag{33}$$

Definition 3.2 *The generalized P^h-four-recurrent space which possess the properties of an affinely connected space satisfies any one of the equations [(32)a], [(32)b],[(33)a] and [(33)b], we called a generalized P^h-four-recurrent affinely connected space and denoted by $P^h - G - FRF_n$-affinely connected space.*

Result 3.3 *It will be sufficient to call Cartan's 2^{th} curvature tensor P^i_{jkh} which possess the property of $P^h - G - FRF_n$-affinely connected space as generalized h-four-recurrent tensor.*

Let us consider $P^h - G - FRF_n$- affinely connected space.
In view of the theorem 2.1 and definition 3.2., we may conclude

Theorem 3.4 *In generalized P^h-recurrent a ffinely connected space, the generalized P^h- birecurrent affinely connected is $P^h - G - FRF_n$-affinely connected space.*

Using [(33)b]in (23), we get

$$\begin{aligned} &P^i_{jkh|l|m|n|s} - [\dot{\partial}_r(P^i_{kh})P^r_{jl}]_{|m|n|s} - [\dot{\partial}_r(P^i_{kh|l})P^r_{jm}]_{|n|s} - [\dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s} \\ &-[(\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} - \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} = (\dot{\partial}_j c_{lmns})P^i_{kh} + c_{lmns}P^i_{jkh} \\ &+(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) + d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big). \end{aligned} \tag{34}$$

This shows that

$$P^i_{jkh|l|m|n|s} = c_{lmns}P^i_{jkh} + d_{lmns}\Big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\Big). \tag{35}$$

If and only if

$$\begin{aligned} &[\dot{\partial}_r(P^i_{kh})P^r_{jl}]_{|m|n|s} + [\dot{\partial}_r(P^i_{kh|l})P^r_{jm}]_{|n|s} + [\dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s} \\ &+[(\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} + \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} + (\dot{\partial}_j c_{lmns})P^i_{kh} \\ &+(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) = 0. \end{aligned} \tag{36}$$

Further, using [(33)b] in (26), we get

$$\begin{aligned} &P_{jpkh|l|m|n|s} - [g_{ip}\dot{\partial}_r(P^i_{kh})P^r_{jl}]_{|m|n|s} - [g_{ip}\dot{\partial}_r(P^i_{kh|l})P^r_{jm}]_{|n|s} \\ &-[g_{ip}\dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s} - [(g_{ip}\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} - g_{ip}\dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} \\ &= (\dot{\partial}_j c_{lmns})P_{kph} + c_{lmns}P_{jpkh} + g_{ip}(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) + d_{lmns}\Big(g_{kp}g_{jh} - g_{hp}g_{jk}\Big). \end{aligned} \tag{37}$$

This shows that

$$P_{jpkh|l|m|n|s} = c_{lmns}P_{jpkh} + d_{lmns}\Big(g_{kp}g_{jh} - g_{hp}g_{jk}\Big). \tag{38}$$

If and only if

$$\begin{aligned} &[g_{ip}\dot{\partial}_r(P^i_{kh})P^r_{jl}]_{|m|n|s} + [g_{ip}\dot{\partial}_r(P^i_{kh|l})P^r_{jm}]_{|n|s} \\ &+[g_{ip}\dot{\partial}_r(P^i_{kh|l|m})P^r_{jn}]_{|s} + [(g_{ip}\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} + g_{ip}\dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_{js} \\ &+(\dot{\partial}_j c_{lmns})P_{kph} + g_{ip}(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big) = 0. \end{aligned} \tag{39}$$

Therefore, we can state the following theorem

Theorem 3.5 *In $P^h - G - FRF_n$-affinely connected space, P^i_{kjh} and its associative P_{jpkh} curvature tensor are generalized trirecurrent tensor if and only if the conditions* (35) *and* (38)*, respectively hold good.*

Transvecting (34) by y^j , using [(5)a], [(11)a], [(11)f] and [(9)b], we get

$$\begin{aligned} &P^i_{kh|l|m|n|s} - [\dot{\partial}_r(P^i_{kh})P^r_l]_{|m|n|s} - [\dot{\partial}_r(P^i_{kh|l})P^r_m]_{|n|s} - [\dot{\partial}_r(P^i_{kh|l|m})P^r_n]_{|s} \\ &-[(\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} - \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_s = (\dot{\partial}_j c_{lmns})P^i_{kh}y^j + c_{lmns}P^i_{kh} \\ &+(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big)y^j + d_{lmns}\Big(\delta^i_k y_h - \delta^i_h y_k\Big). \end{aligned} \tag{40}$$

This shows that

$$P^i_{kh|l|m|n|s} = c_{lmns}P^i_{kh} + d_{lmns}\Big(\delta^i_k y_h - \delta^i_h y_k\Big). \tag{41}$$

If and only if

$$\begin{aligned} &[\dot{\partial}_r(P^i_{kh})P^r_l]_{|m|n|s} + [\dot{\partial}_r(P^i_{kh|l})P^r_m]_{|n|s} + \dot{\partial}_r(P^i_{kh|l|m})P^r_n]_{|s} \\ &+[(\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} + \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_s + (\dot{\partial}_j c_{lmns})P^i_{kh}y^j \\ &+(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big)y^j = 0. \end{aligned} \tag{42}$$

Transvecting (40) by g_{ir}, using [(3)a], [(11)h] and [(9)a], we get

$$P_{krh|l|m|n|s} - g_{ir}[\dot{\partial}_r(P^i_{kh})P^r_l]_{|m|n|s} - g_{ir}[\dot{\partial}_r(P^i_{kh|l})P^r_m]_{|n|s} - [g_{ir}\dot{\partial}_r(P^i_{kh|l|m})P^r_n]_{|s} \quad (43)$$
$$-[g_{ir}(\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} - \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_s = (\dot{\partial}_j c_{lmns})P_{krh}y^j + c_{lmns}P_{krh}$$
$$+g_{ir}(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big)y^j + d_{lmns}\Big(g_{kr}y_h - g_{hr}y_k\Big).$$

This shows that

$$P_{krh|l|m|n|s} = c_{lmns}P_{krh} + d_{lmns}\Big(g_{kr}y_h - g_{hr}y_k\Big). \quad (44)$$

If and only if

$$g_{ir}[\dot{\partial}_r(P^i_{kh})P^r_l]_{|m|n|s} + g_{ir}[\dot{\partial}_r(P^i_{kh|l})P^r_m]_{|n|s} + [g_{ir}\dot{\partial}_r(P^i_{kh|l|m})P^r_n]_{|s} \quad (45)$$
$$+[g_{ir}(\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} - \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_s + (\dot{\partial}_j c_{lmns})P_{krh}y^j$$
$$+g_{ir}(\dot{\partial}_j d_{lmns})\Big(\delta^i_k y_h - \delta^i_h y_k\Big)y^j = 0.$$

Transvecting (40) by y^k , using [(5)a], [(11)f], [(7)a] and [(6)a], we get

$$P^i_{h|l|m|n|s} - y^k[\dot{\partial}_r(P^i_{kh})P^r_l]_{|m|n|s} - y^k[\dot{\partial}_r(P^i_{kh|l})P^r_m]_{|n|s} - y^k[\dot{\partial}_r(P^i_{kh|l|m})P^r_n]_{|s} (46)$$
$$-y^k[(\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} - \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_s = (\dot{\partial}_j c_{lmns})P^i_h y^j + c_{lmns}P^i_h$$
$$+(\dot{\partial}_j d_{lmns})\Big(y^k y_h - \delta^i_h F^2\Big)y^j + d_{lmns}\Big(y^k y_h - \delta^i_h F^2\Big).$$

This shows that

$$P^i_{h|l|m|n|s} = c_{lmns}P^i_h + d_{lmns}\Big(y^k y_h - \delta^i_h F^2\Big). \quad (47)$$

If and only if

$$y^k[\dot{\partial}_r(P^i_{kh})P^r_l]_{|m|n|s} + y^k[\dot{\partial}_r(P^i_{kh|l})P^r_m]_{|n|s} + y^k[\dot{\partial}_r(P^i_{kh|l|m})P^r_n]_{|s} \quad (48)$$
$$+y^k[(\dot{\partial}_r(P^i_{kh|l|m|n}))_{|s} - \dot{\partial}_q(P^i_{kh|l|m})P^q_{rn}]P^r_s + (\dot{\partial}_j c_{lmns})P^i_h y^j$$
$$+(\dot{\partial}_j d_{lmns})\Big(y^k y_h - \delta^i_h F^2\Big)y^j = 0.$$

Consequently, the following theorem can be concluded

Theorem 3.6 *In $P^h\!-\!G\!-\!FRF_n$-affinely connected space, the h-covariant derivative of fourth order for the torsion tensor P^i_{kh} , its associative tensor P_{krh} and the deviation tensor P^i_h given by*(41), (44) *and* (47)*if and only if the conditions* (42),(45) *and* (48), *respectively hold.*

Contracting the indices i and h in (34), using [(11)d], [(11)e], [(7)b] and (4), we get

$$\begin{aligned} &P_{jk|l|m|n|s} - [\dot{\partial}_r(P_k)P^r_{jl}]_{|m|n|s} - [\dot{\partial}_r(P_{k|l})P^r_{jm}]_{|n|s} - [\dot{\partial}_r(P_{k|l|m})P^r_{jn}]_{|s} \\ &-[(\dot{\partial}_r(P_{k|l|m|n}))_{|s} - \dot{\partial}_q(P_{k|l|m})P^q_{rn}]P^r_{js} = (\dot{\partial}_j c_{lmns})P_k + c_{lmns}P_{jk} \\ &+(1-n)(\dot{\partial}_j d_{lmns})y_k + (1-n)d_{lmns}g_{jk}. \end{aligned} \tag{49}$$

This shows that

$$P_{jk|l|m|n|s} = c_{lmns}P_{jk} + (1-n)d_{lmns}g_{jk}. \tag{50}$$

If and only if

$$\begin{aligned} &[\dot{\partial}_r(P_k)P^r_{jl}]_{|m|n|s} + [\dot{\partial}_r(P_{k|l})P^r_{jm}]_{|n|s} + [\dot{\partial}_r(P_{k|l|m})P^r_{jn}]_{|s} \\ &+[(\dot{\partial}_r(P_{k|l|m|n}))_{|s} - \dot{\partial}_q(P_{k|l|m})P^q_{rn}]P^r_{js} + (\dot{\partial}_j c_{lmns})P_k \\ &+(1-n)(\dot{\partial}_j d_{lmns})y_k = 0. \end{aligned} \tag{51}$$

Contracting the indices i and h in (40), using [(11)e], [(7)b] and (4), we get

$$\begin{aligned} &P_{k|l|m|n|s} - [\dot{\partial}_r(P_k)P^r_{l}]_{|m|n|s} - [\dot{\partial}_r(P_{k|l})P^r_{m}]_{|n|s} - [\dot{\partial}_r(P_{k|l|m})P^r_{n}]_{|s} \\ &-[(\dot{\partial}_r(P_{k|l|m|n}))_{|s} - \dot{\partial}_q(P_{k|l|m})P^q_{rn}]P^r_{s} = (\dot{\partial}_j c_{lmns})P_k y^j + c_{lmns}P_k \\ &+(1-n)(\dot{\partial}_j d_{lmns})y_k y^j + (1-n)d_{lmns}y_k. \end{aligned} \tag{52}$$

This shows that

$$P_{k|l|m|n|s} = c_{lmns}P_k + (1-n)d_{lmns}y_k. \tag{53}$$

If and only if

$$\begin{aligned} &[\dot{\partial}_r(P_k)P^r_{l}]_{|m|n|s} + [\dot{\partial}_r(P_{k|l})P^r_{m}]_{|n|s} + [\dot{\partial}_r(P_{k|l|m})P^r_{n}]_{|s} \\ &+[(\dot{\partial}_r(P_{k|l|m|n}))_{|s} - \dot{\partial}_q(P_{k|l|m})P^q_{rn}]P^r_{s} + (\dot{\partial}_j c_{lmns})P_k y^j \\ &+(1-n)(\dot{\partial}_j d_{lmns})y_k y^j = 0. \end{aligned} \tag{54}$$

Transvecting (52) by y^k , using [(5)a], [(11)i] and [(6)a], we get

$$\begin{aligned} &P_{|l|m|n|s} - [\dot{\partial}_r(P)P^r_{l}]_{|m|n|s} - [\dot{\partial}_r(P_{|l})P^r_{m}]_{|n|s} - [\dot{\partial}_r(P_{|l|m})P^r_{n}]_{|s} \\ &-[(\dot{\partial}_r(P_{|l|m|n}))_{|s} - \dot{\partial}_q(P_{|l|m})P^q_{rn}]P^r_{s} = (\dot{\partial}_j c_{lmns})P y^j + c_{lmns}P \\ &+(1-n)(\dot{\partial}_j d_{lmns})F^2 y^j + (1-n)d_{lmns}F^2. \end{aligned} \tag{55}$$

This shows that

$$P_{|l|m|n|s} = c_{lmns}P + (1-n)d_{lmns}F^2. \quad (56)$$

If and only if

$$[\dot{\partial}_r(P)P^r_l]_{|m|n|s} + [\dot{\partial}_r(P_{|l})P^r_m]_{|n|s} + [\dot{\partial}_r(P_{|l|m})P^r_n]_{|s} \quad (57)$$
$$+[(\dot{\partial}_r(P_{|l|m|n}))_{|s} - \dot{\partial}_q(P_{|l|m})P^q_{rn}]P^r_s + (\dot{\partial}_j c_{lmns})Py^j$$
$$+(1-n)(\dot{\partial}_j d_{lmns})F^2y^j = 0.$$

The equations (50), (53) and (56) show that Ricci tensor P_{jk}, curvature vector P_k and scalar curvature P, can't equal to zero, because the vanishing of any one of them would imply d_{lmns}=0 , if and only if (51), (54) and (57), respectively, hold, a contradiction.
Hence, the subsequent theorem is as follows

Theorem 3.7 *In $P^h - G - FRF_n$-affinely connected space, the Ricci tensorP_{jk}, curvature vector P_k and scalar curvature P, are non-vanishing if and only if the conditions* (51)*,* (54) *and* (57)*,respectively hold.*

4 Conclusion

The present study has provided a detailed analysis of fourth-order geometric properties in P-generalized recurrent Finsler manifolds. Our findings have revealed several significant results, including:
The P^h-generalized four-recurrent space is satisfies (15). InP^h-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of fourth order are vanish, then tensor P^i_{jkh} is generalized four recurrent in P^h-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of fourth order are vanish, then torsion tensor P^i_{kh} , deviation tensor P^i_k, curvature vector P_k, curvature scalar P and tensor P_{kph} are all generalized four rrecurrent. In $P^h - G - FRF_n$-affinely connected space, Ricci tensor P_{jk} in sense of Berwald coincide with Ricci tensor P_{jk} of Cartan's second curvature. In $P^h - G - FRF_n$- affinely connected space the associate tensor P_{jpkh} of Berwald tensor coincide with the associate tensor P_{jpkh} Cartan's second curvature tensor.
These results contribute to a deeper understanding of the geometric structure of P-generalized recurrent Finsler manifolds and provide valuable insights into the interplay between curvature and torsion.

5 Recommendations

Based on the results obtained in this study, we recommend the following directions for future research:
The authors we call the need for research and study in generalized p^h- recurrent Finsler spaces and higher order and $P^h - G - FRF_n$-affinely connected space interlard it with the properties of special spaces for Finsler space.
By pursuing these research avenues, we can further advance our understanding of Finsler geometry and its applications.

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