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MINIMUM BOUNDARY DOMINATING PARTITION ENERGY OF CERTAIN GRAPHS

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Abstract. E Sampathkumar et al.[12] introduced the Partition matrix and it's corresponding energy of graph. Motivated by this, in this paper we discuss the concept of minimum boundary dominating partition energy of a graph, $E_p^D(G)$ and compute the minimum boundary dominating partition energy $E_p^D(G)$ of few families of graphs. Also, few properties with respect to many domination parameters are discussed.

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1. Introduction

In chemistry, hydrocarbons can be represented by molecular graph. We represent every carbon atom by a vertex and every carbon-carbon bond by an edge and hydrogen atoms are ignored. The concept of graph energy basically emanated from chemistry and defined as the sum of the absolute value of the eigenvalues of the adjacency matrix of the graph. 21st century witnessed the rapid growth in spectral graph theory. As a result different graph energies are introduced. Many graph theoritical parameters such as domination, coloring, partitioning, covering etc., are linked with spectral analysis.

2. Minimum Boundary Dominating set and Partition Energy of a Graph

The minimum boundary dominating set was introduced by K M. Kathiresan et al[5]., For a simple graph G with vertex set $V = \{v_1, v_2, v_3, ..., v_n\}$ and edge set E, a vertex v is called a boundary neighbor of u if v is a nearest boundary of u. A vertex u boundary dominate a vertex v if v is a boundary neighbor of u. A subset $B \subseteq V$ is a boundary dominating set if every vertex of $V - B$ is boundary dominated by some vertex of V.

Any boundary dominating set with minimum cardinality is called a minimum boundary dominating set. Let B be a minimum boundary dominating set of a graph G, the minimum boundary dominating partition matrix is given by

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i, v_j \in V_r, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent where } v_i, v_j \in V_r, \\ 1 & \text{if } i = j \text{ and } v_i \in B, \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets} \\ & V_r \text{ and } V_s \text{ for } r \neq s, \text{ where } v_i \in V_r \text{ and } v_j \in V_s, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we study minimum boundary dominating partition energy of a graph with respect to different partitions of graph. Further, we implement this to complements also.

Definition 2.1. [11] *The complement of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.*

3. Properties of Minimum Boundary Dominating Partition Energy of a Graph

In this section, we consider the set α, which represents the (general) domination set. As there are different sort of domination sets such as minimum domination, boundary domination, perfect domination etc., we take a domination set α which may be any sort of dominating set. Let $G = (V, E)$ be a graph with n vertices and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of V. For $1 \leq i \leq k$, let b_i denote the total number of edges joining the vertices of V_i and c_i be the total number of edges joining the vertices from V_i to V_j for $i \neq j, 1 \leq j \leq k$ and d_i be the number of non-adjacent pairs of vertices within V_i. Let $m_1 = \sum_{i=1}^{k} b_i$, $m_2 = \sum_{i=1}^{k} c_i$ and $m_3 = \sum_{i=1}^{k} d_i$.

The following proposition determines the first three coefficients of the characteristic polynomial of $P_k^{\alpha}(G)$.

Proposition 3.1. *The initial three coefficients of $\phi_k^{\alpha}(G, \lambda)$ are given as follows:*
(i) $a_0 = 1$,
(ii) $a_1 = -|\alpha|$,
(iii) $a_2 = |\alpha| C_2 - [4m_1 + m_2 + m_3]$. *(Here α is any dominating set viz, minimum dominating set, global dominating set etc.,)*

Proof. *(i)* This will be proven easily by using the definition of characteristic polynomial.

(ii) The sum of determinants of all 1×1 principal submatrices of $P_k^{\alpha}(G)$ is equal to the trace of $P_k^{\alpha}(G)$.
$\Rightarrow a_1 = (-1)^1$ trace of $[P_k^{\alpha}(G)] = -|\alpha|$.

(*iii*)

$$\begin{aligned}
(-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\
&= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\
&= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\
&= |\alpha|C_2 - [(2)^2 m_1 + (1)^2 m_2 + (-1)^2 m_3] = |\alpha|C_2 - [4m_1 + m_2 + m_3].
\end{aligned}$$

Proposition 3.2. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are partition eigenvalues of $P_k^{\alpha}(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = |\alpha| + 2[4m_1 + m_2 + m_3].$$

Proof. We know that

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ii})^2 \\
&= 2\sum_{i<j} (a_{ij})^2 + |\alpha| \\
&= |\alpha| + 2[4m_1 + m_2 + m_3].
\end{aligned}$$

Theorem 3.3. *Let G be a graph with n vertices and m edges with $(4m_1 + 2m_2 + 2m_3) + |\alpha| \ge n$ and $(4m_1 + 2m_2 - 2m_3 + |\alpha|)^2 - (8m_1 n + 2m_2 n + 2m_3 n + n|\alpha|) \ge n$ then,*

$$[E_{P_k}^{\alpha}]^2 \le \lambda_1 + \sqrt{(n-1)(|\alpha| + [8m_1 + 2m_2 + 2m_3] - \lambda_1^2)}$$

Proof.

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Substitute $a_i = 1, b_i = |\lambda_i|$, then,

$$\left(\sum_{i=2}^{n} |\lambda_i|\right)^2 \le \left(\sum_{i=2}^{n} 1\right)\left(\sum_{i=2}^{n} |\lambda_i|^2\right)$$

$$[E_{P_k}^{\alpha}]^2 \le (n-1)(|\alpha| + [8m_1 + 2m_2 + 2m_3] - \lambda_1^2)$$

$$[E_{P_k}^{\alpha}]^2 \le \lambda_1 + \sqrt{(n-1)(|\alpha| + [8m_1 + 2m_2 + 2m_3] - \lambda_1^2)}$$

Theorem 3.4. *Let G be a graph with n vertices and P_k be a partition of G. Then*

$$E^{\alpha}_{P_k}(G) \leq \sqrt{n(|\alpha| + 2[4m_1 + m_2 + m_3])}$$

where m_1, m_2, m_3 are as defined above for G.

Proof. By using Cauchy - Schwartz inequality, and after substitution of $a_i = 1$, $b_i =| \lambda_i |$, we can get the desired bound.

Theorem 3.5. *Let G be a partition graph with n vertices. If R= det $P^{\alpha}_k(G)$, then*

$$E^{\alpha}_{P_k}(G) \geq \sqrt{(|\alpha| + 2[4m_1 + m_2 + m_3]) + n(n-1)R^{\frac{2}{n}}}.$$

Proof. By using the definition of minimum boundary dominating partition energy, and employing the arithmetic mean and geometric mean inequality, we can get the result easily.

Theorem 3.6. *If the minimum boundary dominating partition energy of a graph is a rational number, then it must be a positive even number.*

Proof of this theorem is similar to the proof of Theorem 2.12 in [1].

Theorem 3.7. *If $\lambda_1(G)$ is the largest minimum dominating eigen value of AD(G),then*

$$\lambda_1(G) \geq \frac{2[4m_1 + m_2 + m_3] + k}{n}$$

where k is the domination number.

Proof. Let X be any nonzero vector .Then by Lemma 3.17 of [2], we have,

$$\lambda_1(G) = \max_{X \neq 0} \frac{X^{'}AX}{X^{'}X}$$

$$\lambda_1(G) \geq \max_{X \neq 0} \frac{J^{'}P^{D}J}{J^{'}J} = \frac{2[4m_1 + m_2 + m_3] + k}{n}$$

where J is a unit matrix.

4. Energy of Some Partition Graphs and Complements

Definition 4.1. *A book graph (B_m) will be having of $m-$quadrilaterals sharing a common edge. That is, it is a Cartesian product $K_{1,n-1}$ and P_2, where $K_{1,n-1}$ is a star graph and P_2 is the path graph with two vertices.*

Theorem 4.2. *The minimum boundary dominating* 1*-partition energy of a Book graph B_m is $E^{B}_{P_1}(B_m) = 6m - 6 + \sqrt{1+36m} + \sqrt{1+4m^2}$.*

Proof. Consider all the vertices is in one partition. The minimum boundary dominating set $= B = \{v_1, v_2\}$. The minimum boundary dominating

1-partition matrix is

$$P_1^B(B_m) = \begin{bmatrix} 1 & 2 & 2 & -1 & 2 & -1 & \dots & 2 & -1 \\ 2 & 1 & -1 & 2 & -1 & 2 & \dots & -1 & 2 \\ 2 & -1 & 0 & 2 & -1 & -1 & \dots & -1 & -1 \\ -1 & 2 & 2 & 0 & -1 & -1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ 2 & -1 & -1 & -1 & 0 & 2 & \dots & -1 & -1 \\ -1 & 2 & 2 & 0 & -1 & -1 & \dots & -1 & -1 \\ 2 & -1 & -1 & -1 & -1 & -1 & \dots & 0 & 2 \\ -1 & 2 & -1 & -1 & -1 & -1 & \dots & 0 & 2 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda+2)^{m-1}(\lambda-4)^{m-1}(\lambda^2+3\lambda-(9m-2))(\lambda^2+(2m-7)\lambda-(7m-12))=0$$

and the spectrum is $Spec^D_{P_1}(B_m) =$

$$\begin{pmatrix} -2 & 4 & \frac{-3+\sqrt{1+36m}}{2} & \frac{-(2m-7)+\sqrt{4m^2+1}}{2} & \frac{-3-\sqrt{1+36m}}{2} & \frac{-(2m-7)-\sqrt{4m^2+1}}{2} \\ m-1 & m-1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore, $E^B_{P_1}(B_m) = 6m-6+\sqrt{1+36m}+\sqrt{1+4m^2}$.

Theorem 4.3. *The minimum boundary dominating* 2*-partition energy of a Book graph* B_m *is* $E^B_{P_2}(B_m) = 6m-6+\sqrt{1+4m}+\sqrt{1+4m^2}$.

Proof. Consider the base vertices $\{v_1, v_2\}$ vertices in one partition and remaining vertices in another partition. The minimum boundary dominating set $= B = \{v_1, v_2\}$. The minimum boundary dominating 2-partition matrix is

$$P_2^B(B_m) = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 & \dots & 0 & 1 \\ 1 & 0 & 0 & 2 & -1 & -1 & \dots & -1 & -1 \\ 0 & 1 & 2 & 0 & -1 & -1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ 1 & 0 & -1 & -1 & 0 & 2 & \dots & -1 & -1 \\ 0 & 1 & -1 & -1 & 2 & 0 & \dots & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 & -1 & \dots & 0 & 2 \\ 0 & 1 & -1 & -1 & -1 & -1 & \dots & 2 & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda+2)^{m-1}(\lambda-4)^{m-1}(\lambda^2+3\lambda-(9m-2))(\lambda^2+(2m-7)\lambda-(7m-12))=0$$

and the spectrum is $Spec_{MBP}(B_m) =$

$$\begin{pmatrix} -2 & 4 & \frac{-3+\sqrt{1+4m}}{2} & \frac{-(2m-7)+\sqrt{4m^2+1}}{2} & \frac{-3-\sqrt{1+4m}}{2} & \frac{-(2m-7)-\sqrt{4m^2+1}}{2} \\ m-1 & m-1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore, $E^B_{P_2}(B_m) = 6m-6+\sqrt{1+4m}+\sqrt{1+4m^2}$.

Theorem 4.4. *The minimum boundary dominating 1-partition energy of the complement* $\overline{K_{n\times 2}}$ *of the cocktail party graph of order* $2n$ *is*

$$E^B_{P_1}\overline{(K_{n\times 2})} = 2n-3+(n-1)\sqrt{37}.$$

Proof. Let $\overline{K_{n\times 2}}$ be the complement of the cocktail party graph of order $2n$ with vertex set $\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$. For 1−partition, we consider all the vertices in a single partition. Here $\{u_1,\ u_2,\ \cdots,\ u_n\}$ is the minimum boundary dominating set. Then the minimum boundary dominating 1-partition matrix of $\overline{K_{n\times 2}}$ is

$$P_1^B(\overline{K_{n\times 2}}) = \begin{bmatrix} 1 & -1 & -1 & -1 & \ldots & 2 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & \ldots & -1 & 2 & -1 & -1 \\ -1 & -1 & 1 & -1 & \ldots & 2 & -1 & 2 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & -1 & -1 & -1 & \ldots & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & \ldots & -1 & 0 & -1 & -1 \\ -1 & -1 & 2 & -1 & \ldots & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 2 & \ldots & -1 & -1 & -1 & 0 \end{bmatrix}.$$

Hence the characteristic equation becomes

$$(\lambda^2 - 3\lambda - 7)^{n-1}(\lambda^2 + (2n-3)\lambda + (3n-7)) = 0$$

and therefore the spectrum is

$$Spec_{MBP}(K_{n\times 2}) = \begin{pmatrix} \frac{3+\sqrt{37}}{2} & \frac{3-\sqrt{37}}{2} & \frac{-2n+3+\sqrt{4n^2-24n+37}}{2} & \frac{-2n+3-\sqrt{4n^2-24n+37}}{2} \\ n & & n & \end{pmatrix}.$$

Finally we get $E^B_{P_1}(\overline{K_{n\times 2}}) = 2n - 3 + (n-1)\sqrt{37}$.

Theorem 4.5. *The minimum boundary dominating 2-partition energy of the complement $\overline{K_{n\times 2}}$ of the cocktail party graph of order $2n$ is*

$$E^B_{P_2}\overline{(K_{n\times 2})} = 5n - 6.$$

Proof. Let $\overline{K_{n\times 2}}$ be the complement of the cocktail party graph of order $2n$ with vertex set $\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$. Let $\{u_1,\ u_2,\ \cdots,\ u_n\}$ be in one partition and remaining vertices in another partition. Here $\{u_1,\ u_2,\ \cdots,\ u_n\}$ is the minimum boundary dominating set. Then the minimum boundary dominating 2-partition matrix of $\overline{K_{n\times 2}}$ is

$$P_2^B(\overline{K_{n\times 2}}) = \begin{bmatrix} 1 & -1 & -1 & -1 & \ldots & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & \ldots & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & -1 & \ldots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \ldots & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & \ldots & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & \ldots & -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 & \ldots & -1 & -1 & -1 & 0 \end{bmatrix}.$$

Hence the characteristic equation becomes

$$(\lambda^2 - 3\lambda + 1)^{n-1}(\lambda^2 + (2n-3)\lambda + (n^2 - 3n + 1)) = 0$$

and therefore the spectrum is

$$Spec_{MBP}(K_{n\times 2}) = \begin{pmatrix} \frac{-(2n-3)+\sqrt{5}}{2} & \frac{-(2n-3)-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ n & n & & \end{pmatrix}.$$

Finally we get $E^B_{P_2}(\overline{K_{n\times 2}}) = 5n - 6$.

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