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FRACTION LOGARITHMIC ENERGY OF A GRAPH

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Abstract. This paper deals with the investigation of fraction logarithmic energy $FLE(G)$ of a graph G. Here we present some upper and lower bounds for $FLE(G)$. We calculated the fraction logarithmic energy for several graph classes and also for some graphs with one edge deleted. Also $FLE(G)$ is calculated for some complements of several graphs.

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1. Introduction

In this paper, we use simple graph G without self loops and multiple edges. Let the set of vertices be $\{v_1, v_2, \cdots, v_n\}$ for $i = 1, 2, \cdots, n$, and let v_i and v_j are the i^{th} and the j^{th} vertices respectively. If two vertices v_i and v_j of G are adjacent, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, we denote the degree of the vertex v_i, by d_i.

In mathematical chemistry, energy of graph defined by Ivan Gutman in 1978. The energy $E(G)$ of G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. For more details on energy of a graph, see [2, 3].

In 21st century, topological indices increasingly attracted the researchers due to their various applications in different fields of Science and Technology. K. N Prakasha [6] introduced the fraction logarithmic index in the chemical graph theory, which is defined as

$$FL(G) = \sum_{v_i \sim v_j} \frac{\ln(d_i d_j)}{(d_i + d_j)}.$$

The fraction logarithmic index motivated us to study fraction logarithmic matrix which depends on degree of the vertices. fraction logarithmic matrix is given by

$$F_{ij} = \begin{cases} \frac{\ln(d_i d_j)}{(d_i + d_j)} & \text{if } v_i \sim v_j, \\ 0 & otherwise. \end{cases}$$

The characteristic polynomial of $FL(G)$ is denoted by $\phi_{FL}(G, \lambda)$ and defined by

$$\phi_{FL}(G, \lambda) = det(\lambda I - FL(G)).$$

Since the fraction logarithmic matrix is real and symmetric, its eigenvalues are all real numbers. We label them in non-increasing order as $\lambda_1 \geq \lambda_2 \geq$

$\cdots \geq \lambda_n$. The fraction logarithmic energy is denoted by $FLE(G)$ and is defined by

$$FLE(G) = \sum_{i=1}^{n} |\lambda_i|.$$

2. Some basic properties of the fraction logarithmic energy of a graph

Proposition 2.1. *The first four coefficients of the polynomial $\phi_{FL}(G, \lambda)$ are given as follows:*

(i) $a_0 = 1$,
(ii) $a_1 = 0$,
(iii) $a_2 = -\sum_{i<j} \left[\frac{\ln(d_i d_j)}{(d_i+d_j)}\right]^2$.
(iv) $a_3 = -2 \sum_{v_i \sim v_j \sim v_k \sim v_i} \left[\frac{\ln(d_i d_j)}{(d_i + d_j)} \frac{\ln(d_j d_k)}{(d_j + d_k)} \frac{\ln(d_i d_k)}{(d_i + d_k)}\right]$.

Proof. **(i)** From the definition, $\phi_{FL}(G, \lambda) = det[\lambda I - FL(G)]$ and then we get $a_0 = 1$ after easy calculations.

(ii) The sum of the determinants of all 1×1 principal submatrices of $FL(G)$ is equal to the trace of $FL(G)$. Therefore

$$a_1 = (-1)^1 \cdot \text{trace of } [FL(G)] = 0.$$

(iii) Similarly we have

$$\begin{aligned}
(-1)^2 a_2 &= \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\
&= \sum_{1 \leq i < j \leq n} a_{ii} a_{jj} - a_{ji} a_{ij} \\
&= \sum_{1 \leq i < j \leq n} a_{ii} a_{jj} - \sum_{1 \leq i < j \leq n} a_{ji} a_{ij} \\
&= -\sum_{i<j} \left[\frac{\ln(d_i d_j)}{(d_i + d_j)}\right]^2.
\end{aligned}$$

(iv)

$$a_3 = (-1)^3 \sum_{1 \leq i < j < k \leq n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}$$

$$= -2 \sum_{1 \leq i < j < k \leq n} a_{ij} a_{jk} a_{ki}$$

$$= -2 \sum_{v_i \sim v_j \sim v_k \sim v_i} \left[\frac{\ln(d_i d_j)}{(d_i + d_j)} \frac{\ln(d_j d_k)}{(d_j + d_k)} \frac{\ln(d_i d_k)}{(d_i + d_k)}\right].$$

□

Proposition 2.2. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the fraction logarithmic eigenvalues of $FL(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2\sum_{i<j}\left[\frac{\ln(d_i d_j)}{(d_i+d_j)}\right]^2.$$

Proof. We know that

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji}\\ &= 2\sum_{i<j} a_{ij}^2 + \sum_{i=1}^{n} a_{ii}^2\\ &= 2\sum_{i<j} a_{ij}^2\\ &= 2\sum_{i<j}\left[\frac{\ln(d_i d_j)}{(d_i+d_j)}\right]^2.\end{aligned}$$

□

Theorem 2.3. *Let G be a graph with n vertices. Then*

$$FLE(G) \leq \sqrt{2n\left[\frac{\ln(d_i d_j)}{(d_i+d_j)}\right]^2}$$

.

Proof. Let $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ be the eigenvalues of $FL(G)$. Now by the Cauchy-Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

We let $a_i = 1$ and $b_i = \lambda_i$. Then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

which implies that

$$[FLE(G)]^2 \leq n(2\sum_{i<j}\left[\frac{\ln(d_i d_j)}{(d_i+d_j)}\right]^2)$$

and finally

$$FLE(G) \leq \sqrt{2n\sum_{i<j}\left[\frac{\ln(d_i d_j)}{(d_i+d_j)}\right]^2}$$

which is an upper bound. □

Theorem 2.4. *Let G be a graph with n vertices. If $R=$ det $FL(G)$, then*

$$FLE(G) \geq \sqrt{2\sum_{i<j}\left[\frac{\ln(d_i d_j)}{(d_i+d_j)}\right]^2 + n(n-1)R^{\frac{2}{n}}}.$$

Proof. By definition,

$$\begin{aligned}(FLE(G))^2 &= \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid \sum_{j=1}^{n} \mid \lambda_j \mid \\ &= \left(\sum_{i=1}^{n} \mid \lambda_i \mid^2\right) + \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid .\end{aligned}$$

Using arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \;\; \geq \;\; \left(\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}} .$$

Therefore,

$$\begin{aligned}(FLE(G))^2 &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) \left(\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) \left(\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + n(n-1) R^{\frac{2}{n}} \\ &= 2\sum_{i<j} \left[\frac{\ln{(d_i d_j)}}{(d_i+d_j)}\right]^2 + n(n-1)R^{\frac{2}{n}}.\end{aligned}$$

Thus,

$$FLE(G) \geq \sqrt{2\sum_{i<j} \left[\frac{\ln{(d_i d_j)}}{(d_i+d_j)}\right]^2 + n(n-1)R^{\frac{2}{n}}}.$$

□

Theorem 2.5. *Let G be a regular graph of n vertices with regularity r, then*

$$FLE(G) = \frac{\ln r}{r} E(G)$$

Theorem 2.6. *Let G be a semiregular graph of degrees $r \geq 1$ and $s \geq 1$. Then $FLE(G) = \frac{\ln{(rs)}}{(r+s)} E(G)$.*

Proof. Consider a semiregular graph of degrees $r \geq 1$ and $s \geq 1$, the $FL-$matrix is given by

$$FL(G) = \frac{\ln{(rs)}}{(r+s)} (J-I) \; .$$

$$\lambda_i = \frac{\ln{(rs)}}{(r+s)} \gamma_i.$$

(Here γ_i represents the eigenvalue with respect to adjacency matrix of the corresponding graph.)
Thus the proof follows. $\square$

3. Fraction logarithmic energy of some standard graphs

Theorem 3.1. *The fraction logarithmic energy of a complete graph K_n is*

$$FLE(K_n) = 2\ln(n-1).$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1,\ v_2,\ ...,\ v_n\}$. The fraction logarithmic matrix is

$$FL(K_n) = \begin{bmatrix} 0 & \frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} & \dots & \frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} \\ \frac{\ln(n-1)}{n-1} & 0 & \frac{\ln(n-1)}{n-1} & \dots & \frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} \\ \frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} & 0 & \dots & \frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} & \dots & \frac{\ln(n-1)}{n-1} & 0 & \frac{\ln(n-1)}{n-1} \\ \frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} & \dots & \frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} & 0 \end{bmatrix}.$$

Hence the characteristic equation will be

$$\left(\lambda - \frac{\ln(n-1)}{n-1}\right)^{n-1} (\lambda - \ln(n-1)) = 0$$

and therefore the spectrum becomes

$$Spec_{FL}(K_n) = \begin{pmatrix} \frac{\ln(n-1)}{n-1} & \ln(n-1) \\ n-1 & 1 \end{pmatrix}.$$

Therefore,

$$FL(K_n) = 2\ln(n-1).$$

$\square$

Theorem 3.2. *The fraction logarithmic energy of the crown graph S_n^0 is*

$$FLE(S_n^0) = 4\ln(n-1).$$

Proof. Let S_n^0 be the crown graph of order $2n$, the fraction logarithmic matrix is

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & A & \dots & A & A \\ 0 & 0 & 0 & \dots & 0 & A & 0 & \dots & A & A \\ 0 & 0 & 0 & \dots & 0 & A & A & \dots & 0 & A \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & A & A & \dots & A & 0 \\ 0 & A & A & \dots & A & 0 & 0 & \dots & 0 & 0 \\ A & 0 & A & \dots & A & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & 0 & \dots & A & 0 & 0 & \dots & 0 & 0 \\ A & A & A & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

where $A = \frac{\ln(n-1)}{n-1}$.
Here the characteristic equation is

$$(\lambda - A)^{n-1}(\lambda + A)^{n-1}\left(\lambda + \sqrt{2}(n-2)(n-1)\right)\left(\lambda - \sqrt{2}(n-2)(n-1)\right) = 0$$

implying that the spectrum is

$$Spec_{FL}(S_n^0) = \begin{pmatrix} -\frac{\ln(n-1)}{n-1} & \frac{\ln(n-1)}{n-1} & -\ln(n-1) & \ln(n-1) \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore,

$$FLE(S_n^0) = 4\ln(n-1).$$

□

Theorem 3.3. *The fraction logarithmic energy of the cocktail party graph $K_{n\times 2}$ is*

$$FLE(K_{n\times 2}) = \ln(2n-2).$$

Proof. Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The fraction logarithmic matrix is $FL(K_{n\times 2}) =$

$$\begin{bmatrix} 0 & B & B & B & \dots & 0 & B & B & B \\ B & 0 & B & B & \dots & B & 0 & B & B \\ B & B & 0 & B & \dots & B & B & 0 & B \\ B & B & B & 0 & \dots & B & B & B & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & B & B & B & \dots & 0 & B & B & B \\ B & 0 & B & B & \dots & B & 0 & B & B \\ B & B & 0 & B & \dots & B & B & 0 & B \\ B & B & B & 0 & \dots & B & B & B & 0 \end{bmatrix}.$$

Here $B = \frac{\ln(2n-2)}{2n-2}$ This implies that the characteristic equation becomes

$$\lambda^n\left(\lambda + \frac{\ln(2n-2)}{2n-2}\right)^{n-1}(\lambda - \frac{1}{2}\ln(2n-2)) = 0.$$

Hence, the spectrum is

$$Spec_{FL}(K_{n\times 2}) = \begin{pmatrix} -\frac{\ln(2n-2)}{2n-2} & 0 & \frac{1}{2}\ln(2n-2) \\ n-1 & n & 1 \end{pmatrix}.$$

Therefore,

$$FLE(K_{n\times 2}) = \ln(2n-2).$$

□

Theorem 3.4. *The fraction logarithmic energy of the complete bipartite graph $K_{m,n}$ of order $m \times n$ is*

$$FLE(K_{m,n}) = 2\frac{\sqrt{mn}}{m+n}\ln(mn).$$

Proof. Let $K_{m,n}$ be the complete bipartite graph, the fraction logarithmic matrix is

$$FL(K_{m,n}) = \begin{bmatrix} 0 & 0 & 0 & \dots & C & C & C \\ 0 & 0 & 0 & \dots & C & C & C \\ 0 & 0 & 0 & \dots & C & C & C \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ C & C & C & \dots & 0 & 0 & 0 \\ C & C & C & \dots & 0 & 0 & 0 \\ C & C & C & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Where $C = \frac{\ln(mn)}{m+n}$.

So the characteristic equation is

$$\lambda^{m+n-2}(\lambda - \frac{\sqrt{mn}}{m+n}\ln(mn))(\lambda + \frac{\sqrt{mn}}{m+n}\ln(mn)) = 0$$

and hence, the spectrum will be

$$Spec_{FL}(K_{m,n}) = \begin{pmatrix} \frac{\sqrt{mn}}{m+n}\ln(mn) & 0 & -\frac{\sqrt{mn}}{m+n}\ln(mn) \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

Therefore,

$$FLE(K_{m,n}) = 2\frac{\sqrt{mn}}{m+n}\ln(mn).$$

□

4. FRACTION LOGARITHMIC ENERGY OF COMPLEMENTS

Theorem 4.1. *The fraction logarithmic energy of the complement* $\overline{K_n}$ *of the complete graph is*

$$FLE\overline{(K_n)} = 0.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The fraction logarithmic matrix is

$$FL(\overline{K_n}) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Then the characteristic equation is $\lambda^n = 0$. Therefore,

$$FLE(\overline{K_n}) = 0.$$

□

Theorem 4.2. *The fraction logarithmic energy of the complement* $\overline{K_{1,n-1}}$ *of the star graph is*

$$FLE(\overline{K_{1,n-1}}) = 2\ln(n-2).$$

Proof. Let $\overline{(K_{1,n-1})}$ be the complement of star graph with vertex set $V = \{v_0, v_1 ... v_{n-1}\}$. The fraction logarithmic matrix is

$$FL(\overline{K_{1,n-1}}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{\ln(n-2)}{n-2} & \cdots & \frac{\ln(n-2)}{n-2} & \frac{\ln(n-2)}{n-2} \\ 0 & \frac{\ln(n-2)}{n-2} & 0 & \cdots & \frac{\ln(n-2)}{n-2} & \frac{\ln(n-2)}{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{\ln(n-2)}{n-2} & \frac{\ln(n-2)}{n-2} & \cdots & 0 & \frac{\ln(n-2)}{n-2} \\ 0 & \frac{\ln(n-2)}{n-2} & \frac{\ln(n-2)}{n-2} & \cdots & \frac{\ln(n-2)}{n-2} & 0 \end{bmatrix}.$$

Then the characteristic equation is

$$\lambda^1 \left(\lambda - \frac{\ln(n-2)}{n-2}\right)^{n-2} (\lambda - \ln(n-2)) = 0$$

and therefore the spectrum is

$$Spec_{FL}\overline{K_{1,n-1}} = \begin{pmatrix} \frac{\ln(n-2)}{n-2} & 0 & \ln(n-2) \\ n-2 & 1 & 1 \end{pmatrix}.$$

Therefore,

$$FLE(\overline{K_{1,n-1}}) = 2\ln(n-2).$$

□

Theorem 4.3. *Let e be the edge of complete graph K_n. The fraction logarithmic energy of $K_n - e$ is*

$$\frac{(n-1)(n-3)}{2} + \sqrt{\left(\frac{(n-3)\ln(n-1)}{n-1}\right)^2 - (8n-16)\left(\frac{[\ln(n-2)(n-1)]}{2n-3}\right)^2}.$$

Proof. $FL(K_n - e) = \begin{pmatrix} 0_{2\times 2} & \frac{\ln[(n-2)(n-1)]}{2n-3} J_{2\times(n-2)} \\ \frac{\ln[(n-2)(n-1)]}{2n-3} J_{n-2\times 2} & \frac{\ln(n-1)}{n-1}(J-I)_{(n-2)} \end{pmatrix}$.
Characteristic equation is

$$\lambda \left(\lambda + \frac{\ln(n-1)}{n-1}\right)^{n-3} \left(\lambda^2 - (E)\,\lambda - (2n-4)\left[\frac{\ln[(n-2)(n-1)]}{(2n-3)}\right]^2\right) = 0.$$

Hence, spectrum is
$Spec_{FL}(K_n - e) = \begin{pmatrix} \frac{-(n-1)}{2} & \frac{E+F}{2} & \frac{E-F}{2} & 0 \\ n-3 & 1 & 1 & 1 \end{pmatrix}$.
Where $E = \frac{n-3}{n-1}\ln(n-1)$
$F = \sqrt{\left(\frac{(n-3)\ln(n-1)}{n-1}\right)^2 - (8n-16)\left(\frac{[\ln(n-2)(n-1)]}{2n-3}\right)^2}$
Therefore, $FLE(K_n - e)$ is

$$\frac{(n-1)(n-3)}{2} + \sqrt{\left(\frac{(n-3)\ln(n-1)}{n-1}\right)^2 - (8n-16)\left(\frac{[\ln(n-2)(n-1)]}{2n-3}\right)^2}.$$

□

Theorem 4.4. *Let e be the edge of complete bipartite graph $K_{n,n}$. The fraction logarithmic energy of $K_{n,n} - e$ is*

$$FLE(K_{n,n} - e) = \sqrt{\frac{(n-1)^2}{n}(\ln n)^2 + 4(n-1)\left(\frac{\ln(n^2 - n)}{2n-1}\right)^2}.$$

Proof. $FL(K_{n,n} - e) = \begin{pmatrix} 0_{n\times n} & A \\ A & 0_{n\times n} \end{pmatrix}$.

Where $A = \begin{pmatrix} \frac{\ln(n)}{n} J_{(n-1)\times(n-1)} & \frac{\ln(n(n-1))}{2n-1} J_{(n-1)\times 1} \\ \frac{\ln(n(n-1))}{2n-1} J_{1\times(n-1)} & 0_{(1\times 1)} \end{pmatrix}$.

Therefore, $FLE(K_{n,n} - e) = \sqrt{\frac{(n-1)^2}{n}(\ln n)^2 + 4(n-1)\left(\frac{\ln(n^2-n)}{2n-1}\right)^2}$. □

5. Conclusion

In this paper, we have discussed about fraction logarithmic matrix, and the fraction logarithmic energy $FLE(G)$ of a graph G. Lower and upper bounds are also obtained. The first four coefficients of the polynomial $\phi_{FL}(G, \lambda)$ are studied. Fraction logarithmic energy $FLE(G)$ of various graph structures and complements are investigated. We also mention the relation between energy (based on adjacency matrix) and fraction logarithmic energy of regular graph.

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