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# CERTAIN FREDHOLM TYPE INTEGRAL EQUATIONS AND FRACTIONAL OPERATORS ASSOCIATED WITH MODIFIED INCOMPLETE *I*-FUNCTION

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Abstract. In this article, several interesting properties of modified incomplete $I$-function associated with the Fredholm type integral equation in its kernel have been investigated. Also by using Weyl fractional integral (WFI) and Mellin transform (MT) several integral problem involving modified incomplete $I$-function have been obtained. Fractional integration and fractional differentiation are also derived of modified incomplete $I$-function with the help of Marichev-Saigo-Maeda (MSM) fractional operators. The desired outcomes seem to be very helpful in resolving many real-world problems whose solutions represnt different physical phenomena. Additionally, the results facilitate the solution of problems involving fractional differential, interodifferential, and extended integral equations.

**Keywords:** Fractional calculus operator; Incomplete gamma function; Incomplete $I$-function; Mellin transform; Weyl fractional integral.

**MSC:** 33B20; 33E20; 26A33; 44A20; 45B05.

## 1. Intoduction and Preliminaries

Fractional calculus and special functions have been attracted the attention of mathematicians and scientists over the past four decades due to the wide variety of applications as well as significance in fields like biology, computer science, communication theory, fluid dynamics, viscoelasticity, diffusive transport, electrical finance networks, signal processing, probability theory, ecology, environmental science, and so on [1, 18, 24].

The scope of special functions is extensive, yet it constantly expands due to the development of new issues in engineering and applied science field. In addition, the development of $H$-function and $I$-function is facilitated by dissemination. Jangid et al [8] proposed the incomplete $I$-function and developed various integral transformations for it. A few applications are also presented [9, 16]. The integral equation has been observed in various response-related problems, including diffusion, quening theory, reaction-diffusion, quantam mechanics and a variety of others area of physics, biology, and probability theory [6, 19, 23].

Integer-order calculus is augmented by fractional calculus, which yields more precise results than classical calculus. Therefore, it is widely used in the mathematical modelling of almost all science and engineering, medicine, and education areas

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[11, 2]. Several fractional operators are available to deal with real world problems, such as the Riemann-Liouville integral, Caputo derivative, Caputo-Fabrizo derivatives, Weyl integral, Weyl derivatives, Atangana-Balneau fractional integral, Hilfer fractional derivative, Marichev-Saigo-Maeda (MSM), and many others [20, 17].

Science and engineering frequently benefit from techniques for finding solutions to integral equations. The Fredholm integral equation which incorporates special functions like Hypergeometric functions, Legendre functions, and Fox $H$-functions are presented and explored by many authors [22, 12, 10]. Recently, Bhatter et al [3] have provided the solution of Fredholm-type integral problem involving the incomplete $I$-function and incomplete $\bar{I}$ function in the kernel.

Motivated by the recent research work done by the researchers [3] on incomplete $I$-function, we have defined the modified incomplete $I$-function. Several interesting results have been derived by using Fredholm-type integral equations associated with modified incomplete $I$-function in the kernel. The manuscript is organised in the following manner: In sect.1, some basic definitions of the functions are given. In sect.2, modified incomplete $I$- function is defined. In sect.3, results are derived for Fredholm-type integral equation involving the modified incomplete $I$-function. In sect.4, fractional integration of the modified incomplete $I$-function are derived. In sect.5, fractional differentiation of the modified incomplete $I$-function are established. In sect.6, the conclussion is given.

The classical gamma function $\Gamma(v)$ is expressed as follows [4]:

$$\Gamma(v) = \begin{cases} \int_0^\infty e^{-u}u^{v-1}du, & (\Re(v) > 0), \\ \frac{\Gamma(v+k)}{(v)_k}, & (k \in \mathbb{N}_0; v \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases} \tag{1.1}$$

here, $(v)_k$ denotes the Pochhammer symbol and it is defined as

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} v(v+k)\cdots(v+k-1), & (k \in \mathbb{N}, v \in \mathbb{C}), \\ 1, & (k = 0, v \in \mathbb{C} \setminus \{0\}). \end{cases} \tag{1.2}$$

The usual incomplete gamma functions $\gamma(v, s)$ and $\Gamma(v, s)$ are represented by [7]:

$$\gamma(v, s) = \int_0^s e^{-u}u^{v-1}du, \ (\Re(v) > 0; s \geq 0), \tag{1.3}$$

and

$$\Gamma(v, s) = \int_s^\infty e^{-u}u^{v-1}du, \ (\Re(v) > 0; s \geq 0), \tag{1.4}$$

satisfy the subsequent rule of decompostion

$$\gamma(v, s) + \Gamma(v, s) = \Gamma(v), \ (\Re(v) > 0), \tag{1.5}$$

where $\Re(v)$ stands for real part of the parameter $v$.
Moreover, if we take $s = 0$, then we have $\Gamma(v, s) = \Gamma(v)$.

***H-Function*:** The well-known Fox $H$-function is defined by [14]:

$$H_{r,s}^{u,v}(t) = H_{r,s}^{u,v}\left[ t \left| \begin{matrix} (a_1, \alpha_1), (a_2, \alpha_2), \ldots, (a_r, \alpha_r) \\ (b_1, \beta_1), (b_2, \beta_2), \ldots, (b_s, \beta_s) \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i}\int_{\$}\psi(w)t^{w}dw, \tag{1.6}$$

where

$$\psi(w) = \frac{\prod_{i=1}^{u}\Gamma(b_i-\beta_i w)\prod_{i=1}^{v}\Gamma(1-a_i+\alpha_i w)}{\prod_{i=v+1}^{r}\Gamma(a_i-\alpha_i w)\prod_{i=u+1}^{s}\Gamma(1-b_i-\beta_i w)} \tag{1.7}$$

***Incomplete H-Function*:** The incomplete $H$-function ${}^{\gamma}H_{r,s}^{u,v}(t)$ and ${}^{\Gamma}H_{r,s}^{u,v}(t)$, which is a generalization of $H$-function is given by [14]:

$$\begin{aligned}{}^{\gamma}H_{r,s}^{u,v}(t) = {}^{\gamma}H_{r,s}^{u,v}\left[t\,\middle|\,\begin{matrix}(a_1,\alpha_1:Y),(a_2,\alpha_2),\ldots,(a_r,\alpha_r)\\(b_1,\beta_1),(b_2,\beta_2),\ldots,(b_s,\beta_s)\end{matrix}\right]\\ = \frac{1}{2\pi i}\int_{\$}\psi(w,Y)t^{w}dw,\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}{}^{\Gamma}H_{r,s}^{u,v}(t) = {}^{\Gamma}H_{r,s}^{u,v}\left[t\,\middle|\,\begin{matrix}(a_1,\alpha_1:Y),(a_2,\alpha_2),\ldots,(a_r,\alpha_r)\\(b_1,\beta_1),(b_2,\beta_2),\ldots,(b_s,\beta_s)\end{matrix}\right]\\ = \frac{1}{2\pi i}\int_{\$}\phi(w,Y)t^{w}dw,\end{aligned} \tag{1.9}$$

where

$$\psi(w,Y) = \frac{\gamma(1-a_1+\alpha_1 w,Y)\prod_{i=1}^{u}\Gamma(b_i-\beta_i w)\prod_{i=2}^{v}\Gamma(1-a_i+\alpha_i w)}{\prod_{i=v+1}^{r}\Gamma(a_i-\alpha_i w)\prod_{i=u+1}^{s}\Gamma(1-b_i-\beta_i w)} \tag{1.10}$$

and

$$\phi(w,Y) = \frac{\Gamma(1-a_1+\alpha_1 w,Y)\prod_{i=1}^{u}\Gamma(b_i-\beta_i w)\prod_{i=2}^{v}\Gamma(1-a_i+\alpha_i w)}{\prod_{i=v+1}^{r}\Gamma(a_i-\alpha_i w)\prod_{i=u+1}^{s}\Gamma(1-b_i-\beta_i w)} \tag{1.11}$$

***I-Function*:** In 1997, Raithe [21] discovered the $I$-function, which is defined as:

$$\begin{aligned}I_{r,s}^{u,v}(t) = I_{r,s}^{u,v}\left[t\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1),(a_2,\alpha_2;A_2),\ldots,(a_r,\alpha_r;A_r)\\(b_1,\beta_1;B_1),(b_2,\beta_2;B_2),\ldots,(b_s,\beta_s;B_s)\end{matrix}\right]\\ = \frac{1}{2\pi i}\int_{\$}\psi(w)t^{w}dw,\end{aligned} \tag{1.12}$$

where

$$\psi(w) = \frac{\prod_{i=1}^{u}\{\Gamma(b_i-\beta_i w)\}^{B_i}\prod_{i=1}^{v}\{\Gamma(1-a_i+\alpha_i w)\}^{A_i}}{\prod_{i=v+1}^{r}\{\Gamma(a_i-\alpha_i w)\}^{A_i}\prod_{i=u+1}^{s}\{\Gamma(1-b_i-\beta_i w)\}^{B_i}} \tag{1.13}$$

***Incomplete I-Function*:** The incomplete $I$-function ${}^{\gamma}I_{r,s}^{u,v}(t)$ and ${}^{\Gamma}I_{r,s}^{u,v}(t)$, which is a generalization of $I$-function is given by [23]:

$$\begin{aligned}{}^{\gamma}I_{r,s}^{u,v}(t) = {}^{\gamma}I_{r,s}^{u,v}\left[t\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_2,\alpha_2;A_2),\ldots,(a_r,\alpha_r;A_r)\\(b_1,\beta_1;B_1),(b_2,\beta_2;B_2),\ldots,(b_s,\beta_s;B_s)\end{matrix}\right]\\ = \frac{1}{2\pi i}\int_{\$}\psi(w,Y)t^{w}dw,\end{aligned} \tag{1.14}$$

and

$$\begin{aligned}{}^{\Gamma}I_{r,s}^{u,v}(t) = {}^{\Gamma}I_{r,s}^{u,v}\left[t\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_2,\alpha_2;A_2),\ldots,(a_r,\alpha_r;A_r)\\(b_1,\beta_1;B_1),(b_2,\beta_2;B_2),\ldots,(b_s,\beta_s;B_s)\end{matrix}\right]\\ = \frac{1}{2\pi i}\int_{\$}\phi(w,Y)t^{w}dw,\end{aligned} \tag{1.15}$$

where

$$\psi(w,Y)=\frac{\{\gamma(1-a_1+\alpha_1 w,Y)\}^{A_1}\prod_{i=1}^{u}\{\Gamma(b_i-\beta_i w)\}^{B_i}\prod_{i=2}^{v}\{\Gamma(1-a_i+\alpha_i w)\}^{A_i}}{\prod_{i=v+1}^{r}\{\Gamma(a_i-\alpha_i w)\}^{A_i}\prod_{i=u+1}^{s}\{\Gamma(1-b_i-\beta_i w)\}^{B_i}} \tag{1.16}$$

and

$$\phi(w,Y)=\frac{\{\Gamma(1-a_1+\alpha_1 w,Y)\}^{A_1}\prod_{i=1}^{u}\{\Gamma(b_i-\beta_i w)\}^{B_i}\prod_{i=2}^{v}\{\Gamma(1-a_i+\alpha_i w)\}^{A_i}}{\prod_{i=v+1}^{r}\{\Gamma(a_i-\alpha_i w)\}^{A_i}\prod_{i=u+1}^{s}\{\Gamma(1-b_i-\beta_i w)\}^{B_i}} \tag{1.17}$$

The incomplete $I$-function ${}^{\gamma}I_{r,s}^{u,v}(t)$ and ${}^{\Gamma}I_{r,s}^{u,v}(t)$ identified in (1.14) and (1.15) appear for $Y \geq 0$, according to the family of restrictions provided by Raithe [?], such as

$$\delta > 0,\ |\arg(t)| < \frac{1}{2}\delta\pi,$$

where

$$\delta=\sum_{i=1}^{u}B_i\beta_i-\sum_{i=u+1}^{s}B_i\beta_i+\sum_{i=1}^{v}A_i\alpha_i-\sum_{i=v+1}^{r}A_i\alpha_i \tag{1.18}$$

***Weyl Fractional Integral:*** The standard definition of Weyl fractional integral (WFI) is given by Miller and Ross [15]:

$$W^{-u}\phi(w)=\frac{1}{\Gamma(u)}\int_{w}^{\infty}(\xi-w)^{u-1}\phi(\xi)d\xi \tag{1.19}$$

***Mellin Transform:*** The standard definition of Mellin transform (MT) is given as follows [15]:

$$M[\phi(w);P]=\hat{\phi}(P)=\int_{0}^{\infty}w^{P-1}\phi(w)dw \tag{1.20}$$

and

$$M^{-1}[\hat{\phi}(P);w]=\phi(w)=\frac{1}{2\pi i}\int_{\nabla-i\infty}^{\nabla+i\infty}w^{-P}\hat{\phi}(P)dP \tag{1.21}$$

where $\Re(P)>0$ and $\nabla$ is constant.

Next, we provide a significant results incorporated to our main findings.

The MT of incomplete $I$-function is defined as follows [8]:

$$M\left\{{}^{\Gamma}I_{r,s}^{u,v}\left[c\,t^{\$}\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_2,\alpha_2;A_2),\ldots,(a_r,\alpha_r;A_r)\\(b_1,\beta_1;B_1),(b_2,\beta_2;B_2),\ldots,(b_s,\beta_s;B_s)\end{matrix}\right];P\right\}=\frac{c^{\frac{-P}{\$}}}{\$}\phi\left(\frac{-P}{\$},Y\right) \tag{1.22}$$

where $\phi\left(\frac{-P}{\$},Y\right)$ is given by equation (1.11).

and

$$M\left\{{}^{\gamma}I_{r,s}^{u,v}\left[c\,t^{\$}\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_2,\alpha_2;A_2),\ldots,(a_r,\alpha_r;A_r)\\(b_1,\beta_1;B_1),(b_2,\beta_2;B_2),\ldots,(b_s,\beta_s;B_s)\end{matrix}\right];P\right\}=\frac{c^{\frac{-P}{\$}}}{\$}\psi\left(\frac{-P}{\$},Y\right) \tag{1.23}$$

where $\psi\left(\frac{-P}{\$},Y\right)$ is given by equation (1.10).

***MSM fractional operator:*** For $\mu, \mu', \nu, \nu', \tau \in \mathbb{C}$ and $x > 0$ with $\Re(\tau) > 0$, the left and right hand sided Marichev-Saigo-Maeda (MSM) fractional integral operators are defined as [13]:

$$\big(I_{0^+}^{\mu,\mu',\nu,\nu',\tau} f\big)(x) = \frac{x^{-\mu}}{\Gamma(\tau)} \int_0^x (x-y)^{\tau-1} y^{-\mu'} F_3\Big(\mu, \mu', \nu, \nu'; \tau; 1-\frac{y}{x}, 1-\frac{x}{y}\Big) f(y)dy \tag{1.24}$$

and

$$\big(I_{0^-}^{\mu,\mu',\nu,\nu',\tau} f\big)(x) = \frac{x^{-\mu'}}{\Gamma(\tau)} \int_x^\infty (y-x)^{\tau-1} y^{-\mu} F_3\Big(\mu, \mu', \nu, \nu'; \tau; 1-\frac{x}{y}, 1-\frac{y}{x}\Big) f(y)dy \tag{1.25}$$

In the same manner, the left and right hand sided Marichev-Saigo-Maeda (MSM) fractional differential operators are defined as [13]:

$$\big(D_{0^+}^{\mu,\mu',\nu,\nu',\tau} f\big)(x) = \Big(\frac{d}{dx}\Big)^\alpha \big(I_{0^+}^{\mu',-\mu,-\nu'+\alpha,-\nu,-\tau+\alpha} f\big)(x) \tag{1.26}$$

and

$$\big(D_{0^-}^{\mu,\mu',\nu,\nu',\tau} f\big)(x) = \Big(-\frac{d}{dx}\Big)^\alpha \big(I_{0^-}^{-\mu',-\mu,-\nu',-\nu+\alpha,-\tau+\alpha} f\big)(x) \tag{1.27}$$

where $\alpha = \Re[\tau] + 1$ and $\Re(\tau)$ symbolizes the integer part in regard to $\Re(\tau)$.

If $\max\{|x|, |y|\} < 1$, then the third Appell function $F_3$ is defined as

$$F_3(\mu, \mu', \nu, \nu'; \tau; x; y) = \sum_{i,j=0}^{\infty} \frac{(\mu)_i(\mu')_j(\nu)_i(\nu')_j}{(\tau)_{i+j}} \frac{x^i y^j}{i!j!} \tag{1.28}$$

Here, $(\mu)_n$ is the well-known Pochhammer symbol.

## 2. Modified Incomplete *I*-Fuction

The modified incomplete $I$-function ${}^{\gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}(t)$ and ${}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}(t)$ are defined by

$$\begin{aligned} {}^{\gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}(t) &= {}^{\gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[ t \,\middle|\, \begin{matrix} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X), \\ (c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] \\ &= \frac{1}{2\pi i}\int_{\$} \psi(w,Y,X) t^w dw, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}(t) &= {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[ t \,\middle|\, \begin{matrix} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X), \\ (c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] \\ &= \frac{1}{2\pi i}\int_{\$} \phi(w,Y,X) t^w dw, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \psi(w,Y,X) &= \frac{\{\gamma(1-a_1+\alpha_1 w, Y)\}^{A_1}\prod_{i=1}^{m_1}\{\Gamma(b_i-\beta_i w)\}^{B_i}\prod_{i=2}^{n_1}\{\Gamma(1-a_i+\alpha_i w)\}^{A_i}}{\prod_{i=m_1+1}^{q_1}\{\Gamma(1-b_i+\beta_i w)\}^{B_i}\prod_{i=n_1+1}^{p_1}\{\Gamma(a_i-\alpha_i w)\}^{A_i}} \\ &\times \frac{\{\gamma(1-c_1+\gamma_1 w, X)\}^{A_1}\prod_{i=1}^{m_2}\{\Gamma(d_i-\delta_i w)\}^{D_i}\prod_{i=2}^{n_2}\{\Gamma(1-c_i+\gamma_i w)\}^{C_i}}{\prod_{i=m_2+1}^{q_2}\{\Gamma(1-d_i-\delta_i w)\}^{D_i}\prod_{i=n_2+1}^{p_2}\{\Gamma(c_i-\gamma_i w)\}^{C_i}} \end{aligned} \tag{2.3}$$

and

$$\phi(w,Y,X)=\frac{\{\Gamma(1-a_1+\alpha_1 w,Y)\}^{A_1}\prod_{i=1}^{m_1}\{\Gamma(b_i-\beta_i w)\}^{B_i}\prod_{i=2}^{n_1}\{\Gamma(1-a_i+\alpha_i w)\}^{A_i}}{\prod_{i=m_1+1}^{q_1}\{\Gamma(1-b_i+\beta_i w)\}^{B_i}\prod_{i=n_1+1}^{p_1}\{\Gamma(a_i-\alpha_i w)\}^{A_i}}$$
$$\times\frac{\{\Gamma(1-c_1+\gamma_1 w,X)\}^{C_1}\prod_{i=1}^{m_2}\{\Gamma(d_i-\delta_i w)\}^{D_i}\prod_{i=2}^{n_2}\{\Gamma(1-c_i+\gamma_i w)\}^{C_i}}{\prod_{i=m_2+1}^{q_2}\{\Gamma(1-d_i-\delta_i w)\}^{D_i}\prod_{i=n_2+1}^{p_2}\{\Gamma(c_i-\gamma_i w)\}^{C_i}} \tag{2.4}$$

## 3. Solution of Fredholm type integral equation associated with modified incomplete $I$-function

In this section, we obtain different properties of modified incomplete $I$-fuction by using MT method and the well-recognized Weyl Fractional Integral.

**Lemma 3.1.** *Let,*
$(A)$ $m_1,n_1,m_2,n_2,p_1,q_1,r_1,s_1\in\mathbb{Z}_{0+}$ *such that* $0\le n_1\le p_1,\ 1\le m_1\le q_1,$
$0\le n_2\le p_2,\ 0\le n_2\le q_2,$

$(B)$ $\Re(\wedge-k)>0;\ \Re(k)+\$\Re(\frac{\phi_i}{B_i})>0,\ (i=1,2,...,u),$

$(C)$ $Y,X\ge 0,\ \$>0,$ *and* $\wedge\in\mathbb{C}$

$(D)$ $|arg(C)|<\frac{1}{2}\pi\delta,$ *provided* $\delta$ *is characterized in the relation* (1.18).
*Then,*

$$W^{-\wedge}\Bigg\{(1-kt)^{-\wedge}$$
$$\times\,{}^{\Gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[c\left(\frac{X}{1-kt}\right)^{\$}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg\}$$
$$=t^{\wedge-k}(1-kt)^{-\wedge}$$
$$\times\,{}^{\Gamma}I^{m_1,n_1+1;m_2,n_2}_{p_1+1,q_1+1;p_2,q_2}\left[c\left(\frac{X}{1-kt}\right)^{\$}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(1-k,\$,1)(a_i,\alpha_i;A_i)_{2,p_1};\\(c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1},(1-\wedge,\$,1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]. \tag{3.1}$$

*Proof.* Consider the left hand side of the above equation, i.e

$$W^{k-\wedge}\Bigg\{(1-kt)^{-\wedge}$$
$$\times\,{}^{\Gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[c(\frac{X}{1-kt})^{\$}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg\}$$
$$=W^{k-\wedge}\left\{(1-kt)^{-\wedge}\frac{1}{2\pi i}\int_{\$}c^w\left(\frac{X}{1-kt}\right)^{\$w}\phi(w,Y,X)dw\right\}$$
$$=\frac{1}{\Gamma(\wedge-k)}\int_t^{\infty}(U-t)^{\wedge-k-1}\left((1-kU)^{-\wedge}\frac{1}{2\pi i}\int_{\$}c^w\left(\frac{X}{1-kU}\right)^{\$w}\phi(w,Y,X)dw\right)dU$$
$$=\frac{1}{2\pi i}\frac{1}{\Gamma(\wedge-k)}\int_{\$}c^wX^{\$w}\phi(w,Y,X)\left(\int_t^{\infty}(U-t)^{\wedge-k-1}(1-kU)^{-\wedge-\$w}dU\right)dw$$

$$= \frac{1}{2\pi i}\frac{1}{\Gamma(\wedge - k)}\int_{\$} c^w X^{\$w}\phi(w,Y,X)\left(\int_t^{\infty} t^{\wedge-k-1}\left(\frac{U}{t}-1\right)^{\wedge-k-1}(1-kU)^{-\wedge-\$w}dU\right)dw$$

on taking the subsitution, $\frac{U}{t} = \frac{1}{y}$ and after calculating the terms we get

$$= \frac{1}{2\pi i}\frac{1}{\Gamma(\wedge - k)}\int_{\$} c^w X^{\$w}\phi(w,Y,X)\left(\int_0^1 t^{\wedge-k}(1-kt)^{-\wedge-\$w}(1-y)^{k-1}y^{k+\$w-1}dy\right)dw$$

$$= t^{\wedge-k}(1-kt)^{-\wedge}\frac{1}{2\pi i}\int_{\$} c^w\left(\frac{X}{1-kt}\right)^{\$w}\phi(w,Y,X)\frac{\Gamma(k+\$w)}{\Gamma(\wedge+\$w)}dw$$

$$= t^{\wedge-k}(1-kt)^{-\wedge}$$

$$\times I^{m_1,n_1+1;m_2,n_2}_{p_1+1,q_1+1;p_2,q_2}\left[c\left(\frac{X}{1-kt}\right)^{\$}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(1-k,\$,1)(a_i,\alpha_i;A_i)_{2,p_1};\\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2}\\ (b_i,\beta_i;B_i)_{1,q_1},(1-\wedge,\$,1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]$$

□

**Lemma 3.2.** *Let,*

*(A)* $m_1, n_1, m_2, n_2, p_1, q_1, r_1, s_1 \in \mathbb{Z}_{0+}$ *such that* $0 \le n_1 \le p_1,\ 1 \le m_1 \le q_1,$

$0 \le n_2 \le p_2,\ 0 \le n_2 \le q_2,$

*(B)* $\Re(\wedge - k) > 0;\ \Re(k) + \$\Re(\frac{\phi_i}{B_i}) > 0,\ (i = 1, 2, ..., u),$

*(C)* $Y, X \ge 0,\ \$ > 0,$ *and* $\wedge \in \mathbb{C}$

*(D)* $|arg(C)| < \frac{1}{2}\pi\delta$*, provided* $\delta$ *is characterized in the relation* (1.18)

$$W^{K-\wedge}\Bigg\{(1-kt)^{-\wedge}$$

$$\times\ {}^{\gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[c\left(\frac{X}{1-kt}\right)^{\$}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\ (c_i,\gamma_i,C_i)_{2,p_2}\\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg\}$$

$$= t^{\wedge-k}(1-kt)^{-\wedge}$$

$$\times\ {}^{\gamma}I^{m_1,n_1+1;m_2,n_2}_{p_1+1,q_1+1;p_2,q_2}\left[c\left(\frac{X}{1-kt}\right)^{\$}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(1-k,\$,1)(a_i,\alpha_i;A_i)_{2,p_1};\\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2}\\ (b_i,\beta_i;B_i)_{1,q_1},(1-\wedge,\$,1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right] \tag{3.2}$$

*Proof.* The proof is similar of lemma (3.1). □

**Theorem 3.3.** *Let,*

*(A)* $m_1, n_1, m_2, n_2, p_1, q_1, r_1, s_1 \in \mathbb{Z}_{0+}$ *such that* $0 \le n_1 \le p_1, 1 \le m_1 \le q_1,$

$0 \le n_2 \le p_2, 0 \le n_2 \le q_2,$

*(B)* $a_{p_1},\ \beta_{q_1},\ c_{p_2},\ \delta_{q_2}$ *are positive real numbers,*

*(C)* $\Re(k) + \$\left(\frac{a_{p_i}-1}{A_i}\right) < 0; \Re(k) + \$\left(\frac{b_i}{B_i}\right) > 0,\ (i = 1, 2, \cdots, m_1),\ (i = 1, 2, \cdots, m_2),$

*(D)* $Y, X \ge 0,\ \$ > 0,$ *and* $\wedge \in \mathbb{C}.$

*Then, the following integral relation holds:*

$$\int_0^\infty {}^{\Gamma}I^{m_1,n_1+1;m_2,n_2}_{p_1+1,q_1+1;p_2,q_2}\left[ c\left(\frac{X}{1-kt}\right)^{\$} \middle| \begin{matrix} (a_1,\alpha_1;A_1:Y),(1-k,\$,1)(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1},(1-\wedge,\$,1);(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] \times (1-kt)^{-k} f(t)dt$$

$$= \int_0^\infty {}^{\Gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[ c\left(\frac{X}{1-kt}\right)^{\$} \middle| \begin{matrix} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] \times (1-kt)^{-k} D^{K-\wedge}\{\phi(t)\}dt. \tag{3.3}$$

*Proof.* Let, the left hand side is denoted by $G$. i.e,

$$G = \int_0^\infty {}^{\Gamma}I^{m_1,n_1+1;m_2,n_2}_{p_1+1,q_1+1;p_2,q_2}\left[ c\left(\frac{X}{1-kt}\right)^{\$} \middle| \begin{matrix} (a_1,\alpha_1;A_1:Y),(1-k,\$,1)(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1},(1-\wedge,\$,1);(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] \times (1-kt)^{-k} f(t)dt$$

by using lemma (3.1) and the definition (1.19) we get

$$= \int_0^\infty \phi(t)\Bigg(\int_t^\infty (1-kU)^{-\wedge}\frac{(U-t)^{\wedge-k-1}}{\Gamma(\wedge-k)} \times {}^{\Gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[ c\left(\frac{X}{1-kU}\right)^{\$} \middle| \begin{matrix} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] dU\Bigg)dt$$

by changing the order of integeration under the allowable circumstances, we have

$$= \int_0^\infty (1-kU)^{-\wedge}\, {}^{\Gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[ c\left(\frac{X}{1-kU}\right)^{\$} \middle| \begin{matrix} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] \times \left(\int_0^U \frac{(U-t)^{\wedge-k-1}}{\Gamma(\wedge-k)}\phi(t)dt\right)dU$$

Now by using the Riemann-Liouville fractional derivatives, we get

$$= {}^{\Gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[ c\left(\frac{X}{1-kU}\right)^{\$} \middle| \begin{matrix} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] \times (1-kU)^{-\wedge} D^{k-\wedge}\{\phi(U)\}dU.$$

which is the right hand side of the proof. □

**Theorem 3.4.** *Let,*
*(A)* $m_1, n_1, m_2, n_2, p_1, q_1, r_1, s_1 \in \mathbb{Z}_{0+}$ *such that* $0 \le n_1 \le p_1, 1 \le m_1 \le q_1$, $0 \le n_2 \le p_2, 0 \le n_2 \le q_2$,

*(B)* $a_{p_1}$, $\beta_{q_1}$, $c_{p_2}$, $\delta_{q_2}$ *are positive real numbers,*

*(C)* $\Re(k) + \$\big(\frac{a_{p_i}-1}{A_i}\big) < 0$; $\Re(k) + \$\big(\frac{b_i}{B_i}\big) > 0$, $(i = 1, 2, \cdots, m_1)$, $(i = 1, 2, \cdots, m_2)$,

*(D)* $Y, X \geq 0,\ \$ > 0,$ *and* $\wedge \in \mathbb{C}$.

*then, the following integral relation holds:*

$$\int_0^\infty {}_{\gamma}I^{m_1,n_1+1;m_2,n_2}_{p_1+1,q_1+1;p_2,q_2}\left[ c\left(\frac{X}{1-kt}\right)^{\$} \middle| \begin{array}{c} (a_1,\alpha_1;A_1:Y),(1-k,\$,1)(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1},(1-\wedge,\$,1);(d_i,\delta_i;D_i)_{1,q_2} \end{array} \right] \times (1-kt)^{-k}\phi(t)dt$$

$$= \int_0^\infty {}_{\gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[ c\left(\frac{X}{1-kt}\right)^{\$} \middle| \begin{array}{c} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{array} \right] \times (1-kt)^{-k}D^{K-\wedge}\{\phi(t)\}dt \quad (3.4)$$

*Proof.* The proof is similar of theorem (3.3). □

**Theorem 3.5.** *Let,*
*(A)* $m_1, n_1, m_2, n_2, p_1, q_1, r_1, s_1 \in \mathbb{Z}_{0+}$ *such that* $0 \leq n_1 \leq p_1, 1 \leq m_1 \leq q_1$,
$0 \leq n_2 \leq p_2, 0 \leq n_2 \leq q_2$,

*(B)* $a_{p_1}$, $\beta_{q_1}$, $c_{p_2}$, $\delta_{q_2}$ *are positive real numbers,*

*(C)* $\Re(k)+\$\left(\frac{a_{p_i}-1}{A_i}\right) < 0$; $\Re(k)+\$\left(\frac{b_i}{B_i}\right) > 0$, $(i = 1, 2, \cdots, m_1)$, $(i = 1, 2, \cdots, m_2)$,

*(D)* $Y, X \geq 0,\ \$ > 0,$ *and* $\wedge \in \mathbb{C}$.

*Then, the consequent IE:*

$$\int_0^\infty {}_{\Gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[ c\left(\frac{X}{1-kt}\right)^{\$} \middle| \begin{array}{c} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{array} \right] \times (1-kt)^{-k}\{\phi(t)\}dt = \psi(X) \quad (3.5)$$

*has a solution given by*

$$\phi(X) = \frac{\$c^{\frac{P}{\$}(1-kX)^{\wedge-1}}}{2\pi i}\int_{\$}(1-kX)^{-P}\left[\phi\left(\frac{-P}{\$},Y\right)\right]^{-1}E(P)dP,$$

*where*

$$E(P) = \int_0^\infty X^{P-1}\psi(X)dx$$

*and* $\phi\left(\frac{-P}{\$}, Y, X\right)$ *is shown in equation* (1.17).

*Proof.* To find the solution of integral equation $f$ is replaced by $D^{\wedge-k}f$ in equation (3.3), we have

$$\int_0^\infty (1-kt)^{-k}\, {}_{\Gamma}I^{m_1,n_1+1;m_2,n_2}_{p_1+1,q_1+1;p_2,q_2}\left[ c\left(\frac{X}{1-kt}\right)^{\$} \middle| \begin{array}{c} (a_1,\alpha_1;A_1:Y),(1-k,\$,1)(a_i,\alpha_i;A_i)_{2,p_1}; \\ (c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1},(1-\wedge,\$,1);(d_i,\delta_i;D_i)_{1,q_2} \end{array} \right] \times D^{k-\wedge}\{\phi(t)\}dt = \psi(X)$$

On multiplying both sides by $X^{P-1}$, integrating from 0 to $\infty$ with respect to X, and thereupon altering the order of integration together with the allowable circumstances, we have

$$E(P)=\int_0^\infty X^{P-1}\psi(X)dx=\int_0^\infty (1-kt)^{-k}D^{k-\wedge}\{\phi(t)\}$$

$$\times\left(X^{P-1}\int_0^\infty {}^{\Gamma}I^{m_1,n_1+1;m_2,n_2}_{p_1+1,q_1+1;p_2,q_2}\left[c\left(\frac{X}{1-kt}\right)^{\$}\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(1-k,\$,1)(a_i,\alpha_i;A_i)_{2,p_1};\\(c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1},(1-\wedge,\$,1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]dX\right)dt$$

Now by using equation (1.22), we obtain

$$E(P)=\frac{\Gamma(k-P)}{\Gamma(\wedge-P)}\frac{c^{\frac{-P}{\$}}}{\$}\phi\left(\frac{-P}{\$},Y\right)\int_0^\infty (1-kt)^{P-k}D^{\wedge-k}\phi\{t\}dt$$

Moreover, by using Mellin inversion theorem, we obtain

$$D^{\wedge-k}\{\phi(t)\}=\frac{\$}{2\pi i}\int_{\$}(1-kt)^{P-k-1}\frac{\Gamma(\wedge-P)}{\Gamma(k-P)}c^{\frac{P}{\$}}\left[\phi\left(\frac{-P}{\$},Y\right)\right]^{-1}E(P)dP$$

Now, by operating on each side with $D^{k-\wedge}$, we obtain

$$\phi(t)=\frac{\$}{2\pi i}D^{k-\wedge}\left\{\int_{\$}(1-kt)^{P-k-1}\frac{\Gamma(\wedge-P)}{\Gamma(k-P)}c^{\frac{P}{\$}}\left[\phi\left(\frac{-P}{\$},Y\right)\right]^{-1}E(P)dP\right\}$$

which eventually gives

$$\phi(X)=\frac{\$c^{\frac{P}{\$}(1-kX)^{\wedge-1}}}{2\pi i}\int_{\$}(1-kX)^{-P}\left[\phi\left(\frac{-P}{\$},Y\right)\right]^{-1}E(P)dP$$

□

**Theorem 3.6.** *Let,*
*(A)* $m_1,n_1,m_2,n_2,p_1,q_1,r_1,s_1\in\mathbb{Z}_{0+}$ *such that* $0\le n_1\le p_1, 1\le m_1\le q_1,$
$0\le n_2\le p_2, 0\le n_2\le q_2,$

*(B)* $a_{p_1}$, $\beta_{q_1}$, $c_{p_2}$, $\delta_{q_2}$ *are positive real numbers,*

*(C)* $\Re(k)+\$(\frac{a_{p_i}-1}{A_i})<0$; $\Re(k)+\$(\frac{b_i}{B_i})>0$, $(i=1,2,\cdots,m_1)$, $(i=1,2,\cdots,m_2)$,

*(D)* $Y,X\ge 0$, $\$>0$, *and* $\wedge\in\mathbb{C}$.

*Then, the consequent IE:*

$$\int_0^\infty {}^{\gamma}I^{m_1,n_1;m_2,n_2}_{p_1,q_1;p_2,q_2}\left[c\left(\frac{X}{1-kt}\right)^{\$}\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};\\(c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\times(1-kt)^{-k}\{\phi(t)\}dt=\psi(X) \quad (3.6)$$

*has a solution given by*

$$\phi(X)=\frac{\$c^{\frac{P}{\$}(1-kX)^{\wedge-1}}}{2\pi i}\int_{\$}(1-kX)^{-P}\left[\phi\left(\frac{-P}{\$},Y\right)\right]^{-1}E(P)dP,$$

*where*

$$E(P)=\int_0^\infty X^{P-1}\psi(X)dx$$

*and* $\phi\left(\frac{-P}{\$}, Y, X\right)$ *is shown in equation* (1.10).

*Proof.* The proof is similar of theorem (3.3). □

## 4. Fractional Integration of Modified Incomplete *I*-function

In this section, we shall investigate the left and right hand sided MSM fractional order integrals of modified incomplete $I$-function.

The followings are well-known results (see [5]) and will be needed in proving the subsequent theorems.

**Lemma 4.1.** *Let* $\mu, \mu', \nu, \nu', \tau, \lambda \in \mathbb{C}$ *and* $\tau > 0$.

(1) *If* $\Re(\lambda) > \max\{0, \Re(\mu' - \nu'), \Re(\nu + \nu' + \mu - \tau)\}$, *then*

$$\left(I_{0+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}\right)(x) = (1-kt)^{-\mu-\mu'+\tau+\lambda-1}\frac{\Gamma(\lambda)\Gamma(-\mu'+\nu'+\lambda)\Gamma(-\mu-\mu'-\nu+\tau+\lambda)}{\Gamma(\nu'+\lambda)\Gamma(-\mu-\mu'+\tau+\lambda)\Gamma(\mu'-\nu+\tau+\lambda)} \tag{4.1}$$

(2) *If* $\Re(\lambda) > \max\{\Re(\nu), \Re(-\mu-\mu'+\tau), \Re(-\mu-\nu'+\tau)\}$, *then*

$$\left(I_{0-}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda}\right)(x) = (1-kt)^{-\mu-\mu'+\tau-\lambda}\frac{\Gamma(-\nu+\lambda)\Gamma(\mu+\mu'-\tau+\lambda)\Gamma(\mu+\nu'-\tau+\lambda)}{\Gamma(\lambda)\Gamma(\mu-\nu+\lambda)\Gamma(\mu+\mu'+\nu'-\tau+\lambda)} \tag{4.2}$$

**Theorem 4.2.** *Let* $\mu, \mu', \nu, \nu', \tau, \lambda, c \in \mathbb{C}$ *be so that* $\Re(\tau), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(\mu'-\nu'), \Re(\nu+\nu'+\mu-\tau)\}$. *Then for* $x > 0$ *the following relation holds:*

$$\begin{aligned}&\Bigg(I_{0+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}\\ &\times {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{\mu}\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg)(x)\\ =\,&(1-kt)^{-\mu-\mu'+\tau+\lambda-1}\\ &\times {}^{\Gamma}I_{p_1+3,q_1+3;p_2,q_2}^{m_1,n_1+3;m_2,n_2}\left[c(1-kt)^{\mu}\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(1-\lambda,\mu;1),(1+\mu'-\nu'-\lambda,\mu;1),\\(1+\mu+\mu'+\nu-\tau-\lambda,\mu;1)(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1},(1-\nu'-\lambda,\mu;1),(1+\mu+\mu'-\tau-\lambda,\mu;1),\\(1+\mu+\mu'+\nu-\tau-\lambda,\mu;1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\end{aligned} \tag{4.3}$$

*Proof.* Let the left hand side of equation (4.3) is denoted by H. i.e,

$$H = \Bigg( I_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}$$

$$\times\, {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[ c(1-kt)^{\mu} \;\middle|\; \begin{matrix} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X), \\ (c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right]\Bigg)(x)$$

$$= \left( I_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}\left(\frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\; c^w(1-kt)^{\mu w}dw\right)\right)(x)$$

where $\phi(w,Y,X)$ is given in equation (2.4). On interchanging the order of integration, we have

$$= \frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\; c^w\left(I_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda+\mu w-1}\right)(x)dx$$

$$= \frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\; c^w\Bigg((1-kt)^{-\mu-\mu'+\tau+\lambda+\mu w-1}$$

$$\times\frac{\Gamma(\lambda+\mu w)\Gamma(-\mu'+\nu'+\lambda+\mu w)\Gamma(-\mu-\mu'-\nu+\tau+\lambda+\mu w)}{\Gamma(\nu'+\lambda+\mu w)\Gamma(-\mu-\mu'+\tau+\lambda+\mu w)\Gamma(\mu'-\nu+\tau+\lambda+\mu w)}\Bigg)dw$$

$$= \frac{(1-kt)^{-\mu-\mu'+\tau+\lambda-1}}{2\pi i}\int_{\$}\phi(w,Y,X)\big(c(1-kt)^{\mu}\big)^{w}$$

$$\times\left(\frac{\Gamma(\lambda+\mu w)\Gamma(-\mu'+\nu'+\lambda+\mu w)\Gamma(-\mu-\mu'-\nu+\tau+\lambda+\mu w)}{\Gamma(\nu'+\lambda+\mu w)\Gamma(-\mu-\mu'+\tau+\lambda+\mu w)\Gamma(\mu'-\nu+\tau+\lambda+\mu w)}\right)dw$$

after calculating we get the right hand side of the equation (4.3). □

**Theorem 4.3.** *Let $\mu,\mu',\nu,\nu',\tau,\lambda,c\in\mathbb{C}$ be so that $\Re(\tau),\mu>0$ and $\Re(\lambda)>max\{0,\Re(\mu'-\nu'),\Re(\nu+\nu'+\mu-\tau)\}$. Then for $x>0$ the following relation holds:*

$$\Bigg( I_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}$$

$$\times\, {}^{\gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[ c(1-kt)^{\mu} \;\middle|\; \begin{matrix} (a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X), \\ (c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right]\Bigg)(x)$$

$$= (1-kt)^{-\mu-\mu'+\tau+\lambda-1}$$

$$\times\, {}^{\gamma}I_{p_1+3,q_1+3;p_2,q_2}^{m_1,n_1+3;m_2,n_2}\left[ c(1-kt)^{\mu} \;\middle|\; \begin{matrix} (a_1,\alpha_1;A_1:Y),(1-\lambda,\mu;1),(1+\mu'-\nu'-\lambda,\mu;1), \\ (1+\mu+\mu'+\nu-\tau-\lambda,\mu;1)(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X), \\ (c_i,\gamma_i,C_i)_{2,p_2} \\ (b_i,\beta_i;B_i)_{1,q_1},(1-\nu'-\lambda,\mu;1),(1+\mu+\mu'-\tau-\lambda,\mu;1), \\ (1+\mu+\mu'+\nu-\tau-\lambda,\mu;1);(d_i,\delta_i;D_i)_{1,q_2} \end{matrix} \right] \tag{4.4}$$

*Proof.* The proof is similar of theorem (4.1). □

**Theorem 4.4.** *Let $\mu,\mu',\nu,\nu',\tau,\lambda,c\in\mathbb{C}$ be so that $\Re(\tau),\mu>0$ and $\Re(\lambda)>\max\{\Re(\nu),\Re(-\mu-\mu'+\tau),\Re(-\mu-\nu'+\tau)\}$. Then for $x>0$ the following*

*relation holds:*

$$\left(I_{0^-}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda} \times {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{-\mu}\left|\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right.\right]\right)(x)$$
$$= (1-kt)^{-\mu-\mu'+\tau-\lambda} \times {}^{\Gamma}I_{p_1+3,q_1+3;p_2,q_2}^{m_1,n_1+3;m_2,n_2}\left[c(1-kt)^{-\mu}\left|\begin{matrix}(a_1,\alpha_1;A_1:Y),(1+\nu-\lambda,\mu;1),(1-\mu-\mu'+\tau-\lambda,\mu;1),\\(1-\mu-\nu'+\tau-\lambda,\mu;1)(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1},(1-\lambda,\mu;1),(1-\mu+\nu-\lambda,\mu;1),\\(1-\mu-\mu'-\nu'+\tau-\lambda,\mu;1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right.\right] \tag{4.5}$$

*Proof.* Let the left hand side of the above equation is denoted by G. i.e,

$$G = \left(I_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda} \times {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{-\mu}\left|\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right.\right]\right)(x)$$
$$= \left(I_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda}\left(\frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\ c^w(1-kt)^{-\mu w}dw\right)\right)(x)$$

On interchanging the order of integration, we have

$$= \frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\ c^w\left(I_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda-\mu w}\right)(x)dx$$

$$= \frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\ c^w\Bigg((1-kt)^{-\mu-\mu'+\tau-\lambda-\mu w} \times \frac{\Gamma(-\nu+\lambda+\mu w)\Gamma(\mu+\mu'-\tau+\lambda+\mu w)\Gamma(\mu+\nu'-\tau+\lambda+\mu w)}{\Gamma(\lambda+\mu w)\Gamma(\mu-\nu+\lambda+\mu w)\Gamma(\mu+\mu'+\nu'-\tau+\lambda+\mu w)}\Bigg)dw$$

$$= \frac{(1-kt)^{-\mu-\mu'+\tau-\lambda}}{2\pi i}\int_{\$}\phi(w,Y,X)\big(c(1-kt)^{-\mu}\big)^w \times \left(\frac{\Gamma(-\nu+\lambda+\mu w)\Gamma(\mu+\mu'-\tau+\lambda+\mu w)\Gamma(\mu+\nu'-\tau+\lambda+\mu w)}{\Gamma(\lambda+\mu w)\Gamma(\mu-\nu+\lambda+\mu w)\Gamma(\mu+\mu'+\nu'-\tau+\lambda+\mu w)}\right)dw$$

after calculating we get the right hand side of the equation (4.5). □

**Theorem 4.5.** *Let $\mu,\mu',\nu,\nu',\tau,\lambda,c\in\mathbb{C}$ be so that $\Re(\tau),\mu>0$ and $\Re(\lambda)>\max\{\Re(\nu),\Re(-\mu-\mu'+\tau),\Re(-\mu-\nu'+\tau)\}$. Then for $x>0$ the following*

*relation holds:*

$$\begin{aligned}&\Bigg(I_{0^-}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda}\\ &\times {}^{\gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{-\mu}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg)(x)\\ =&(1-kt)^{-\mu-\mu'+\tau-\lambda}\\ &\times {}^{\gamma}I_{p_1+3,q_1+3;p_2,q_2}^{m_1,n_1+3;m_2,n_2}\left[c(1-kt)^{-\mu}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(1+\nu-\lambda,\mu;1),(1-\mu-\mu'+\tau-\lambda,\mu;1),\\(1-\mu-\nu'+\tau-\lambda,\mu;1)(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1},(1-\lambda,\mu;1),(1-\mu+\nu-\lambda,\mu;1),\\(1-\mu-\mu'-\nu'+\tau-\lambda,\mu;1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\end{aligned} \tag{4.6}$$

*Proof.* The proof is similar of theorem (4.3). □

## 5. Fractional Differentiation of Modified Incomplete *I*-function

In this section, we shall investigate the left and right hand sided MSM fractional order derivative of modified incomplete $I$-function.

The followings are well-known results (see [5]) and it will be needed in proving the subsequent theorems.

**Lemma 5.1.** *Let* $\mu,\mu',\nu,\nu',\tau,\lambda\in\mathbb{C}$.

(1) *If* $\Re(\lambda)>\max\{0,\Re(-\mu+\nu),\Re(-\mu-\mu'-\nu'+\tau)\}$, *then*

$$\left(D_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}\right)(x)=(1-kt)^{\mu+\mu'-\tau+\lambda-1}\frac{\Gamma(\lambda)\Gamma(\mu-\nu+\lambda)\Gamma(\mu+\mu'+\nu'-\tau+\lambda)}{\Gamma(-\nu+\lambda)\Gamma(\mu+\mu'-\tau+\lambda)\Gamma(\mu+\nu'-\tau+\lambda)} \tag{5.1}$$

(2) *If* $\Re(\lambda)>\max\{\Re(-\nu'),\Re(\mu'+\nu-\tau),\Re(\mu+\mu'-\tau)+[\Re(\tau)]+1\}$, *then*

$$\left(D_{0^-}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda}\right)(x)=(1-kt)^{\mu+\mu'-\tau-\lambda}\frac{\Gamma(\nu'+\lambda)\Gamma(-\mu-\mu'+\tau+\lambda)\Gamma(-\mu'-\nu'+\tau+\lambda)}{\Gamma(\lambda)\Gamma(-\mu'+\nu'+\lambda)\Gamma(-\mu-\mu'-\nu+\tau+\lambda)} \tag{5.2}$$

**Theorem 5.2.** *Let* $\mu,\mu',\nu,\nu',\tau,\lambda,c\in\mathbb{C}$ *be so that* $\Re(\tau),\mu>0$ *and* $\Re(\lambda)>\max\{0,\Re(-\mu+\nu),\Re(-\mu-\mu'-\nu'+\tau)\}$, *then for* $x>0$ *the following*

*relation holds:*

$$\Bigg(D_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}$$

$$\times\, {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{\mu}\,\middle|\, \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\ (c_i,\gamma_i,C_i)_{2,p_2}\\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg)(x)$$

$$=(1-kt)^{\mu+\mu'-\tau+\lambda-1}$$

$$\times\, {}^{\Gamma}I_{p_1+3,q_1+3;p_2,q_2}^{m_1,n_1+3;m_2,n_2}\left[c(1-kt)^{\mu}\,\middle|\, \begin{matrix}(a_1,\alpha_1;A_1:Y),(1-\lambda,\mu;1),(1-\mu+\nu-\lambda,\mu;1),\\ (1-\mu-\mu'-\nu'+\tau-\lambda,\mu;1)(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\ (c_i,\gamma_i,C_i)_{2,p_2}\\ (b_i,\beta_i;B_i)_{1,q_1},(1+\nu-\lambda,\mu;1),(1-\mu-\mu'+\tau-\lambda,\mu;1),\\ (1-\mu-\nu'+\tau-\lambda,\mu;1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right] \tag{5.3}$$

*Proof.* Let the left hand side of the above equation is denoted by $R$. i.e,

$$R=\Bigg(D_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}$$

$$\times\, {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{\mu}\,\middle|\, \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\ (c_i,\gamma_i,C_i)_{2,p_2}\\ (b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg)(x)$$

$$=\left(D_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1}\left(\frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)c^{w}(1-kt)^{\mu w}dw\right)\right)(x)$$

On interchanging the order of integeration, we have

$$=\frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\ c^{w}\left(D_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda+\mu w-1}\right)(x)dx$$

$$=\frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\ c^{w}\Bigg((1-kt)^{\mu+\mu'-\tau+\lambda+\mu w-1}$$

$$\times\frac{\Gamma(\lambda+\mu w)\Gamma(\mu-\nu+\lambda+\mu w)\Gamma(\mu+\mu'+\nu'-\tau+\lambda+\mu w)}{\Gamma(-\nu+\lambda+\mu w)\Gamma(\mu+\mu'-\tau+\lambda+\mu w)\Gamma(\mu+\nu'-\tau+\lambda+\mu w)}\Bigg)dw$$

$$=\frac{(1-kt)^{\mu+\mu'-\tau+\lambda-1}}{2\pi i}\int_{\$}\phi(w,Y,X)\big(c(1-kt)^{\mu}\big)^{w}$$

$$\times\frac{\Gamma(\lambda+\mu w)\Gamma(\mu-\nu+\lambda+\mu w)\Gamma(\mu+\mu'+\nu'-\tau+\lambda+\mu w)}{\Gamma(-\nu+\lambda+\mu w)\Gamma(\mu+\mu'-\tau+\lambda+\mu w)\Gamma(\mu+\nu'-\tau+\lambda+\mu w)}\Bigg)dw$$

after calculating we get the right hand side of the equation (5.3). □

**Theorem 5.3.** *Let $\mu,\mu',\nu,\nu',\tau,\lambda,c\in\mathbb{C}$ be so that $\Re(\tau),\mu>0$ and $\Re(\lambda)>\max\{0,\Re(-\mu+\nu),\Re(-\mu-\mu'-\nu'+\tau)\}$, then for $x>0$, the following*

*relation holds:*

$$\Bigg(D_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{\lambda-1} \times {}^{\gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{\mu}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg)(x)$$
$$= (1-kt)^{\mu+\mu'-\tau+\lambda-1} \times {}^{\gamma}I_{p_1+3,q_1+3;p_2,q_2}^{m_1,n_1+3;m_2,n_2}\left[c(1-kt)^{\mu}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(1-\lambda,\mu;1),(1-\mu+\nu-\lambda,\mu;1),\\(1-\mu-\mu'-\nu'+\tau-\lambda,\mu;1)(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1},(1+\nu-\lambda,\mu;1),(1-\mu-\mu'+\tau-\lambda,\mu;1),\\(1-\mu-\nu'+\tau-\lambda,\mu;1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right] \tag{5.4}$$

*Proof.* The proof is similar of theorem (5.1). □

**Theorem 5.4.** *Let $\mu,\mu',\nu,\nu',\tau,\lambda,c\in\mathbb{C}$ be such that $\Re(\tau),\mu>0$ and $\Re(\lambda)>\max\{\Re(-\nu'),\Re(\mu'+\nu-\tau),\Re(\nu+\nu'-\tau)+[\Re(\tau)]+1\}$, then for $x>0$ the following relation holds:*

$$\Bigg(D_{0^-}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda} \times {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{-\mu}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg)(x)$$
$$= (1-kt)^{\mu+\mu'-\tau-\lambda-1} \times {}^{\Gamma}I_{p_1+3,q_1+3;p_2,q_2}^{m_1,n_1+3;m_2,n_2}\left[c(1-kt)^{-\mu}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(1-\nu'-\lambda,\mu;1),(1+\mu+\mu'-\tau-\lambda,\mu;1),\\(1+\mu'+\nu-\tau-\lambda,\mu;1)(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1},(1-\lambda,\mu;1),(1+\mu'-\nu'-\lambda,\mu;1),\\(1+\mu+\mu'+\nu-\tau-\lambda,\mu;1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right] \tag{5.5}$$

*Proof.* Let the left hand side of the above equation is denoted by $S$. i.e,

$$S = \Bigg(D_{0^-}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda} \times {}^{\Gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{-\mu}\ \middle|\ \begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg)(x)$$
$$= \left(D_{0^-}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda}\left(\frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\ c^w(1-kt)^{-\mu w}dw\right)\right)(x)$$

On interchanging the order of integeration, we have

$$= \frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\ c^w\left(D_{0^+}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda-\mu w}\right)(x)dx$$

$$= \frac{1}{2\pi i}\int_{\$}\phi(w,Y,X)\; c^{w}\Bigg((1-kt)^{\mu+\mu'-\tau-\lambda-\mu w}$$
$$\times \frac{\Gamma(\nu'+\lambda+\mu w)\Gamma(-\mu-\mu'+\tau+\lambda+\mu w)\Gamma(-\mu'-\nu+\tau+\lambda+\mu w)}{\Gamma(\lambda+\mu w)\Gamma(-\mu'+\nu'+\lambda+\mu w)\Gamma(-\mu-\mu'-\nu+\tau+\lambda+\mu w)}\Bigg)dw$$

$$= \frac{(1-kt)^{\mu+\mu'-\tau-\lambda}}{2\pi i}\int_{\$}\phi(w,Y,X)\big(c(1-kt)^{-\mu}\big)^{w}$$
$$\times\left(\frac{\Gamma(\nu'+\lambda+\mu w)\Gamma(-\mu-\mu'+\tau+\lambda+\mu w)\Gamma(-\mu'-\nu+\tau+\lambda+\mu w)}{\Gamma(\lambda+\mu w)\Gamma(-\mu'+\nu'+\lambda+\mu w)\Gamma(-\mu-\mu'-\nu+\tau+\lambda+\mu w)}\right)dw$$

after calculating we get the right hand side of the equation (5.5). □

**Theorem 5.5.** *Let $\mu,\mu',\nu,\nu',\tau,\lambda,c\in\mathbb{C}$ be so that $\Re(\tau),\mu>0$ and $\Re(\lambda)>\max\{\Re(-\nu'),\Re(\mu'+\nu-\tau),\Re(\mu+\mu-\tau)+[\Re(\tau)]+1\}$, then for $x>0$ the following relation holds:*

$$\Bigg(D_{0^-}^{\mu,\mu',\nu,\nu',\tau}(1-kt)^{-\lambda}$$
$$\times\,{}^{\gamma}I_{p_1,q_1;p_2,q_2}^{m_1,n_1;m_2,n_2}\left[c(1-kt)^{-\mu}\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1};(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\Bigg)(x)$$
$$=(1-kt)^{\mu+\mu'-\tau-\lambda-1}$$
$$\times\,{}^{\gamma}I_{p_1+3,q_1+3;p_2,q_2}^{m_1,n_1+3;m_2,n_2}\left[c(1-kt)^{-\mu}\,\middle|\,\begin{matrix}(a_1,\alpha_1;A_1:Y),(1-\nu'-\lambda,\mu;1),(1+\mu+\mu'-\tau-\lambda,\mu;1),\\(1+\mu'+\nu-\tau-\lambda,\mu;1)(a_i,\alpha_i;A_i)_{2,p_1};(c_1,\gamma_1,C_1,X),\\(c_i,\gamma_i,C_i)_{2,p_2}\\(b_i,\beta_i;B_i)_{1,q_1},(1-\lambda,\mu;1),(1+\mu'-\nu'-\lambda,\mu;1),\\(1+\mu+\mu'+\nu-\tau-\lambda,\mu;1);(d_i,\delta_i;D_i)_{1,q_2}\end{matrix}\right]\tag{5.6}$$

*Proof.* The proof is similar of theorem (5.3). □

## 6. **Conclusions**

This paper introduces the Fredholm-type integral equation involving the modified incomplete $I$-function in the kernel. Several integral problem of this function have been obtained by using Weyl fractional integral and Mellin transform. Marichev-Saigo-Maeda (MSM) fractional operators are also used to find the fractional integration and fractional differentiation of modified incomplete $I$-function. Our conclussions are crucial in many different fields. With their aid, a wide range of fascinating and useful fractional integral equations with applications in engineering, communication theory, probability theory, and science can be created. In the near future work, the solutions of the other differential and integral equations may be obtained by considering the modified incomplete $I$-function in the kernel for more generalization to transcendental problems and these solutions can be represented in terms of modified Incomplete $I$ -function.

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