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ARITHMETIC PROPERTIES OF OPOND PARTITIONS

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Abstract. In 2023, Ballantine and Welch have considered various generalizations of POD and PED, wherein odd parts distinct (in case of POD) and even parts distinct (in case of PED) respectively. In the process, they led to two new classes of POND and PEND partitions which are integer partitions with odd parts can not be distinct (in case of POND) and even parts can not be distinct (in case of PEND) respectively. In this paper, we obtain some infinite families of congruences for a new class of patrition function called as $[j,k]-$ pond overpartition or $OPOND$ partition, an integer partition of n wherein odd parts not distinct (even parts unrestricted) and first occurrence of each part congruent to j modulo k may be overlined.

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1. Introduction

Throughout this paper, we let $|q| < 1$. We use the standard notation

$$f_k := (q^k; q^k)_\infty.$$

Following Ramanujan, we define

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{1}$$

$$f^3(-q) = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n(n+1)}{2}}, \tag{2}$$

which are special cases of Ramanujan's general theta function [1], [4]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{3}$$

In Ramanujan's notation, Jacobi's famous triple product identity becomes,

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \tag{4}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $P(n)$ denotes the number of partitions of n

with $P(0) = 1$ and the generating function for $P(n)$ is

$$\sum_{n=0}^{\infty} P(n) = \frac{1}{(q,q)_\infty}.$$

In 2004, Corteel and Lovejoy [5] introduced the notion of an overpartitions, which are partition of a positive integer n in which first occurrence of each part may be overlined. The generating function is

$$\sum_{n=0}^{\infty} \overline{P}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{5}$$

For example, there are 8 partitions for $\overline{P}(3)$, namely

$3,\quad \overline{3},\quad 2{+}1,\quad \overline{2}{+}1,\quad 2{+}\overline{1},\quad \overline{2}{+}\overline{1},\quad 1{+}1{+}1,\quad \overline{1}{+}1{+}1$

In 2022, Mahadeva Naika, Harishkumar and Veeranayaka [10],[11] introduced notion of $[j,k]$- overpartition of a positive integer n is a non-increasing sequence of positive integers whose sum is n in which first occurrence of a part congruent to j modulo k may be overlined. Let $\overline{P}_{j,k}(n)$ denote the number of such partitions of n with $\overline{P}_{j,k}(0) = 1$. The generating function is given by

$$\sum_{n=0}^{\infty} \overline{P}_{j,k}(n)q^n = \frac{(-q^j;q^k)_\infty}{(q,q)_\infty}. \tag{6}$$

For example, there are 5 partitions for $\overline{P}_{1,3}(3)$, namely

$3,\quad 2{+}1,\quad 2{+}\overline{1},\quad 1{+}1{+}1,\quad \overline{1}{+}1{+}1$

In [2], Ballantine and Welch have considered various generalizations and refinements of POD and PED partitions. In the process they led to two new classes of POND and PEND partitions which are integer partitions with odd parts can not be distinct (in case of POND) and even parts can not be distinct (in case of PEND) respectively.

Motivated by the above works, in this paper we introduce and obtain some infinite families of congruences for a new class of partition function called as $[j,k]-$ pond overpartition or $OPOND$ partitions which enumerates the number of partition of a positive integer n in which the first occurrence of each part congruent to j modulo k may be overlined and odd parts are not distinct (even parts are unrestricted). Let $opond_{j,k}(n)$ denotes the number of such partition of a positive integer n and the generating function is given by

$$\sum_{n=0}^{\infty} opond_{j,k}(n)q^n = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}}(-q^j;q^k)_\infty. \tag{7}$$

For example. There are 10 partitions for $opoond_{3,3}(6)$, namely

$6,\quad \overline{6},\quad 4{+}2,\quad 4{+}1{+}1,\quad 3{+}3,\quad \overline{3}{+}3,\quad 2{+}2{+}2,\quad 2{+}2{+}1{+}1,$
$2{+}1{+}1{+}1{+}1,\quad 1{+}1{+}1{+}1{+}1{+}1.$

2. Preliminary results

Lemma 2.1. *The following* 2*-dissections hold*

$$\frac{1}{f_1^2}=\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}, \tag{8}$$

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}, \tag{9}$$

$$\frac{1}{f_1^8}=\frac{f_4^{28}}{f_2^{28}f_8^8}+8q\frac{f_4^{16}}{f_2^{24}}+16q^2\frac{f_4^4f_8^8}{f_2^{20}}, \tag{10}$$

$$\frac{f_3^3}{f_1}=\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}, \tag{11}$$

$$\frac{f_3}{f_1}=\frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{24}}+q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}, \tag{12}$$

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}, \tag{13}$$

$$\frac{1}{f_1f_3}=\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}} \tag{14}$$

and

$$\frac{f_1}{f_3}=\frac{f_2f_{16}f_{24}^2}{f_6^2f_8f_{48}}-q\frac{f_2f_8^2f_{12}f_{48}}{f_4f_6^2f_{16}f_{24}}. \tag{15}$$

The equations (8) and (9) can be found in [4, Entry 25, p.40]. The equations (10) and (14) can be found in [6]. The equation(11) can be found in [7]. The equations (12),(13) and (15) can be found in [12].

Lemma 2.2. *The following* 3*-dissections hold*

$$f_1^3=\frac{f_6f_9^6}{f_3f_{18}^3}+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^2}-3qf_9^3, \tag{16}$$

$$\frac{f_2}{f_1^2}=\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}, \tag{17}$$

$$\frac{f_4}{f_1}=\frac{f_{12}f_{18}^4}{f_3^3f_{36}^2}+q\frac{f_6^2f_9^3f_{36}}{f_3^4f_{18}^2}+2q^2\frac{f_6f_{18}f_{36}}{f_3^3}, \tag{18}$$

The equation (16) can be found in [4, Entry 25, p.40]. The equation (17) can be found in [8]. The equation (18) can be found in [3].

Lemma 2.3. *The following 5-dissection holds*

$$f_1=f_{25}\left(a-q-q^2/a\right), \tag{19}$$

where

$$a:=a(q^5)=\frac{f(-q^{10},-q^{15})}{f(-q^5,q^{20})}.$$

The equation (19) can be found in [4].

Lemma 2.4. *The following 7-dissection holds*

$$f_1 = f_{49}\left(\frac{B}{C} - q\frac{A}{B} - q^2 + q^5\frac{C}{A}\right), \tag{20}$$

where $A := A(q^7) = f(-q^{21}, -q^{28}), B := B(q^7) = f(-q^{14}, q^{35})$ *and* $C := C(q^7) = f(-q^7, -q^{42})$.

The equation(20) is an exercise in [6, (10.5.1)] and also in [4, Entry 17(v), p.303].

3. Generating function for $\mathbf{OPOND_{j,k}(n)}$

In this section, we provide proof for a generating function of a partition $\mathbf{opond_{j,k}(n)}$.

Theorem 3.1. *For* $1 \leq j \leq k$, *we have*

$$\sum_{n=0}^{\infty} opond_{j,k}(n)q^n = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}}(-q^j; q^k)_\infty. \tag{21}$$

Proof. By the definition,

$$\begin{aligned}
\sum_{n=0}^{\infty} opond_{j,k}(n)q^n &= \frac{(-q^j; q^k)_\infty}{(q^2; q^2)_\infty}\prod_{i=1}^{\infty}\left(\frac{1}{1-q^{2i-1}} - q^{2i-1}\right)\\
&= \frac{(-q^j; q^k)_\infty}{(q^2; q^2)_\infty}\prod_{i=1}^{\infty}\left(\frac{1-q^{2i-1}+(q^{2i-1})^2}{(1-q^{2i-1})}\right)\\
&= \frac{(-q^j; q^k)_\infty}{(q^2; q^2)_\infty}\prod_{i=1}^{\infty}\left(\frac{(1+q^{3(2i-1)})}{(1-q^{2i-1})(1+q^{2i-1})}\right)\\
&= \frac{(-q^j; q^k)_\infty}{(q^2; q^2)_\infty}\frac{(-q^3; q^6)_\infty}{(q^2; q^4)_\infty} = \frac{f_4 f_6^2}{f_2^2 f_3 f_{12}}(-q^j, ; q^k)_\infty.
\end{aligned}$$

□

4. Congruences for $\mathbf{opond_{3,3}(n)}$

Theorem 4.1. *For* $n \geq 0$, *then we have*

$$opond_{3,3}(12n+8) \equiv 0 \pmod{18} \tag{22}$$

and

$$opond_{3,3}(12n+11) \equiv 0 \pmod{36}. \tag{23}$$

Proof. Putting $j = 3$ and $k = 3$ in (21), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(n)q^n = \frac{f_4 f_6^3}{f_2^2 f_3^2 f_{12}}. \tag{24}$$

Using (17) in (24), we get

$$\sum_{n=0}^{\infty} opond_{3,3}(3n)q^n = \frac{f_4^3 f_6^6}{f_1^2 f_2^5 f_{12}^3}, \tag{25}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(3n+1)q^n = 4\frac{qf_4f_{12}^3}{f_1^2f_2^3} \tag{26}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(3n+2)q^n = 2\frac{f_4^2f_6^3}{f_1^2f_2^4}. \tag{27}$$

Using (8) in (27), we get

$$\sum_{n=0}^{\infty} opond_{3,3}(6n+2)q^n = 2\frac{f_2^2f_4^5}{f_8^2} \pmod{18} \tag{28}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(6n+5)q^n = 4\frac{f_2^4f_8^2}{f_4} \pmod{36}. \tag{29}$$

From the equations (28) and (29), we obtain respectively (22) and (23). □

Theorem 4.2. *For $\alpha, n \geq 0$, then we have for* $\pmod{\mathbf{128}}$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha}(24n+13))q^n \equiv 64f_1^{13}, \tag{30}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha+1}(24n+17))q^n \equiv 64q^2f_5^{13}, \tag{31}$$

$$opond_{3,3}(5^{2\alpha+1}(120n+b_1)) \equiv 0, \tag{32}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha}(48n+34))q^n \equiv 64f_1^{17}, \tag{33}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha+1}(48n+46))q^n \equiv 64q^4f_7^{17}, \tag{34}$$

$$opond_{3,3}(7^{2\alpha+1}(336n+b_2)) \equiv 0, \tag{35}$$

where $b_1 \in \{17, 41, 89, 113\}$ and $b_2 \in \{46, 94, 142, 190, 286, 334\}$.

Proof. Using (9) in (26), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(6n+1)q^n = 8\frac{qf_2^3f_6^3f_8^2}{f_1^8f_4} \tag{36}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(6n+4)q^n = 4\frac{f_2f_4^5f_6^3}{f_1^8f_8^2}. \tag{37}$$

Using (10) in (36), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(12n+1)q^n = 64\frac{qf_2^9f_3^3}{f_1} \pmod{256} \tag{38}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(12n+7)q^n = 8\frac{f_2^3 f_3^3}{f_1} \pmod{16}. \tag{39}$$

Employing (10) in (37), we get

$$\sum_{n=0}^{\infty} opond_{3,3}(12n+4)q^n = 4\frac{f_2^9 f_3^3}{f_1^{11} f_4^2} \pmod{64} \tag{40}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(12n+10)q^n = 32\frac{f_1 f_2^5 f_6^2}{f_3} \pmod{128}. \tag{41}$$

Using (11) in (38), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(24n+1)q^n = 64q f_1 f_2^3 f_6^3 \pmod{128} \tag{42}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(24n+13)q^n = 64 f_1^{13} \pmod{128}. \tag{43}$$

Using (11) in (39), we get

$$\sum_{n=0}^{\infty} opond_{3,3}(24n+7)q^n = 8 f_1^7 \pmod{16} \tag{44}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(24n+19)q^n = 8 f_1 f_6^3 \pmod{16}. \tag{45}$$

Using (15) in (41) and extracting the terms involving q^{2n} and q^{2n+1}, we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(24n+10)q^n = 32\frac{f_4 f_8 f_{12}^2}{f_1^2 f_{24}} \pmod{128} \tag{46}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(24n+22)q^n = 96\frac{f_2^7 f_6 f_{24}}{f_1^2 f_8 f_{12}} \pmod{128}. \tag{47}$$

Using (8) in (46) and extracting the terms involving q^{2n} and q^{2n+1}, we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(48n+10)q^n \equiv 32 f_1^5 \pmod{64} \tag{48}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(48n+34)q^n \equiv 64 f_1^{17} \pmod{128}. \tag{49}$$

Using (8) in (47) and extracting the terms involving q^{2n} and q^{2n+1}, we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(48n+22)q^n \equiv 32f_2f_3^3 \pmod{64} \tag{50}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(48n+46)q^n \equiv 64f_2f_3^3f_4^3 \pmod{128}. \tag{51}$$

The equation (43) is $\alpha=0$ case of (30).
Suppose that the equation (30) is true for all $\alpha \geq 0$.
Using (19) in (30) and then extracting the term involving q^{5n+3} on both sides, we obtain (31).
From the equation (31), we obtain (32) and

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha+2}(24n+13))q^n \equiv 64f_1^{13}, \tag{52}$$

which is $\alpha+1$ case of (30). By mathematical induction, the equation (30) hold for all $\alpha \geq 0$.
Since the proofs of (33)-(35) are similar to the proofs of (30)-(32), we omit the details. □

Theorem 4.3. *For $\alpha, \beta, n \geq 0$, then we have for* $\pmod{\mathbf{64}}$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3\cdot 2^{2\alpha+7}n+2^{2\alpha+8}+96\right)q^n \equiv 4\frac{f_2f_4^5f_6^3}{f_1^8f_8^2}+56\frac{f_2^3f_3^2f_8^2}{f_1^2f_4f_6}+32f_{16}, \tag{53}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha}(48n+10))q^n \equiv 32f_1^5, \tag{54}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha+1}(48n+22))q^n \equiv 32qf_7^5, \tag{55}$$

$$opond_{3,3}(7^{2\alpha+1}(336n+b_3)) \equiv 0, \tag{56}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha}(48n+22))q^n \equiv 32f_2f_3^3, \tag{57}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha+1}(48n+14))q^n \equiv 32q^2f_{10}f_{15}^3, \tag{58}$$

$$opond_{3,3}(5^{2\alpha+1}(240n+b_4)) \equiv 0, \tag{59}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha}(192n+136))q^n \equiv 32f_1^{17}, \tag{60}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha+1}(192n+184))q^n \equiv 32q^4 f_7^{17}, \tag{61}$$

$$opond_{3,3}(7^{2\alpha+1}(1344n+b_5)) \equiv 0, \tag{62}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha}(768n+544))q^n \equiv 32 f_1^{17}, \tag{63}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha+1}(768n+736))q^n \equiv 32q^4 f_7^{17}, \tag{64}$$

$$opond_{3,3}(7^{2\alpha+1}(5376n+b_6)) \equiv 0, \tag{65}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta}n + 17\cdot 2^{2\alpha+7}7^{2\beta}+96)q^n \equiv 32 f_1^{17}, \tag{66}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta+1}n + 23\cdot 2^{2\alpha+7}7^{2\beta}+96)q^n \equiv 32q^4 f_7^{17} \tag{67}$$

and

$$opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta+2}n + b_8\cdot 2^{2\alpha+7}7^{2\beta}+96) \equiv 0, \tag{68}$$

where $b_3 \in \{22, 118, 166, 214, 262, 310\}, b_4 \in \{14, 62, 158, 206\}$,
$b_5 \in \{184, 376, 568, 760, 1144, 1336\}, b_6 \in \{736, 1504, 2272, 3040, 4576, 5344\}$,
$b_8 \in \{23, 47, 71, 95, 143, 167\}$.

Proof. Using (8), (10) and (11) in the equation (40), we get for $\pmod{\mathbf{64}}$

$$\sum_{n=0}^{\infty} opond_{3,3}(24n+4)q^n \equiv 4\frac{f_2^5 f_3^2 f_4^5}{f_1^{10} f_6 f_8^2} + 40\frac{q f_6^3 f_8^2}{f_2 f_4} \tag{69}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(24n+16)q^n \equiv 4\frac{f_2 f_4^5 f_6^3}{f_1^8 f_8^2} + 8\frac{f_2^3 f_3^2 f_8^2}{f_1^2 f_4 f_6} + 32 f_{16}. \tag{70}$$

Using (10) and (13) in (70), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(48n+16)q^n \equiv 4\frac{f_2^9 f_3^3}{f_1^{11} f_4^2} + 8\frac{f_2^3 f_4 f_6^3}{f_1^2 f_{12}} + 32 f_8 \tag{71}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(48n+40)q^n \equiv 48\frac{f_3 f_4^3 f_{12}}{f_1 f_6}. \tag{72}$$

Employing (8), (10) and (11) in the equation (71), we get

$$\sum_{n=0}^{\infty} opond_{3,3}(96n+16)q^n \equiv 4\frac{f_2^5 f_3^2 f_4^5}{f_1^{10} f_6 f_8} + 40\frac{q f_6^3 f_8^2}{f_2 f_4} + 32 f_4 \tag{73}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(96n+64)q^n \equiv 4\frac{f_2 f_4^5 f_6^3}{f_1^8 f_8} + 24\frac{f_2^3 f_3^2 f_8^2}{f_1^2 f_4 f_6} + 32f_{16}. \tag{74}$$

Using (10) and (13) in the equation (74), we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(192n+64)q^n \equiv 4\frac{f_2^9 f_3^3}{f_1^{11} f_4^2} + 24\frac{f_2^3 f_4 f_6^2}{f_1^2 f_{12}} + 32f_8 \tag{75}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(192n+160)q^n \equiv 16\frac{f_3 f_4^3 f_{12}}{f_1 f_6}. \tag{76}$$

Using (8), (10) and (11) in the equation (75), we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(384n+64)q^n \equiv 28\frac{f_2^5 f_3^2 f_4^5}{f_1^{10} f_6 f_8^2} + 40\frac{q f_6^3 f_8^2}{f_2 f_4} + 32f_4 \tag{77}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(384n+352)q^n \equiv 4\frac{f_2 f_4^5 f_6^3}{f_1^8 f_8^2} + 56\frac{f_2^3 f_3^2 f_8^2}{f_1^2 f_4 f_6} + 32f_{16}, \tag{78}$$

which is $\alpha = 0$ case of (53).
Suppose that the identity (53) hold for all $\alpha \geq 0$.
Using (10) and (13) in the equation (53), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} opond_{3,3}\left(3 \cdot 2^{2\alpha+8} n + 2^{2\alpha+8} + 96\right) q^n \\ &\equiv 4\frac{f_2^9 f_3^3}{f_1^{11} f_4^2} + 56\frac{f_2^3 f_4 f_6^2}{f_1^2 f_{12}} + 32f_8 \end{aligned} \tag{79}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3 \cdot 2^{2\alpha+8} n + 5 \cdot 2^{2\alpha+7} + 96\right) q^n \equiv 16\frac{f_3 f_4^3 f_{12}}{f_1 f_6}. \tag{80}$$

Employing (8), (10) and (11) in the equation(79), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} opond_{3,3}\left(3 \cdot 2^{2\alpha+9} n + 2^{2\alpha+8} + 96\right) q^n \\ &\equiv 60\frac{f_2^5 f_3^2 f_4^5}{f_1^{10} f_6 f_8^2} + 40\frac{f_6^3 f_8^2}{f_2 f_4} + 32f_4 \end{aligned} \tag{81}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3 \cdot 2^{2\alpha+9} n + 2^{2\alpha+10} + 96\right) q^n \equiv 4\frac{f_2 f_4^5 f_6^3}{f_1^8 f_8^2} + 56\frac{f_2^3 f_3^2 f_8^2}{f_1^2 f_4 f_6} + 32f_{16}, \tag{82}$$

which is $\alpha + 1$ case of (53). By mathematical induction, the equation (53) hold for all $\alpha \geq 0$.
The equation (48) is $\alpha = 0$ case of (54).
Suppose that the equation (54) is true for all $\alpha \geq 0$.

Employing (20) in (54), we obtain (55).
From the equation (55), we obtain (56) and

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha+2}(48n+10))q^n \equiv 32f_1^5, \tag{83}$$

which is $\alpha+1$ case of (54). By mathematical induction, the equation (54) hold for all $\alpha \geq 0$.
Using (12) in (72), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(96n+40)q^n \equiv 48\frac{f_2^4 f_8 f_{12}^2}{f_1^2 f_4 f_{24}} \pmod{64} \tag{84}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(96n+88)q^n \equiv 48\frac{f_2^3 f_4^2 f_6 f_{24}}{f_1^2 f_8 f_{12}} \pmod{64}. \tag{85}$$

Using (8) in (84), we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(192n+40)q^n \equiv 16f_1^5 \pmod{32} \tag{86}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(192n+136)q^n \equiv 32f_1^{17} \pmod{64}. \tag{87}$$

Using (8) in (85), we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(192n+40)q^n \equiv 16f_2 f_3^3 \pmod{32} \tag{88}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(192n+136)q^n \equiv 32f_2 f_3^3 f_4^3 \pmod{64}. \tag{89}$$

Using (12) in (76), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(384n+160)q^n \equiv 16\frac{f_4 f_8 f_{12}^2}{f_1^2 f_{24}} \pmod{64} \tag{90}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(384n+352)q^n \equiv 16\frac{f_6 f_8 f_{24}}{f_1^2 f_2 f_{12}} \pmod{64}. \tag{91}$$

Using (8) in (90), we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(768n+160)q^n \equiv 16f_1^5 \pmod{32} \tag{92}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(768n+544)q^n \equiv 32f_1^{17} \pmod{64}. \tag{93}$$

Using (8) in (91), we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(768n+352)q^n \equiv 16f_2f_3^3 \pmod{32} \tag{94}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(768n+736)q^n \equiv 32f_2f_3^3f_4^3 \pmod{64}. \tag{95}$$

Employing (12) in (80), we find that

$$\sum_{n=0}^{\infty} \left(3\cdot 2^{2\alpha+9}n + 5\cdot 2^{2\alpha+7} + 96\right) q^n \equiv 16\frac{f_4f_8f_{12}^2}{f_1^2f_{24}} \pmod{64} \tag{96}$$

and

$$\sum_{n=0}^{\infty} \left(3\cdot 2^{2\alpha+9}n + 11\cdot 2^{2\alpha+7} + 96\right) q^n \equiv 16\frac{f_6f_8f_{24}}{f_1^2f_2f_{12}} \pmod{64}. \tag{97}$$

Using (8) in (96), we get

$$\sum_{n=0}^{\infty} \left(3\cdot 2^{2\alpha+10}n + 5\cdot 2^{2\alpha+7} + 96\right) q^n \equiv 16f_1^5 \pmod{32} \tag{98}$$

and

$$\sum_{n=0}^{\infty} \left(3\cdot 2^{2\alpha+10}n + 17\cdot 2^{2\alpha+7} + 96\right) q^n \equiv 32f_1^{17} \pmod{64}. \tag{99}$$

Using (8) in (97), we get

$$\sum_{n=0}^{\infty} \left(3\cdot 2^{2\alpha+10}n + 11\cdot 2^{2\alpha+7} + 96\right) q^n \equiv 16f_2f_3^3 \pmod{32} \tag{100}$$

and

$$\sum_{n=0}^{\infty} \left(3\cdot 2^{2\alpha+10}n + 23\cdot 2^{2\alpha+7} + 96\right) q^n \equiv 32f_2f_3^3f_4^3 \pmod{64}. \tag{101}$$

Since the proofs of identities (57)-(68) are similar to the proofs of the identities (54)-(56). So, we omit the details. □

Theorem 4.4. *For all* $\alpha, \beta, n \geq 0$, *then we have for* $\pmod{\mathbf{32}}$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha}(96n+52))q^n \equiv 16f_1^{13}, \tag{102}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha+1}(96n+68))q^n \equiv 16q^2f_5^{13}, \tag{103}$$

$$opond_{3,3}(5^{2\alpha+1}(480n+b_9)) \equiv 0, \tag{104}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha}(192n+40))q^n \equiv 16f_1^5, \tag{105}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha+1}(192n+88))q^n \equiv 16qf_7^5, \tag{106}$$

$$opond_{3,3}(7^{2\alpha+1}(1344n+b_{10})) \equiv 0, \tag{107}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha}(192n+88))q^n \equiv 16f_2f_3^3, \tag{108}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha+1}(192n+56))q^n \equiv 16q^2f_{10}f_{15}^3, \tag{109}$$

$$opond_{3,3}(5^{2\alpha+1}(960n+b_{11})) \equiv 0, \tag{110}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha}(768n+160))q^n \equiv 16f_1^5, \tag{111}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha+1}(768n+352))q^n \equiv 16qf_7^5, \tag{112}$$

$$opond_{3,3}(7^{2\alpha+1}(5376n+b_{12})) \equiv 0, \tag{113}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha}(768n+352))q^n \equiv 16f_2f_3^3, \tag{114}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha+1}(768n+224))q^n \equiv 16q^2f_{10}f_{15}^3, \tag{115}$$

$$opond_{3,3}(5^{2\alpha+1}(3840n+b_{13})) \equiv 0, \tag{116}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta}n + 5\cdot 2^{2\alpha+7}7^{2\beta}+96)q^n \equiv 16f_1^5, \tag{117}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta+1}n + 11\cdot 2^{2\alpha+7}7^{2\beta+1}+96)q^n \equiv 16qf_7^5, \tag{118}$$

$$opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta+2}n + b_{14}\cdot 2^{2\alpha+7}7^{2\beta+1}+96) \equiv 0, \tag{119}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(3\cdot 2^{2\alpha+10}5^{2\beta}n + 11\cdot 2^{2\alpha+7}5^{2\beta}+96)q^n \equiv 16f_2f_3^3, \tag{120}$$

$$\sum_{n=0}^{\infty} opond_{3,3}(3\cdot 2^{2\alpha+10}5^{2\beta+1}n + 7\cdot 2^{2\alpha+7}5^{2\beta+1}+96)q^n \equiv 16q^2f_{10}f_{15}^3, \tag{121}$$

$$opond_{3,3}(3\cdot 2^{2\alpha+10}5^{2\beta+2}n + b_{15}\cdot 2^{2\alpha+7}5^{2\beta+1}+96) \equiv 0, \tag{122}$$

where $b_9 \in \{68, 164, 356, 452\}, b_{10} \in \{88, 472, 664, 856, 1048, 1240\}$,
$b_{11} \in \{56, 248, 632, 824\}, b_{12} \in \{352, 1888, 2656, 3424, 4192, 4960\}$

$b_{13} \in \{224, 992, 2528, 3296\}, b_{14} \in \{11, 35, 83, 107, 131, 155\}$,
$b_{15} \in \{7, 31, 79, 103\}$.

Proof. From the equation (69), we deduce that

$$\sum_{n=0}^{\infty} opond_{3,3}(24+4)q^n \equiv 4\frac{f_2 f_4^5 f_3^2}{f_1^2 f_6 f_8^2} + 8\frac{q f_6^3 f_8^2}{f_2 f_4} \pmod{32}. \tag{123}$$

Using (13) in the above equation(123), we get

$$\sum_{n=0}^{\infty} opond_{3,3}(48+4)q^n \equiv 4\frac{f_2^9 f_6^2}{f_1^4 f_4^3 f_{12}} \pmod{32}. \tag{124}$$

Utilizing (9) in (124), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}(96+52)q^n \equiv 16 f_1^{13}, \tag{125}$$

which is $\alpha = 0$ case of (102).
Suppose that the equation (102) is true for all $\alpha \geq 0$.
Using (19) in (102) and then extracting the term involving q^{5n+3}, we obtain (103).
From the equation (103), we obtain (104) and

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha+2}(96n+52))q^n \equiv 16 f_1^{13} \pmod{32}, \tag{126}$$

which is $\alpha+1$ case of (102). Hence, by mathematical induction, the equation (102) hold for all $\alpha \geq 0$.
Since the proofs of the identities (105)-(122) are similar to the proofs of (102)-(104). So, we omit the details. □

Theorem 4.5. *For $\alpha, n \geq 0$, then we have* $\pmod{\mathbf{16}}$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+1}(4n+3)\right) q^n \equiv 4 f_1^3 f_2^3 f_3^3, \tag{127}$$

$$opond_{3,3}\left(3^{4\alpha+2}(24n+13)\right) \equiv 0, \tag{128}$$

$$opond_{3,3}\left(3^{4\alpha+4}(12n+1)\right) \equiv 0, \tag{129}$$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+1}(24n+23)\right) q^n \equiv 8 f_1 f_4 f_6^3, \tag{130}$$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+2}(24n+17)\right) q^n \equiv 8 f_8 f_3^3, \tag{131}$$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+3}(24n+23)\right) q^n \equiv 8 f_1 f_4 f_6^3, \tag{132}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+4}(24n+17)\right) q^n \equiv 8 f_8 f_3^3. \tag{133}$$

Proof. Using (8) in (25) and then extracting the term involving q^{2n+1}, we see that

$$\sum_{n=0}^{\infty} opond_{3,3}(6n+3)q^n = 2\frac{f_2 f_6 f_8^2}{f_1^2 f_3^2 f_4} \pmod{16}. \tag{134}$$

Using (14) in (134), we deduce that

$$\sum_{n=0}^{\infty} opond_{3,3}(12n+3)q^n = 2\frac{f_4^6 f_3}{f_1^3 f_2^3 f_6^2} + 2q\frac{f_1 f_2^5 f_6^6}{f_3^3 f_4^2} \pmod{16} \tag{135}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}(12n+9)q^n \equiv 4f_1^3 f_2^3 f_3^3 \pmod{16}, \tag{136}$$

which is $\alpha = 0$ case of (127).
Suppose that the equation (127) hold for all $\alpha \geq 0$.
Using (16) in (127), we get for $\pmod{\mathbf{16}}$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+2}(4n+1)\right)q^n \equiv 4f_2^3 + 12qf_1^3 f_3^3 f_6^3, \tag{137}$$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+1}(12n+7)\right)q^n \equiv 4f_1 f_2^2 f_3 f_6 + 8qf_2 f_6^6 \tag{138}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+1}(12n+11)\right)q^n \equiv 4\frac{f_2^3 f_3^2 f_6^2}{f_1^2}. \tag{139}$$

Employing (16) in (137), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+2}(12n+1)\right)q^n \equiv 4\frac{f_4 f_6^2}{f_2 f_{12}}, \tag{140}$$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+2}(12n+5)\right)q^n \equiv 12\frac{f_2^6 f_3^2}{f_1^2 f_6} \tag{141}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+3}(4n+3)\right)q^n \equiv 4f_6^3 + 12f_1^3 f_2^3 f_3^3. \tag{142}$$

From the equation (140), we obtain (128).
Employing (16) in (142), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+4}(4n+1)\right)q^n \equiv 4qf_1^3 f_3^3 f_6^3, \tag{143}$$

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+3}(12n+7)\right)q^n \equiv 12f_1 f_3 f_2^2 f_6 + 8qf_2 f_6^6 \tag{144}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+3}(12n+11)\right)q^n \equiv 12\frac{f_2^3 f_3^2 f_6^2}{f_1^2}. \tag{145}$$

Employing (16) in (143), we obtain (129),

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+4}(12n+5)\right)q^n \equiv 4\frac{f_2^6 f_3^2}{f_1^2 f_6} \tag{146}$$

and

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+5}(4n+3)\right)q^n \equiv 4f_1^3 f_2^3 f_3^3, \tag{147}$$

which is $\alpha+1$ case of (127). By the mathematical induction, the equation (127) hold for all $\alpha \geq 0$.
Using (14) in(139) and then extracting the term q^{2n+1}, we obtain (130).
Employing (14) in (141), we obtain (131).
Using (14) in (145), we obtain (132).
Employing (14) in (146), we obtain (133). □

5. Ramanujan-type congruence for $\mathbf{opond_{3,3}(n)}$

In this section, we prove several infinite families of Ramanujan-type congruences for $\mathbf{opond_{3,3}(n)}$.

Theorem 5.1. *Let p be a prime with $p \equiv 3 \pmod 4$. Then for all $\alpha, \beta, k, m \geq 0$ such that $p \nmid m$, we have*

$$opond_{3,3}(7^{2\alpha}\cdot p^{2k+1}(48m+10p)) \equiv 0 \pmod{64}, \tag{148}$$

$$opond_{3,3}(7^{2\alpha+1}\cdot p^{2k+1}(48m+22p)) \equiv 0 \pmod{64}, \tag{149}$$

$$opond_{3,3}(7^{2\alpha}\cdot p^{2k+1}(192m+40p)) \equiv 0 \pmod{32}, \tag{150}$$

$$opond_{3,3}(7^{2\alpha+1}\cdot p^{2k+1}(192m+88p)) \equiv 0 \pmod{32}, \tag{151}$$

$$opond_{3,3}(7^{2\alpha}\cdot p^{2k+1}(768m+160p)) \equiv 0 \pmod{32}, \tag{152}$$

$$opond_{3,3}(7^{2\alpha+1}\cdot p^{2k+1}(768m+352p)) \equiv 0 \pmod{32}, \tag{153}$$

$$opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta}p^{2k+1}m+5\cdot 2^{2\alpha+7}7^{2\beta}p^{2k+2}+96) \equiv 0 \pmod{32} \tag{154}$$

and

$$opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta+1}p^{2k+1}m+11\cdot 2^{2\alpha+7}7^{2\beta+1}p^{2k+2}+96) \equiv 0 \pmod{32}. \tag{155}$$

Proof. Utilizing (1) in (54), we find that for $\pmod{64}$

$$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha}(48n+10))q^n \equiv 32\sum_{s,t=-\infty}^{\infty} q^{\frac{s(3s-1)}{2}+2t(3t-1)}, \tag{156}$$

which implies $n = \frac{s(3s-1)}{2} + 2t(3t-1)$. It is equivalent to

$$24n+5 = (6s-1)^2 + (12t-2)^2 = x^2 + y^2.$$

Thus, if $24n+5 \neq (6s-1)^2 + (12t-2)^2 = x^2 + y^2$, then

$$opond_{3,3}(7^{2\alpha}(48n+10)) \equiv 0 \pmod{64}. \tag{157}$$

Since $24n+5 = p^{2k+1}(24m+5p) \neq x^2+y^2$, from which (148) follows. Since the proofs of (149)-(155) are similar to the proof of (148). So, we omit the details. □

Theorem 5.2. *Let p be a prime with $p \equiv 5$ or $7 \pmod 8$. Then for all $\alpha, \beta, k, m \geq 0$ such that $p \nmid m$, we have*

$$opond_{3,3}(5^{2\alpha} \cdot p^{2k+1}(48m+22p)) \equiv 0 \pmod{64}, \tag{158}$$

$$opond_{3,3}(5^{2\alpha+1} \cdot p^{2k+1}(48m+14p)) \equiv 0 \pmod{64}, \tag{159}$$

$$opond_{3,3}(5^{2\alpha+1} \cdot p^{2k+1}(192m+56p)) \equiv 0 \pmod{32}, \tag{160}$$

$$opond_{3,3}(5^{2\alpha} \cdot p^{2k+1}(768m+352p)) \equiv 0 \pmod{32}, \tag{161}$$

$$opond_{3,3}(5^{2\alpha+1} \cdot p^{2k+1}(768m+224p)) \equiv 0 \pmod{32}, \tag{162}$$

$$opond_{3,3}(3 \cdot 2^{2\alpha+10} 5^{2\beta} p^{2k+1} m + 11 \cdot 2^{2\alpha+7} 5^{2\beta} p^{2k+2} + 96) \equiv 0 \pmod{32} \tag{163}$$

and

$$opond_{3,3}(3 \cdot 2^{2\alpha+10} 5^{2\beta+1} p^{2k+1} m + 11 \cdot 2^{2\alpha+7} 5^{2\beta+1} p^{2k+2} + 96) \equiv 0 \pmod{32}. \tag{164}$$

Proof. Employing (1) and (2) in (33), we find that for $\pmod{\mathbf{64}}$

$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha}(48n+22))q^n \equiv 32 \sum_{s,t=-\infty}^{\infty} \sum_{t=0}^{\infty} q^{s(3s-1)+\frac{3t(t+1)}{2}}, \tag{165}$$

which implies $n = s(3s-1) + \frac{3t(t+1)}{2}$. It is equivalent to

$$24n+11 = 2(6s-1)^2 + (6t+3)^2 = 2x^2 + y^2.$$

Thus, if $24n+11 \neq 2(6s-1)^2 + (6t+3)^2 = 2x^2 + y^2$, then

$$opond_{3,3}(5^{2\alpha}(48n+22)) \equiv 0 \pmod{64}. \tag{166}$$

Since $24n+11 = p^{2k+1}(24m+11p) \neq 2x^2+y^2$, from which (158) follows. Since the proofs of (159)-(164) are similar to the proof of (158). So, we omit the details. □

Theorem 5.3. *Let p be a prime with $p \equiv 5$ or $11 \pmod{12}$. Then for all $\alpha, k, m \geq 0$ such that $p \nmid m$, we have*

$$opond_{3,3}(5^{2\alpha} \cdot p^{2k+1}(24m+13p)) \equiv 0 \pmod{128}, \tag{167}$$

$$opond_{3,3}(5^{2\alpha+1} \cdot p^{2k+1}(24m+17p)) \equiv 0 \pmod{128}, \tag{168}$$

$$opond_{3,3}(5^{2\alpha} \cdot p^{2k+1}(96m+52p)) \equiv 0 \pmod{32} \tag{169}$$

and

$$opond_{3,3}(5^{2\alpha+1} \cdot p^{2k+1}(96m+68p)) \equiv 0 \pmod{32}. \tag{170}$$

Proof. Utilizing (1) and (2) in (30), we find that

(171)
$$\sum_{n=0}^{\infty} opond_{3,3}(5^{2\alpha}(24n+13))q^n \equiv 64 \sum_{s=-\infty}^{\infty}\sum_{t=0}^{\infty} q^{\frac{s(3s-1)}{2}+2t(t+1)} \pmod{128},$$

which implies $n = \frac{s(3s-1)}{2} + 2t(t+1)$. It is equivalent to

$$24n+13 = (6s-1)^2 + 3(8t+4)^2 = x^2 + 3y^2.$$

Thus, if $24n+13 \neq (6s-1)^2 + 3(8t+4)^2 = x^2+3y^2$, then

(172) $$opond_{3,3}(5^{2\alpha}(24n+13)) \equiv 0 \pmod{128}.$$

Since $24n+13 = p^{2k+1}(24m+13p) \neq x^2+3y^2$, from which (167) follows. Since the proofs of (168)-(170) are similar to the proof of (167). So, we omit the details. □

Theorem 5.4. *Let p be a prime with $p \equiv 3 \pmod 4$. Then for all $\alpha, \beta, k, m \geq 0$ such that $p \nmid m$, we have*

(173) $$opond_{3,3}(7^{2\alpha} \cdot p^{2k+1}(48m+34p)) \equiv 0 \pmod{128},$$

(174) $$opond_{3,3}(7^{2\alpha+1} \cdot p^{2k+1}(48m+46p)) \equiv 0 \pmod{128},$$

(175) $$opond_{3,3}(7^{2\alpha} \cdot p^{2k+1}(192m+136p)) \equiv 0 \pmod{64},$$

(176) $$opond_{3,3}(7^{2\alpha+1} \cdot p^{2k+1}(192m+184p)) \equiv 0 \pmod{64},$$

(177) $$opond_{3,3}(7^{2\alpha} \cdot p^{2k+1}(768m+544p)) \equiv 0 \pmod{64},$$

(178) $$opond_{3,3}(7^{2\alpha+1} \cdot p^{2k+1}(768m+736p)) \equiv 0 \pmod{64},$$

(179) $$opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta}p^{2k+1}m + 17\cdot 2^{2\alpha+7}7^{2\beta}p^{2k+2}+96) \equiv 0 \pmod{64}$$

and

(180)
$$opond_{3,3}(3\cdot 2^{2\alpha+10}7^{2\beta+1}p^{2k+1}m + 23\cdot 2^{2\alpha+7}7^{2\beta+1}p^{2k+2}+96) \equiv 0 \pmod{64}.$$

Proof. Utilizing (1) in (33), we find that for $\pmod{\mathbf{128}}$

(181) $$\sum_{n=0}^{\infty} opond_{3,3}(7^{2\alpha}(48n+34))q^n \equiv 64 \sum_{s,t=-\infty}^{\infty} q^{\frac{s(3s-1)}{2}+8t(3t-1)},$$

which implies $n = \frac{s(3s-1)}{2} + 8t(3t-1)$. It is equivalent to

$$24n+17 = (6s-1)^2 + (24t-4)^2 = x^2 + y^2.$$

Thus, if $24n+17 \neq (6s-1)^2 + (24t-4)^2 = x^2+y^2$, then

(182) $$opond_{3,3}(7^{2\alpha}(24n+13)) \equiv 0 \pmod{128}.$$

Since $24n+17 = p^{2k+1}(24m+17p) \neq x^2+3y^2$, from which (173) follows. Since the proofs of (174)-(180) are similar to the proof of (173). So, we omit the details. □

Theorem 5.5. *If n is not a sum of a triangular number, twice a triangular and three times a triangular number. Then we have*

$$opond_{3,3}\left(3^{4\alpha+1}(4n+3)\right) \equiv 0 \pmod{16}. \tag{183}$$

Proof. Using (2) in (127), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+1}(4n+3)\right) q^n \equiv \sum_{r,s,t=0}^{\infty} q^{\frac{r(r+1)}{2}+\frac{2s(s+1)}{2}+\frac{3t(t+1)}{2}} \pmod{16}, \tag{184}$$

which implies $n = \frac{r(r+1)}{2} + \frac{2s(s+1)}{2} + \frac{3t(t+1)}{2}$.
Thus, if $n \neq \frac{r(r+1)}{2} + \frac{2s(s+1)}{2} + \frac{3t(t+1)}{2}$, we obtain (183). □

Theorem 5.6. *If n is not a sum of a pentagonal number, four times a pentagonal number and six times a triangular number. Then we have*

$$opond_{3,3}\left(3^{4\alpha+1}(24n+23)\right) \equiv 0 \pmod{16} \tag{185}$$

and

$$opond_{3,3}\left(3^{4\alpha+3}(24n+23)\right) \equiv 0 \pmod{16}. \tag{186}$$

Proof. Using (1) and (2) in (130), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+1}(24n+23)\right) q^n \equiv \sum_{r,s,=-\infty}^{\infty} \sum_{t=0}^{\infty} q^{\frac{r(3r-1)}{2}+\frac{4s(3s-1)}{2}+\frac{6t(t+1)}{2}} \pmod{16}, \tag{187}$$

which implies $n = \frac{r(3r-1)}{2} + \frac{4s(3s-1)}{2} + \frac{6t(t+1)}{2}$.
Thus, if $n \neq \frac{r(3r-1)}{2} + \frac{4s(3s-1)}{2} + \frac{6t(t+1)}{2}$, we obtain (185).
Since the proof of (186) is similar to the proof of (185). So, we omit the details. □

Theorem 5.7. *If n is not a sum of eight times a pentagonal number and three times a triangular number. Then we have*

$$opond_{3,3}\left(3^{4\alpha+2}(24n+17)\right) \equiv 0 \pmod{16} \tag{188}$$

and

$$opond_{3,3}\left(3^{4\alpha+4}(24n+17)\right) \equiv 0 \pmod{16}. \tag{189}$$

Proof. Using (1) and (2) in (132), we find that

$$\sum_{n=0}^{\infty} opond_{3,3}\left(3^{4\alpha+2}(24n+17)\right) q^n \equiv \sum_{r,=-\infty}^{\infty} \sum_{t=0}^{\infty} q^{2r(3r-1)+\frac{3t(t+1)}{2}} \pmod{16}, \tag{190}$$

which implies $n = 2r(3r-1) + \frac{3t(t+1)}{2}$.
Thus, if $n \neq 2r(3r-1) + \frac{3t(t+1)}{2}$, we obtain (188).

Since the proof of (189) is similar to the proof of (188). So, we omit the details. □

6. Congruence for $\mathbf{opond_{3,6}(n)}$

Theorem 6.1. *For $\alpha, n \geq 0$, then we have for* $\pmod{\mathbf{32}}$

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+1}n+\frac{3^{16\alpha+1}+1}{4}\right)q^n \equiv 8\frac{qf_1^9f_4f_6^3}{f_2^2}, \tag{191}$$

$$opond_{3,6}\left(2\cdot 3^{16\alpha+9}n+\frac{19\cdot 3^{16\alpha+8}+1}{4}\right) \equiv 0, \tag{192}$$

$$opond_{3,6}\left(2\cdot 3^{16\alpha+17}n+\frac{19\cdot 3^{16\alpha+16}+1}{4}\right) \equiv 0, \tag{193}$$

$$opond_{3,6}\left(2\cdot 3^{16\alpha+10}n+\frac{17\cdot 3^{16\alpha+9}+1}{4}\right) \equiv opond_{3,6}(18n+13), \tag{194}$$

$$opond_{3,6}\left(2\cdot 3^{16\alpha+11}n+\frac{11\cdot 3^{16\alpha+10}+1}{4}\right) \equiv opond_{3,6}(54n+25) \tag{195}$$

and

$$opond_{3,6}\left(2\cdot 3^{16\alpha+11}n+\frac{19\cdot 3^{16\alpha+10}+1}{4}\right) \equiv opond_{3,6}(54n+43). \tag{196}$$

Proof. Putting $j=3$ and $k=6$ in (21), we find that

$$\sum_{n=0}^{\infty} opond_{3,6}(n)q^n = \frac{f_4f_6^4}{f_2^2f_3^2f_{12}^2}. \tag{197}$$

Using (17) in (197) and extracting the term involving q^{3n+1} on both side of the equation, we get

$$\sum_{n=0}^{\infty} opond_{3,6}(3n+1)q^n = 4\frac{qf_{12}^3}{f_1^2f_2^2}. \tag{198}$$

Using (8) in (198) and extracting the term involving q^{2n} on both side of the equation, we find that

$$\sum_{n=0}^{\infty} opond_{3,6}(6n+1)q^n = 8\frac{qf_1^9f_2^2f_6^3}{f_4} \pmod{32}, \tag{199}$$

which is $\alpha=0$ case of (191).
Suppose that the equation (191) is true for all $\alpha \geq 0$.
Employing (16) and (17) in (191) and extracting the term involving q^{3n+1} on both side of the equation, we find that for $\pmod{\mathbf{32}}$

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+2}n+\frac{3^{16\alpha+3}+1}{4}\right)q^n \equiv 8\frac{f_1^9f_3^2f_6}{f_{12}}+24\frac{qf_2^3f_3^9f_6^2}{f_{12}}. \tag{200}$$

Employing (16) in (200) and extracting the term involving q^{3n} on both side of the equation, we find that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+3}n+\frac{3^{16\alpha+3}+1}{4}\right)q^n \equiv 8\frac{f_4f_3^2}{f_1f_6}+8\frac{qf_1^2f_2f_3^9}{f_4}+24\frac{qf_1^9f_4f_6^3}{f_2^2}. \tag{201}$$

Using (16), (17) and (18) in (201) and extracting the term involving q^{3n+1} on both side of the equation, we deduce that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+4}n+\frac{3^{16\alpha+5}+1}{4}\right)q^n \equiv 8\frac{qf_2^3f_3^9f_6^2}{f_{12}}+8\frac{f_2f_{12}}{f_1^2f_3}. \tag{202}$$

Using (16) and (18) in (202) and extracting the term involving q^{3n} on both side of the equation, we deduce that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+5}n+\frac{3^{16\alpha+5}+1}{4}\right)q^n \equiv 8\frac{qf_1^9f_4f_6^3}{f_2^2}+8\frac{f_4f_3^2}{f_1f_6}. \tag{203}$$

Using (16) and (18) in (203) and extracting the term involving q^{3n+1} on both side of the equation, we deduce that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+6}n+\frac{3^{16\alpha+7}+1}{4}\right)q^n \equiv 8\frac{qf_1^9f_3^2f_6}{f_{12}}+8\frac{f_2f_{12}}{f_1^2f_3}+24\frac{qf_2^3f_3^9f_6^2}{f_{12}}. \tag{204}$$

Employing (16) and (18) in (203) and extracting the term involving q^{3n} on both side of the equation, we deduce that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+7}n+\frac{3^{16\alpha+7}+1}{4}\right)q^n \equiv 24\frac{qf_1^9f_4f_6^3}{f_2^2}+8\frac{qf_4f_3^9}{f_1^2f_2}+16f_1^3. \tag{205}$$

Utilizing (16) and (18) in (203) and extracting the term involving q^{3n+1} on both side of the equation, we deduce that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+8}n+\frac{3^{16\alpha+9}+1}{4}\right)q^n \equiv 8\frac{qf_2^3f_3^9f_6^2}{f_{12}}+16f_3^3. \tag{206}$$

Employing (16) in (206) and extracting the term involving q^{3n} on both side of the equation, we obtain (192) and

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+9}n+\frac{3^{16\alpha+9}+1}{4}\right)q^n \equiv 8\frac{qf_1^9f_4f_6^2}{f_2^2}+16f_1^3. \tag{207}$$

Utilizing (16) in (207) and extracting the term involving q^{3n+1} on both side of the equation, we see that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+10}n+\frac{3^{16\alpha+11}+1}{4}\right)q^n \equiv 24q\frac{f_2^3f_3^9f_6^2}{f_{12}}+8\frac{f_1^9f_3^2f_6}{f_{12}}+16f_3^3. \tag{208}$$

Utilizing (16), (17) and (18) in (208) and extracting the term involving q^{3n} on both side of the equation, we see that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+11}n+\frac{3^{16\alpha+11}+1}{4}\right)q^n \equiv 24q\frac{f_1^9f_4f_6^3}{f_2^2}+8\frac{f_4f_3^2}{f_1f_6}+8q\frac{f_1^2f_2f_3^9}{f_4}+16f_1^3. \tag{209}$$

Using (16), (17) and (18) in (209) and extracting the term involving q^{3n+1} on both side of the equation, we find that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+14}n+\frac{3^{16\alpha+15}+1}{4}\right)q^n \equiv 24q\frac{f_2^3f_3^9f_6^2}{f_{12}}+8\frac{f_1^9f_3^2f_6}{f_{12}}+8\frac{f_2f_{12}}{f_1^2f_3}+16f_3^3. \tag{210}$$

Employing (16) and (17) in (210) and extracting the term involving q^{3n} on both side of the equation, we deduce that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+15}n+\frac{3^{16\alpha+15}+1}{4}\right)q^n \equiv 24q\frac{f_1^9f_4f_6^3}{f_2^2}+8q\frac{f_4f_3^9}{f_1^2f_2}. \tag{211}$$

Using (16) and (17) in (211) and extracting the term involving q^{3n+1} on both side of the equation, we find that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+16}n+\frac{3^{16\alpha+17}+1}{4}\right)q^n \equiv 8q\frac{f_2^3f_3^9f_6^2}{f_{12}}. \tag{212}$$

Employing (16) in (212) and extracting the term involving q^{3n} on both side of the equation, we obtain (193) and

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+15}n+\frac{3^{16\alpha+15}+1}{4}\right)q^n \equiv 8q\frac{f_1^9f_4f_6^3}{f_2^2}, \tag{213}$$

which is $\alpha+1$ case of (191). By mathematical induction, the equation (191) hold for all $\alpha\geq 0$.

Using (199) in (207), we find that

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+9}n+\frac{3^{16\alpha+9}+1}{4}\right)q^n \equiv \sum_{n=0}^{\infty} opond_{3,6}(6n+1)q^n+16f_1^3. \tag{214}$$

Using (16) in (214) and then extracting the terms involving q^{3n+1} and q^{3n+2}, we obtain (194) and

$$\sum_{n=0}^{\infty} opond_{3,6}\left(2\cdot 3^{16\alpha+10}n+\frac{3^{16\alpha+11}+1}{4}\right)q^n \equiv \sum_{n=0}^{\infty} opond_{3,6}(18n+7)q^n+16f_3^3, \tag{215}$$

which implies (195) and (196). □

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