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# CONTINUITY CONDITIONS FOR HOMOMORPHISMS OF CONNECTED LOCALLY COMPACT GROUPS INTO LIE GROUPS

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Abstract. Using recent results concerning homomorphisms between connected Lie groups, we obtain some results concerning the automatic continuity of some simplest homomorphisms of connected locally compact groups into connected Lie groups and consider examples showing that such a homomorphism can be discontinuous everywhere.

## § 1. Introduction

Let $G$ be a connected locally compact group, let $H$ be a connected Lie group, and let $f\colon G \to H$ be a homomorphism. We apply the known results [1, 2] concerning the automatic continuity properties of homomorphisms between Lie groups to obtain the simplest automatic continuity conditions for a homomorphism of the form $f$ and to give conditions under which a homomorphism of the form $f$ can be discontinuous at any point of $G$.

## § 2. Preliminaries

Let us recall the results of [1, 2].

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**Theorem 1.** [1] *The discontinuity group of every locally bounded homomorphism of a Lie group into a Lie group is commutative.*

**Theorem 2.** [2] *Every locally bounded homomorphism of a Lie group $G$ into a Lie group is continuous on the commutator subgroup $G'$ of the group $G$.*

## § 3. Main Results

**Theorem 3.** *Let $G$ be a connected locally compact group, let $H$ be a connected Lie group, and let*

$$f\colon G \to H$$

*be a homomorphism. If $f$ is locally bounded and the restriction of $f$ to some normal subgroup $N$ of $G$ for which the quotient group is a Lie group is continuous, then the homomorphism $f$ defines a homomorphism $F_f$ of the quotient group $G/N_1$, where $N_1 \subset N$ is a normal subgroup of $G$ for which $G/N_1$ is a Lie group, into the group $H$, and this homomorphism $F_f$ is automatically continuous on the commutator subgroup of $G/N_1$.*

*Proof.* The connected locally compact group $G$ is isomorphic to the projective limit of the quotient Lie groups $G/N_2$, where $N_2$ ranges over the compact normal subgroups of $G$ for which the quotient groups $G/N_2$ are Lie groups (see Proposition 1.33 and Lemma 9.1 of [3]).

For every neighborhood $U$ of the identity element of $N$, there is a compact normal subgroup $N_1$ that is contained in this neighborhood $U$ and for which the quotient group $G/N_1$ is a Lie group.

If the neighborhood $U$ is chosen in such a way that $f$ is bounded on this neighborhood, then, for some $N_1$, $f$ takes on $N_1$ the values in a small neighborhood of the identity element in $H$ that contains no nontrivial subgroups, and hence takes $N_1$ to the identity element of $H$ [4].

Then $f$ is constant on $N_1$-cosets and hence is defined by a homomorphism of $G/N_1$ into $H$. The rest follows from Theorem 2.

**Example 1.** Let

$$K = \prod_{k=1}^{\infty} G_k,$$

where every compact Lie group $G_k$ is a counterpart of some simple compact Lie group represented in the matrix form, for instance, $G_k = \mathrm{SU}(2)$, $k = 1, 2, \dots$

Every matrix entry $f_{ij}$, $i,j=1,2$, of every element $f=\{f_{ij}\}_{i,j=1}^2$ of the group $K$ defines a (bounded) function on the set of positive integers,

$$f_{ij}\colon \mathbb{N}\to\mathbb{C}, \qquad i,j=1,2.$$

Applying a fixed character $\chi$ of the Banach algebra $m=\mathrm{B}\,(\mathbb{N})$ of bounded sequences which differs from the mapping defined by taking the value of the sequence at some point of the set $\mathbb{N}$, we obtain a finite-dimensional unitary matrix representation

$$\pi\colon f=\{f_{ij}\}_{i,j=1}^2\to\{\chi(f_{ij})\}_{i,j=1}^2, \qquad f\in K,$$

of the group $K$.

The representation $\pi$ is obviously discontinuous, because the set of values of the representation $\pi$ on every neighborhood of the identity element is the entire group SU(2).

## § 4. Discussion

It looks provable that, if a compact normal subgroup $N$ of $G$ for which the quotient group is a Lie group, is a quotient of the product of a connected compact group and a totally disconnected compact group, for which the connected part is a quotient of a (possibly infinite) product of compact simple Lie groups [3], and at least one of the compact simple Lie groups in this product repeats infinitely, then the scheme of the above example can be used to construct a discontinuous finite-dimensional representation of $N$ and extend it to a discontinuous finite-dimensional representation of $G$ [5, 6].

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