Advanced Studies in Contemporary Mathematics
35 (2025), No. 1, pp. 57 - 65

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2025.35.1.57

A note on an identity involving the number of colored partitions

Mateus Alegri
University of Sergipe, Brazil, e-mail:
allegri.mateus@gmail.com

José L. López-Bonilla
Instituto Politécnico Nacional, México, e-mail:
jlopezb@ipn.mx

Taekyun Kim
Kwangwoon University, Republic of Korea, e-mail:
tkkim@kw.ac.kr

Received: 11 October 2024 **Accepted:** 07 December 2024

Abstract: One of our aims in this paper is to discuss some identities of the type *sum to product* as follows.

$$1+\sum_{l=1}^{\infty}\frac{1}{l!}\left(\sum_{n=1}^{\infty}\frac{kq^n}{n(1-q^n)}\right)^l=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^k}.$$

In the final result, we establish a congruence for $p_k(n)$, the number of colored partitions of the integer n, which is related to the aforementioned identity.

Keywords: Colored integer partitions, Partition function, Compositions, Lambert series, Congruences.
2010 Mathematics Subject Classification: Primary 05A17, Secondary 11P82, 11P84.

1 Introduction

In this article we use the q-Pochhammer symbol, defined by:

$$(a;q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n>0; \\ 1, & \text{if } n=0. \end{cases}$$

Taking the limit $n \to \infty$, we obtain:

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Let $p_k(n)$denote the number of integer partitions of n into k colors. The generating function for the sequence $(p_k(n))_{n\geq 0}$ is given by

$$\sum_{n=0}^{\infty} p_k(n) q^n = \frac{1}{(q;q)_\infty^k},$$

valid for $|q|<1$.

For the case $k=1$, the following identities are well-known:

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} = \frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2}. \tag{1}$$

The first equation is attributed to Euler [6], while the second is due to Jacobi [3, 4, 8]. Euler's pioneering work on partitions primarily utilized generating functions. In addition to (1), we provide the following equation:

$$1 + \sum_{l=1}^{\infty} \frac{1}{l!} \left(\sum_{n=1}^{\infty} \frac{q^n}{n(1-q^n)} \right)^l = \frac{1}{(q;q)_\infty}$$

Although identities of the type ”sum to product,” as in (1), are well-studied for partitions, relatively few are known for colored partitions. In the subsequent section, we demonstrate that

$$1+\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{n=1}^{\infty}\left(L(\frac{k}{n};q)\right)^{l}=\frac{1}{(q;q)_{\infty}^{k}}.$$

Here, we utilize a modified Lambert series given by

$$L\left(a_n;q\right)=\sum_{n=1}^{\infty}\frac{a_nq^n}{(1-q^n)^{b_n}},$$

where a_n and b_n are terms in sequences of complex numbers, as discussed by Berndt [5] and Merca [9].

To prove the above equation, we employ a formula obtained by Alegri for $P_n(z)$, $[1,2]$, along with the concept of integer compositions. A composition of an integer n is a way of writing n as a sum of positive integers. The set of compositions of n is denoted by $C(n)$. Considering

$$\left(q^{-\frac{1}{24}}\eta(\tau)\right)^{-z}=\prod_{n=1}^{\infty}(1-q^n)^{-z}, \tag{2}$$

where $\tau\in H=\{b\in\mathbb{C}|Im\{b\}>0\}$, $z\in\mathbb{C}$, and η is the Dedekind eta function as defined in Ono [10]. The Taylor expansion of (2) is

$$\left(q^{-\frac{1}{24}}\eta(\tau)\right)^{-z}=\sum_{n=0}^{\infty}P_n(z)q^n,$$

where

$$P_n(z) = \sum_{l=1}^{n} \frac{z^l}{l!} \sum_{w_1+w_2+\ldots+w_l \in C(n)} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2 \cdots w_l}, \tag{3}$$

and the sum of divisor function σ_x is defined as

$$\sigma_x(n) = \sum_{\substack{d|n \\ d>0}} d^x.$$

Specifically for $z = k$, a positive integer, $P_n(z) = p_k(n)$, the number of k-colored integer partitions of n[1].

2 Results

Theorem 1. *For k a positive integer, and $|q| < 1$, the following identity holds:*

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k} = 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \left(L\left(\frac{k}{n}; q\right)\right)^l.$$

Proof. For $|q| < 1$, note that

$$L\left(\frac{k}{n}; q\right) = \sum_{n=1}^{\infty} \frac{kq^n}{n(1-q^n)}.$$

Simplifying further,

$$L\left(\frac{k}{n}; q\right) = k \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} q^n.$$

[1]Several results in partitions and colored partitions can be found in Andrews [3, 4] and Fu and Tang, [7].

To find the coefficient of q^n, for $n > 0$, in $\left(L\left(\frac{k}{n};q\right)\right)^j$, we utilize integer compositions:

$$\left(L\left(\frac{k}{n};q\right)\right)^j = \sum_{n=j}^{\infty} k^j \left(\sum_{(w_1,w_2,\dots,w_j)\in C_n} \frac{\sigma_1(w_1)\cdots\sigma_1(w_j)}{w_1\cdots w_j} \right) q^n.$$

Thus,

$$\sum_{l=1}^{\infty} \frac{1}{l!} \left(L\left(\frac{k}{n};q\right)\right)^l$$

$$= \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{k^l}{l!} \sum_{(w_1,w_2,\dots,w_j)\in C_n} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2\cdots w_l} \right) q^n.$$

By the equation (3), for $z = k$, we know:

$$p_k(n) = \sum_{l=1}^{n} \frac{k^l}{l!} \sum_{(w_1,w_2,\dots,w_j)\in C_n} \frac{\sigma_1(w_1)\sigma_1(w_2)\cdots\sigma_1(w_l)}{w_1 w_2\cdots w_l}.$$

Therefore, the theorem is proved.

□

We can generalize the previous theorem, albeit without providing explicit combinatorial interpretations, as stated below.

Theorem 2. *If $b(k)$ is an arbitrary arithmetic function and $|q| < 1$, the following identity holds:*

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{b(k)}} = 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \left(L\left(\frac{b(k)}{n};q\right)\right)^l.$$

Proof. Define

$$Q = 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \left(L\left(\frac{b(k)}{n}; q \right) \right)^{l},$$

and

$$R = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{b(k)}}.$$

Given

$$L\left(\frac{b(k)}{n}; q \right) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{b(d)}{d} \right) q^n,$$

denote $c_n = \sum_{d|n} \frac{b(d)}{d}$. Then:

$$\left(L\left(\frac{b(k)}{n}; q \right) \right)^{l} = \sum_{n=l}^{\infty} \left(\sum_{(w_1, w_2, \ldots, w_l) \in C_n} c_{w_1} c_{w_2} \cdots c_{w_l} \right) q^n.$$

Thus,

$$\ln(Q) = \ln\left(\exp\left(L\left(\frac{b(k)}{n}; q \right) \right) \right) = \sum_{n=1}^{\infty} c_n q^n.$$

For R,

$$\ln(R) = \sum_{m=1}^{\infty} (-b(k)) \ln(1-q^m) = b(k) \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{q^{ml}}{l} = \sum_{n=1}^{\infty} c_n q^n,$$

Since $\ln Q = \ln R$, it follows that $R = Q$. This completes the proof.

$\square$

In the next theorem we explore the polynomial structure of $p_k(n)$. Since

$$\left(\sum_{n=0}^{\infty} p(n)q^n\right)^k = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k},$$

it follows that

$$p_k(n) = \sum_{l=1}^{k} \sum_{(w_1,\ldots,w_l)\in C_n} p(w_1)\cdots p(w_l),$$

or equivalently,

$$p_k(n) = \sum_{l=1}^{k} l! \sum_{w_1+\ldots+w_l\in P_n} p(w_1)\cdots p(w_l),$$

where P_n denotes the set of partitions of n.

From the above equation, we derive the following result.

Theorem 3. *For $1 \le j < k$, the following congruence holds:*

$$p_k(n) - \sum_{l=1}^{j} l! \sum_{w_1+\ldots+w_l\in P_n} p(w_1)\cdots p(w_l) \equiv 0 \pmod{(j+1)!}.$$

3 Concluding Remarks

In particular, in the previous theorem if $j = 1$ we find that

$$p_k(n) \equiv p_1(n) \pmod{(j+1)!},$$

i.e., $p_k(n)$ and $p_1(n)$ share the same parity.

We believe that equation (3) and Theorems 1 and 3 have significant potential for discovering new congruences for $p_k(n)$, especially for specific values of n. Moreover, a proof of Theorem 1 using purely combinatorial arguments could offer additional insights and introduce interesting elements into the study of partition theory.

Conflicts of Interest: The authors have no conflict of interest.

References

[1] Alegri, M. *Identities involving sum of divisors, integer partitions and compositions*, Online Journal of Analytic Combinatorics, 2023.

[2] Alegri, M. *Identities involving sum of divisors, compositions and integer partitions hiding in the q-binomial theorem*. Sao Paulo J. Math. Sci., 2023.

[3] Andrews, G.E., Askey, R., Roy, R., *Special Functions*, Cambridge University Press, London, 1999.

[4] Andrews, G.E., Eriksson, K., *Integer Partitions*, Cambridge University Press, London, 2004.

[5] Berndt, B. C, *Number theory in the spirit of Ramanujan*, Student mathematical library, AMS, 2006.

[6] Euler, L., *De partitione numerorum, Introductio in Analysin Infinitorum, Caput XVI (1753), in: Leonardi Euleri Opera Omnia*. A. Kratzer, F. Rudio (Eds.), Teubner, Leipzig, (1911).

[7] Fu, S., Tang, D. *On a generalized crank for k-colored partitions.* J.Number Theory 184, 485–497, 2018.

[8] Jacobi, C. G. J., *Fundamenta Nova Theoriae Functionum Ellipticarum*, Borntrager, Regiomonti, 1829.

[9] Merca, M. *The Lambert series factorization theorem*, Ramanujan J 44, 417-435 (2017). https://doi.org/10.1007/s11139-016-9856-3

[10] Ono, K., *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, CBMS Regional Conference Series in Mathematics, Vol. 102, American Mathematical Society, 2004.