Proceedings of the Jangjeon Mathematical Society
28 (2025), No. 1, pp. 143 - 146
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2025.28.1.143

CENTER OF A PSEUDOCHARACTER

A. I. Shtern

Abstract. We introduce the notion of center of a pseudocharacter on a group, prove that this set is a normal subgroup of the group, and show that the center of a pseudocharacter contains the maximal normal subgroup in the kernel of the pseudocharacter.

§ 1. Introduction

Let G be a group and let f be a pseudocharacter on G. In this note, we study the properties of the so-called center of the pseudocharacter f defined by $Z_f = \{u \in G \mid f(gu) = f(g) + f(u) \text{ for all } g \in G\}$. For the generalities concerning pseudocharacters, see [1–4].

In particular, it turns out that Z_f is a normal subgroup of G containing the maximal normal subgroup in the kernel of f (see [5]).

§ 2. Preliminaries

Lemma. *Let G be a group, let N be a normal subgroup of G, and let π be the canonical epimorphism of G onto G/N. If a pseudocharacter f on G (such that $f(gh) - f(g) - f(h)| \le c$ for $g, h \in G$) vanishes on N, then there exists a pseudocharacter φ on the group G/N such that $f = \psi \circ \varphi$. If G is a topological group, N is closed, and f is continuous, then φ is continuous.*

2010 *Mathematics Subject Classification.* Primary 22A25, Secondary 22E25.

Submitted November 20, 2024.

Key words and phrases. pseudocharacter, center of a pseudocharacter, maximal normal subgroup in the kernel of a pseudocharacter.

Proof. Let G be a group, let N be a normal subgroup of G, let $g \in G$, $n \in N$, and let f be a pseudocharacter on G vanishing on N. Let $m \in \mathbb{N}$. Then

$$m|f(gn)-f(g)| = |f(gn)^m)-mf(g)| = |f(g^m(\prod_{k=m-1}^{1} g^{-k}ng^k)n)-f(g^m)| \leq c,$$

since

$$(\prod_{k=m-1}^{1} g^{-k}ng^k)n \in N;$$

this implies that $f(gn) = f(g)$ for all $g \in G$ and all $n \in N$. Therefore, f is constant on every coset of N in G. Define a real-valued function φ on G/N by setting $\varphi(gN) = f(g)$ (since f is constant on the cosets of N, it follows that this definition is correct). The above formula for $m|f(gn)-f(g)|$, together with a similar formula for $|f(gn)^{-m}) - f(g)^{-m}|$, shows that φ is a pseudocharacter on G/N, and that $\varphi = \psi \circ \pi$, where π is the canonical epimorphism of G onto G/N. The continuity assertion follows immediately from the last formula.

§ 3. Main Result

Theorem. *Let G be a group, let f be a pseudocharacter on G, and let $N = \ker f$, i.e., $N = \{g \in G : f(g) = 0\}$. Consider the set N_0 of all elements $n \in N$ such that $f(gn) = f(g) + f(n)$ for all $g \in G$. Then*

(1) $N_0^{-1} \subset N_0$;

(2) $f(ng) = f(n) + f(g)$ for all $n \in N_0$ and all $g \in G$;

(3) N_0 contains the products of its elements, i.e., $f(gn_1n_2) = f(g)f(n_1n_2)$ for every $n_1, n_2 \in N_0$ and all $g \in G$;

(4) N_0 is invariant under the inner automorphisms of G;

(5) N_0 is a normal subgroup of G;

(6) N_0 contains the maximal normal subgroup in the kernel of f.

Note that N_0 is nonempty since the identity element $e \in N$ belongs to N_0.

Proof. (1) We see that

$$f(g) = f(gn^{-1}n) = f(gn^{-1}) + f(n)$$

for every $n \in N_0$ and all $g \in G$; since

$$0 = f(e) = f(n^{-1}n) = f(n^{-1}) + f(n),$$

we have $f(gn^{-1}) = f(g) + f(n^{-1})$, which implies that $n^{-1} \in N_0$ for all $n \in N_0$.

(2) Let $g \in G$ and $n \in N_0$. Then $f(g) = f(ngn^{-1}) = f(ng) - f(n)$ by (1). Therefore, $f(ng) = f(n) + f(g)$ for all $n \in N_0$ and all $g \in G$.

(3)

$$f(gn_1n_2) = f(gn_1) + f(n_2) = f(g) + f(n_1) + f(n_2) = f(g) + f(n_1n_2)$$

for all $g \in G$ and all $n_1, n_2 \in N_0$,

(4) Let $g, h \in G$ and $n \in N_0$. Then

$$\begin{aligned} f(ghuh^{-1}) &= f(hh^{-1}ghh^{-1}huh^{-1}) = f(hh^{-1}ghuh^{-1}) = f(h^{-1}ghu) \\ &= f(h^{-1}gh) + f(u) = f(g) + f(u) = f(g) + f(huh^{-1}). \end{aligned}$$

(5) N_0 contains inverses (1) and products (3) and is inner invariant (4).

(6) By the lemma, N_0 contains every normal subgroup which is contained in the kernel of f. Hence the normal subgroup N_0 (5) contains the maximal normal subgroup in the kernel of f (see [5]).

This completes the proof of the theorem,

§ 4. Discussion

Obviously, the center of a pseudocharacter coincides with the whole group if and only if this pseudocharacter is an ordinary real character of the group.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Advanced Studies of Contemporary Mathematics.

Funding

The research was supported by the Scientific Research Institute for System Analysis of the National Research Centre "Kurchatov Institute" according to the project FNEF-2024-0001.

References

1. A. I. Shtern, Quasireprezentations and pseudorepresentations, Funktsional. Anal. i Prilozhen. **25** (2), 70–73 (1991) [Funct. Anal. Appl. **25** (2), 140–143 (1991)].
2. A. I. Shtern, Quasisymmetry. I. Russ. J. Math. Phys. **2** (3), 353–382 (1994).

3. A. I. Shtern, A version of van der Waerden's theorem and a proof of Mishchenko's conjecture on homomorphisms of locally compact groups, Izv. Ross. Akad. Nauk Ser. Mat. **72** (1), 183–224 (2008) [Izv. Math. **72** (1), 169–205 (2008)].
4. A. I. Shtern, Finite-dimensional quasirepresentations of connected Lie groups and Mishchenko's conjecture, Fundam. Prikl. Mat. **13** (7), 85–225 (2007) [J. Math. Sci. (N. Y.) **159** (5), 653–751 (2009)].
5. A. I. Shtern, *Maximal normal subgroup in the kernel of a pseudocharacter on a group*, Adv. Stud. Contemp. Math. **32** (2025), no. 1 (to appear).

Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991 Russia
Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia
Scientific Research Institute for System Analysis of the National Research Centre "Kurchatov Institute",
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru