Proceedings of the Jangjeon Mathematical Society www.jangjeon.or.kr
28 (2025), No. 1, pp. 109 - 118 http://dx.doi.org/10.17777/pjms2025.28.1.109

# INVESTIGATING BOUNDS ON VERTEX *SR* CORONA OF SIMPLE CONNECTED GRAPHS

V. LOKESHA[1], NIRUPADI.K[2], A.S.MARAGADAM[3], MANJUNATH.M[4]

ABSTRACT. Topological indices serve as quantitative measures that reflect the structural characteristics of finite graphs, making them invaluable tools in quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) studies. This paper introduces a novel operator graph, denoted as the $\xi$-graph and explores its associated structural properties. Specifically, the paper establishes the boundaries for several key topological indices of the $\xi$-graph, including the inverse sum indeg index, first Zagreb index, first reformulated Zagreb index, Sombar index and Nirmala index.

**2020 Mathematics Subject Classification:** 05C07, 05C09, 05C90.
**Keywords:** *SR*-corona graph; graph operators; topological indices; lower and upper bounds.

## 1. INTRODUCTION

The topology of a molecule is inherently rooted in non-numerical mathematical concepts, yet many measurable characteristics of molecules are expressed as numerical values [3]. Bridging the gap between molecular topology and real chemical attributes requires converting the intrinsic details of chemical structures into numeric representations. This conversion process is pivotal as it underpins the development of topological indices, which serve as numerical descriptors of molecular structure [11, 14]. These indices play a vital role in elucidating the intricate relationship between molecular topology and various chemical properties, thereby aiding in the interpretation and analysis of complex molecular structures in practical applications such as drug design and environmental studies [7, 8].

In 1947, H. Wiener introduced the concept of the topological index, originally known as the path number, during his investigation into the boiling point of paraffin [18]. This index, now widely recognized as the Wiener index, serves as a cornerstone in the field of chemical graph theory. Over time, numerous other topological indices have been formulated and extensively researched. These indices, derived from the topological structure of compounds, are numerical invariants that contribute significantly to characterizing various properties of chemical compounds, encompassing toxicological, physicochemical, and pharmacological attributes [5]. Consequently, comprehending and analyzing topological indices offer valuable insights into the diverse properties and behaviors exhibited by chemical compounds, fostering advancements across disciplines from drug design to environmental chemistry [2].

***Date of submission: March 12, 2024., Accepted date: April 7, 2024.***
[1,3] Department of Studies in Mathematics, Vijayanagara Sri Krishnadevaraya University, Ballari, Karnataka.
e-mail: v.lokesha@gmail.com, Orcid Id:0000-0003-2468-9511,
e-mail: maragadamvijay@gmail.com, Orcid Id:0000-0001-9703-2161.
[2] Department of Mathematics, Shree Annadaneshwar Arts, Science and Commerce college, Naregal. India
e-mail: nirupadik80@gmail.com., Orcid Id: 0000-0002-3992-1531.
[4]Department of Mathematics, Ballari Institute of Technology and Management, Ballari, Karnataka, India
e-mail: manju3479@gmail.com, Orcid Id: 0000-0003-1328-6215.

Let's recall some well-established topological indices.

- The first Zagreb index [3] was introduced by Gutman and Trinajstic N in 1972, is defined as,
$$M_1[G] = \sum_{uv \in E(G)} [d_u + d_v].$$
- The first reformulated Zagreb index [15] was introduced by A. Milicevic et al. in 2004, is defined as,
$$EM_1[G] = \sum_{uv \in E(G)} [d_u + d_v - 2]^2.$$
- The Sombor index [4] was defined by Gutman in 2021, is defined as,
$$SO[G] = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$
- The Nirmala index [6] was defined by Kulli in 2021, is defined as,
$$N[G] = \sum_{uv \in E(G)} \sqrt{d_G(u) + d_G(v)}.$$
- The Bond-Additive topological index [16] namely, Inverse sum indeg index was introduced by Vukicevic and Gasperov in 2010, is defined as,
$$ISI[G] = \sum_{uv \in E(G)} \left( \frac{d_u d_v}{d_u + d_v} \right).$$

## 2. PRELIMINARIES

Here, we define useful definitions which are essential to develop a results.

**Definition 2.1.** [1] *The subdivision graph $S(G)$ of a graph $G$ is formed by replacing every edge in $G$ with a path of length two, effectively inserting a new vertex into each edge of $G$. Alternatively, it can be described as a graph obtained from $G$ by adding an extra vertex into the middle of every edge.*

**Definition 2.2.** [1] *The graph $R(H)$ is constructed from $H$ by introducing a new vertex for each edge in $H$ and then connecting each new vertex to the endpoints of the corresponding edge in $H$.*

Motivated from [12, 13] we set up the following definition.

**Definition 2.3.** *Let $G$ and $H$ denote two simple connected graphs with $n_1$, $n_2$ and $m_1$, $m_2$ are vertices and edges respectively. The vertex SR-corona of $G$ and $H$ is a graph formed by incorporating one instance of the graph $S(G)$ and $n_1$ instances of the graph $R(H)$. In this construction, each vertex of $G$ (for $1 \leq i \leq n_1$) is connected to every vertex of the $i^{th}$ copy of $R(H)$. This resulting composite graph $G \odot_{SR} H$, denoted as $\xi$-graph, encapsulates the vertex SR-corona of $G$ and $H$.*

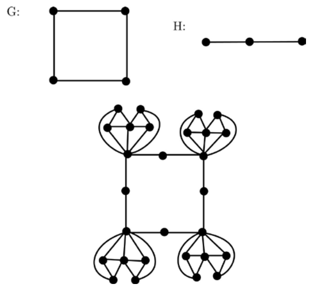


FIGURE 1. $\xi$-graph

**General Edge Partition of $\xi$-graph:**

| $(d_u, d_v)$ | $(d_G+n_2+m_2, 2)$ | $(2d_H+1, 3)$ | $(2d_H+1, 2d_H+1)$ | $(d_G+n_2+m_2, 2d_H+1)$ | $(d_G+n_2+m_2, 3)$ |
|---|---|---|---|---|---|
| freq | $2m_1$ | $2m_2n_1$ | $n_1m_2$ | $n_1n_2$ | $n_1m_2$ |

TABLE 1. Edge Partition of $\xi$-graph

Here, $d_G$ and $d_H$ represent the degrees of vertices in graphs $G$ and $H$ respectively.

## 3. METHODOLOGY

The approach involves examining the structural attributes of the $\xi$-graph, generated through the application of the vertex SR-corona operation to simple connected graphs $G$ and $H$ [13]. It establishes both lower and upper bounds for commonly utilized topological indices of the $\xi$-graph, facilitating a comprehensive grasp of its configuration [9, 10, 12]. The paper consolidates the findings on the $\xi$-graph and its indices, exploring their pertinence in QSAR, QSPR, and various domains.

## 4. RESULTS AND DISCUSSION

In this section establish the bounds for five significant topological indices of the $\xi$-graph: the first Zagreb index, first reformulated Zagreb index, Sombor index, Nirmala index, and Inverse sum indeg index. Throughout the article, $\Delta_G$ and $\delta_G$ represents maximum and minimum degree of graph $G$. Every result is bear by an illustrative example.

**Theorem 4.1.** *The bounds for the first Zagreb index of the $\xi$-graph are determined by,*

$$M_1[\xi] \leq 2m_1[\Delta_G+2+n_2+m_2]+2n_1m_2[4\Delta_H+5]+n_1n_2[\Delta_G+2\Delta_H+n_2+m_2+1] + n_1m_2[\Delta_G+n_2+m_2+3].$$

*and*

$$M_1[\xi] \geq 2m_1[\delta_G+2+n_2+m_2]+2n_1m_2[4\delta_H+5]+n_1n_2[\delta_G+2\delta_H+n_2+m_2+1] + n_1m_2[\delta_G+n_2+m_2+3].$$

*Proof.* Consider,

$$M_1[\xi] = 2m_1[d_G+2+n_2+m_2]+4n_1m_2[d_H+2]+2n_1m_2[2d_H+1]$$

$$+n_1n_2[d_G+2d_H+1+n_2+m_2]+n_1m_2[d_G+3+n_2+m_2].$$
$$=2m_1[d_G+2+n_2+m_2]+2n_1m_2[2d_H+4+2d_H+1]+n_1n_2[d_G+2d_H$$
$$+1+n_2+m_2]+n_1m_2[d_G+3+n_2+m_2].$$
$$M_1[\xi]\leq 2m_1[\Delta_G+2+n_2+m_2]+2n_1m_2[4\Delta_H+5]+n_1n_2[\Delta_G+2\Delta_H+n_2+m_2+1]$$
$$+n_1m_2[\Delta_G+n_2+m_2+3].$$

*Similarly,*

$$M_1[\xi]\geq 2m_1[\delta_G+2+n_2+m_2]+2n_1m_2[4\delta_H+5]+n_1n_2[\delta_G+2\delta_H+n_2+m_2+1]$$
$$+n_1m_2[\delta_G+n_2+m_2+3].$$

□

**Illustrative example 4.1.**: Consider two simple connected graphs $G$ and $H$, then

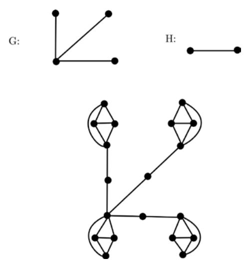


FIGURE 2

$$M_1[\xi]=\sum_{uv\in E(G)}[d_u+d_v]=12(6)+9(7)+3(8)+3(6)+3(9)=204.$$

$$M_1[\xi]\leq 2m_1[\Delta_G+2+n_2+m_2]+2n_1m_2[4\Delta_H+5]+n_1n_2[\Delta_G+2\Delta_H+n_2+m_2+1]$$
$$+n_1m_2[\Delta_G+n_2+m_2+3]=232.$$
$$M_1[\xi]\geq 2m_1[\delta_G+2+n_2+m_2]+2n_1m_2[4\delta_H+5]+n_1n_2[\delta_G+2\delta_H+n_2+m_2+1]$$
$$+n_1m_2[\delta_G+n_2+m_2+3]=192.$$

Hence Theorem 4.1 verified.

**Theorem 4.2.** *The bounds for the first reformulated Zagreb index of the $\xi$-graph are determined by,*

$$EM_1[\xi]\leq 2m_1[\Delta_G+n_2+m_2]^2+8n_1m_2[\Delta_H+1]^2+16n_1m_2\Delta_H^2$$
$$+n_1n_2[\Delta_G+2\Delta_H+n_2+m_2-1]^2+n_1m_2[\Delta_G+n_2+m_2+1]^2.$$

*and*

$$EM_1[\xi]\geq 2m_1[\delta_G+n_2+m_2]^2+8n_1m_2[\delta_H+1]^2+16n_1m_2\delta_H^2$$
$$+n_1n_2[\delta_G+2\delta_H+n_2+m_2-1]^2+n_1m_2[\delta_G+n_2+m_2+1]^2.$$

*Proof.* Consider,

$$EM_1[\xi]=2m_1[d_G+n_2+m_2+2-2]^2+2n_1m_2[2d_H+3-2]^2$$
$$+n_1m_2[2d_H+1+2d_H+1-2]^2+n_1n_2[d_G+n_2+m_2+2d_H+1-2]^2$$

$$
\begin{aligned}
&+n_1m_2[d_G+n_2+m_2+3-2]^2.\\
&=2m_1[d_G+n_2+m_2]^2+8n_1m_2[d_H+1]^2+16n_1m_2d_H^2\\
&+n_1n_2[d_G+n_2+m_2+2d_H-1]^2+n_1m_2[d_G+n_2+m_2+1]^2.\\
EM_1[\xi] &\leq 2m_1[\Delta_G+n_2+m_2]^2+8n_1m_2[\Delta_H+1]^2+16n_1m_2\Delta_H^2\\
&+n_1n_2[\Delta_G+2\Delta_H+n_2+m_2-1]^2+n_1m_2[\Delta_G+n_2+m_2+1]^2.
\end{aligned}
$$

*Similarly,*

$$
\begin{aligned}
EM_1[\xi] &\geq 2m_1[\delta_G+n_2+m_2]^2+8n_1m_2[\delta_H+1]^2+16n_1m_2\delta_H^2\\
&+n_1n_2[\delta_G+2\delta_H+n_2+m_2-1]^2+n_1m_2[\delta_G+n_2+m_2+1]^2.
\end{aligned}
$$

□

**Illustrative example 4.2.**: Consider two simple connected graphs $G$ and $H$, then

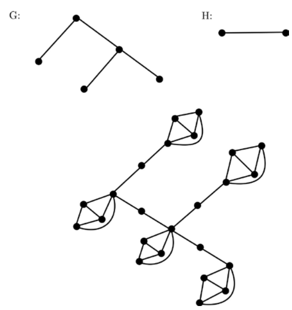


FIGURE 3

$$
\begin{aligned}
EM_1[\xi] &= \sum_{uv\in E(G)}[d_u+d_v-2]^2.\\
&=15(3+3-2)^2+9(3+4-2)^2+3(3+5-2)^2+3(3+6-2)^2=926.\\
EM_1[\xi] &\leq 2m_1[\Delta_G+n_2+m_2]^2+8n_1m_2[\Delta_H+1]^2+16n_1m_2\Delta_H^2+n_1n_2[\Delta_G+2\Delta_H+n_2+m_2-1]^2\\
&+n_1m_2[\Delta_G+n_2+m_2+1]^2=1263.\\
EM_1[\xi] &\geq 2m_1[\delta_G+n_2+m_2]^2+8n_1m_2[\delta_H+1]^2+16n_1m_2\delta_H^2+n_1n_2[\delta_G+2\delta_H+n_2+m_2-1]^2\\
&+n_1m_2[\delta_G+n_2+m_2+1]^2=743.
\end{aligned}
$$

Hence Theorem 4.2 verified.

**Theorem 4.3.** *The bounds for the Sombor index of $\xi$-graph are determined by,*

$$
\begin{aligned}
SO[\xi] &\leq 2m_1\sqrt{(\Delta_G+n_2+m_2)^2+4}+2n_1m_2\sqrt{4\Delta_H^2+4\Delta_H+10}+\sqrt{2}n_1m_2(2\Delta_H+1)\\
&+n_1n_2\sqrt{(\Delta_G+n_2+m_2)^2+(2\Delta_H+1)^2}+n_1m_2\sqrt{(\Delta_G+n_2+m_2)^2+9}.
\end{aligned}
$$

*and*

$$
\begin{aligned}
SO[\xi] &\geq 2m_1\sqrt{(\delta_G+n_2+m_2)^2+4}+2n_1m_2\sqrt{4\delta_H^2+4\delta_H+10}+\sqrt{2}n_1m_2(2\delta_H+1)\\
&+n_1n_2\sqrt{(\delta_G+n_2+m_2)^2+(2\delta_H+1)^2}+n_1m_2\sqrt{(\delta_G+n_2+m_2)^2+9}.
\end{aligned}
$$

*Proof.* Consider,

$$\begin{aligned}SO[\xi] &= 2m_1\sqrt{(d_G+n_2+m_2)^2+2^2}+2n_1m_2\sqrt{(2d_H+1)^2+3^2}+n_1m_2\sqrt{2(2d_H+1)^2}\\ &\quad +n_1n_2\sqrt{(d_G+n_2+m_2)^2+(2d_H+1)^2}+n_1m_2\sqrt{(d_G+n_2+m_2)^2+3^2}.\\ &= 2m_1\sqrt{(d_G+n_2+m_2)^2+4}+2n_1m_2\sqrt{4d_H^2+4d_H+10}+\sqrt{2}n_1m_2(2d_H+1)\\ &\quad +n_1n_2\sqrt{(d_G+n_2+m_2)^2+(2d_H+1)^2}+n_1m_2\sqrt{(d_G+n_2+m_2)^2+9}.\\ SO[\xi] &\leq 2m_1\sqrt{(\Delta_G+n_2+m_2)^2+4}+2n_1m_2\sqrt{4\Delta_H^2+4\Delta_H+10}+\sqrt{2}n_1m_2(2\Delta_H+1)\\ &\quad +n_1n_2\sqrt{(\Delta_G+n_2+m_2)^2+(2\Delta_H+1)^2}+n_1m_2\sqrt{(\Delta_G+n_2+m_2)^2+9}.\end{aligned}$$

*Similarly,*

$$\begin{aligned}SO[\xi] &\geq 2m_1\sqrt{(\delta_G+n_2+m_2)^2+4}+2n_1m_2\sqrt{4\delta_H^2+4\delta_H+10}+\sqrt{2}n_1m_2(2\delta_H+1)\\ &\quad +n_1n_2\sqrt{(\delta_G+n_2+m_2)^2+(2\delta_H+1)^2}+n_1m_2\sqrt{(\delta_G+n_2+m_2)^2+9}.\end{aligned}$$

□

**Illustrative example 4.3.**: Consider two simple connected graphs $G$ and $H$, then

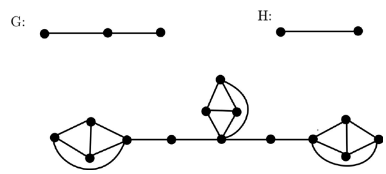


FIGURE 4

$$SO[\xi] = \sum_{uv\in E(G)}\sqrt{d_G(u)^2+d_G(v)^2} = 105.39.$$

$$\begin{aligned}SO[\xi] &\leq 2m_1\sqrt{(\Delta_G+n_2+m_2)^2+4}+2n_1m_2\sqrt{4\Delta_H^2+4\Delta_H+10}+\sqrt{2}n_1m_2(2\Delta_H+1)\\ &\quad +n_1n_2\sqrt{(\Delta_G+n_2+m_2)^2+(2\Delta_H+1)^2}+n_1m_2\sqrt{(\Delta_G+n_2+m_2)^2+9} = 112.20.\\ SO[\xi] &\geq 2m_1\sqrt{(\delta_G+n_2+m_2)^2+4}+2n_1m_2\sqrt{4\delta_H^2+4\delta_H+10}+\sqrt{2}n_1m_2(2\delta_H+1)\\ &\quad +n_1n_2\sqrt{(\delta_G+n_2+m_2)^2+(2\delta_H+1)^2}+n_1m_2\sqrt{(\delta_G+n_2+m_2)^2+9}.\\ &= 101.07.\end{aligned}$$

Hence Theorem 4.3 verified.

**Theorem 4.4.** *The bounds for the Nirmal index of $\xi$-graph are determined by,*

$$\begin{aligned}N[\xi] &\leq 2m_1\sqrt{\Delta_G+2+n_2+m_2}+n_1m_2\left[2\sqrt{2\Delta_H+4}+\sqrt{4\Delta_H+2}+\sqrt{\Delta_G+n_2+m_2+3}\right]\\ &\quad +n_1n_2\sqrt{\Delta_G+2\Delta_H+n_2+m_2+1}.\end{aligned}$$

*and*

$$N[\xi] \geq 2m_1\sqrt{\delta_G+2+n_2+m_2}+n_1m_2\left[2\sqrt{2\delta_H+4}+\sqrt{4\delta_H+2}+\sqrt{\delta_G+n_2+m_2+3}\right]$$
$$+n_1n_2\sqrt{\delta_G+2\delta_H+n_2+m_2+1}.$$

*Proof.* Consider,

$$\begin{aligned}
N[\xi] &= 2m_1\sqrt{d_G+2+n_2+m_2}+2n_1m_2\sqrt{2d_H+1+3}+n_1m_2\sqrt{2d_H+1+2d_H+1} \\
&\quad +n_1n_2\sqrt{d_G+n_2+m_2+2d_H+1}+n_1m_2\sqrt{d_G+n_2+m_2+3}. \\
&= 2m_1\sqrt{d_G+2+n_2+m_2}+2n_1m_2\sqrt{2d_H+4}+n_1m_2\sqrt{4d_H+2} \\
&\quad +n_1n_2\sqrt{d_G+n_2+m_2+2d_H+1}+n_1m_2\sqrt{d_G+n_2+m_2+3}. \\
&= 2m_1\sqrt{d_G+2+n_2+m_2}+n_1m_2\left[2\sqrt{2d_H+4}+\sqrt{4d_H+2}+\sqrt{d_G+n_2+m_2+3}\right] \\
&\quad +n_1n_2\sqrt{d_G+n_2+m_2+2d_H+1}. \\
N[\xi] &\leq 2m_1\sqrt{\Delta_G+2+n_2+m_2}+n_1m_2\left[2\sqrt{2\Delta_H+4}+\sqrt{4\Delta_H+2}+\sqrt{\Delta_G+n_2+m_2+3}\right] \\
&\quad +n_1n_2\sqrt{\Delta_G+n_2+m_2+2\Delta_H+1}.
\end{aligned}$$

*Similarly,*

$$N[\xi] \geq 2m_1\sqrt{\delta_G+2+n_2+m_2}+n_1m_2\left[2\sqrt{2\delta_H+4}+\sqrt{4\delta_H+2}+\sqrt{\delta_G+n_2+m_2+3}\right]$$
$$+n_1n_2\sqrt{\delta_G+n_2+m_2+2\delta_H+1}.$$

□

**Illustrative example 4.4.**: Consider two simple connected graphs $G$ and $H$, then

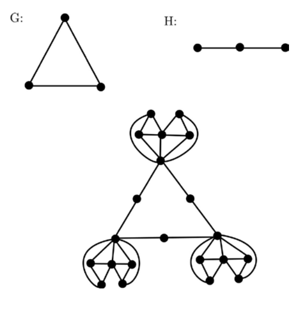


FIGURE 5

$$N[\xi] = \sum_{uv \in E(G)} \sqrt{d_G(u) + d_G(v)}.$$
$$= 6\sqrt{3+3} + 12\sqrt{5+3} + 12\sqrt{7+3} + 3\sqrt{7+5} + 6\sqrt{7+2} = 114.97.$$
$$N[\xi] \leq 2m_1\sqrt{\Delta_G + 2 + n_2 + m_2} + m_1m_2\left[\sqrt{4\Delta_H + 2} + 2\sqrt{2\Delta_H + 4} + \sqrt{5 + n_2 + m_2}\right]$$
$$+ m_1n_2\sqrt{2\Delta_H + 3 + n_2 + m_2} = 121.06.$$
$$N[\xi] \geq 2m_1\sqrt{\delta_G + 2 + n_2 + m_2} + m_1m_2\{\sqrt{4\delta_H + 2} + 2\sqrt{2\delta_H + 4} + \sqrt{5 + n_2 + m_2}\}$$
$$+ m_1n_2\sqrt{2\delta_H + 3 + n_2 + m_2} = 109.52.$$

Hence Theorem 4.4 verified.

**Theorem 4.5.** *The bounds for the Inverse sum indeg index of $\xi$-graph are determined by,*

$$ISI[\xi] \leq 2m_1\left[\frac{2(\Delta_G + n_2 + m_2)}{\Delta_G + 2 + n_2 + m_2}\right] + n_1m_2\left[\left[\frac{3(2\Delta_H + 1)}{2 + \Delta_H}\right] + \left[\frac{2\Delta_H + 1}{2}\right] + \left[\frac{3(\Delta_G + n_2 + m_2)}{\Delta_G + n_2 + m_2 + 3}\right]\right]$$
$$+ n_1n_2\left[\frac{(\Delta_G + n_2 + m_2)(2\Delta_H + 1)}{\Delta_G + n_2 + m_2 + 2\Delta_H + 1}\right].$$

*and*

$$ISI[\xi] \geq 2m_1\left[\frac{2(\delta_G + n_2 + m_2)}{\delta_G + 2 + n_2 + m_2}\right] + n_1m_2\left[\left[\frac{3(2\delta_H + 1)}{2 + \delta_H}\right] + \left[\frac{2\delta_H + 1}{2}\right] + \left[\frac{3(\delta_G + n_2 + m_2)}{\delta_G + n_2 + m_2 + 3}\right]\right]$$
$$+ n_1n_2\left[\frac{(\delta_G + n_2 + m_2)(2\delta_H + 1)}{\delta_G + n_2 + m_2 + 2\delta_H + 1}\right].$$

*Proof.* Consider,

$$ISI[\xi] = 2m_1\left[\frac{2(d_G + n_2 + m_2)}{d_G + n_2 + m_2 + 2}\right] + 2n_1m_2\left[\frac{3(2d_H + 1)}{3 + 2d_H + 1}\right] + n_1m_2\left[\frac{(2d_H + 1)^2}{2d_H + 1 + 2d_H + 1}\right]$$
$$+ n_1n_2\left[\frac{(2d_H + 1)(d_G + n_2 + m_2)}{d_G + n_2 + m_2 + 2d_H + 1}\right] + n_1m_2\left[\frac{3(d_G + n_2 + m_2)}{3 + d_G + n_2 + m_2}\right].$$
$$= 2m_1\left[\frac{2(d_G + n_2 + m_2)}{d_G + n_2 + m_2 + 2}\right] + 2n_1m_2\left[\frac{3(2d_H + 1)}{2d_H + 4}\right] + n_1m_2\left[\frac{(2d_H + 1)^2}{4d_H + 2}\right]$$
$$+ n_1n_2\left[\frac{(2d_H + 1)(d_G + n_2 + m_2)}{d_G + n_2 + m_2 + 2d_H + 1}\right] + n_1m_2\left[\frac{3(d_G + n_2 + m_2)}{3 + d_G + n_2 + m_2}\right].$$
$$= 2m_1\left[\frac{2(d_G + n_2 + m_2)}{d_G + 2 + n_2 + m_2}\right] + n_1m_2\left[\left[\frac{3(2d_H + 1)}{2 + d_H}\right] + \left[\frac{2d_H + 1}{2}\right] + \left[\frac{3(d_G + n_2 + m_2)}{d_G + n_2 + m_2 + 3}\right]\right]$$
$$+ n_1n_2\left[\frac{(d_G + n_2 + m_2)(2d_H + 1)}{d_G + n_2 + m_2 + 2d_H + 1}\right].$$
$$ISI[\xi] \leq 2m_1\left[\frac{2(\Delta_G + n_2 + m_2)}{\Delta_G + 2 + n_2 + m_2}\right] + n_1m_2\left[\left[\frac{3(2\Delta_H + 1)}{2 + \Delta_H}\right] + \left[\frac{2\Delta_H + 1}{2}\right] + \left[\frac{3(\Delta_G + n_2 + m_2)}{\Delta_G + n_2 + m_2 + 3}\right]\right]$$
$$+ n_1n_2\left[\frac{(\Delta_G + n_2 + m_2)(2\Delta_H + 1)}{\Delta_G + n_2 + m_2 + 2\Delta_H + 1}\right].$$

*Similarly,*

$$ISI[\xi] \geq 2m_1\left[\frac{2(\delta_G + n_2 + m_2)}{\delta_G + 2 + n_2 + m_2}\right] + n_1m_2\left[\left[\frac{3(2\delta_H + 1)}{2 + \delta_H}\right] + \left[\frac{2\delta_H + 1}{2}\right] + \left[\frac{3(\delta_G + n_2 + m_2)}{\delta_G + n_2 + m_2 + 3}\right]\right]$$

$$+n_1n_2\left[\frac{(\delta_G+n_2+m_2)(2\delta_H+1)}{\delta_G+n_2+m_2+2\delta_H+1}\right].$$

□

**Illustrative example 4.5.**: Consider two simple connected graphs $G$ and $H$, then

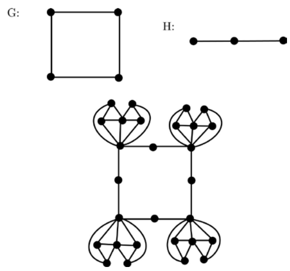


FIGURE 6. $\xi$-graph

$$ISI[\xi]=\sum_{uv\in E(G)}\left(\frac{d_ud_v}{d_u+d_v}\right)=8\left(\frac{9}{6}\right)+16\left(\frac{15}{8}\right)+16\left(\frac{21}{10}\right)+8\left(\frac{14}{9}\right)+4\left(\frac{35}{12}\right)=99.71.$$

$$ISI[\xi]\leq 2m_1\left[\frac{2(\Delta_G+n_2+m_2)}{\Delta_G+2+n_2+m_2}\right]+n_1m_2\left[\left[\frac{3(2\Delta_H+1)}{2+\Delta_H}\right]+\left[\frac{2\Delta_H+1}{2}\right]+\left[\frac{3(\Delta_G+n_2+m_2)}{\Delta_G+n_2+m_2+3}\right]\right]$$
$$+n_1n_2\left[\frac{(\Delta_G+n_2+m_2)(2\Delta_H+1)}{\Delta_G+n_2+m_2+2\Delta_H+1}\right]=114.20.$$
$$ISI[\xi]\geq 2m_1\left[\frac{2(\delta_G+n_2+m_2)}{\delta_G+2+n_2+m_2}\right]+n_1m_2\left[\left[\frac{3(2\delta_H+1)}{2+\delta_H}\right]+\left[\frac{2\delta_H+1}{2}\right]+\left[\frac{3(\delta_G+n_2+m_2)}{\delta_G+n_2+m_2+3}\right]\right]$$
$$+n_1n_2\left[\frac{(\delta_G+n_2+m_2)(2\delta_H+1)}{\delta_G+n_2+m_2+2\delta_H+1}\right]=90.44.$$

Hence Theorem 4.5. verified.

## 5. CONCLUSION

In this study, we focused on the $\xi$-graph and analyzed five significant topological indices to establish their bounds. Our exploration paves the way for future investigations into diverse classes of topological indices and their bounds for the $\xi$-graph.

## 6. ACKNOWLEDGEMENT

We express our thanks to anonymous referees for valuable comments that helped the quality of the article.

## REFERENCES

[1] Dafik, V. Lokesha, A. S. Maragadam, M. Manjunath, Ika Hesti Agustin, Semi-Total point Graph of Neighbourhood Edge Corona of two Graphs, European Journal of Pure and Applied Mahematics, 16(2), 2023, 1094-1109.

[2] E. Estrada, L. Torres, L. Rodriguez, I. Gutman, *An atom-bond connectivity index: modelling the enthalpy of formation of alkanes*, Indian Journal of Chemistry, 37A, 1998, 849–855.

[3] I.Gutman and N.Trinajstic, *Graph theory and molecular orbitals, total $\phi-$electron energy of alternant hydrocarbons*, Chemical Physics letters, 17(4), 1972, 535-538.

[4] I. Gutman, *Geometric approach to degree based topological indices*, MATCH Communications in Mathematical and in Computer Chemistry, 86(1), 2021, 11-16.

[5] V. R. Kulli, V. Lokesha, Sushmitha Jain, A.S. Maragadam, *Certain topological indices and related polynomials for Polysaccharides, TWMS Journal of pure and applied Mathematics*, 13(3), 2023, 990-997.

[6] V. R. Kulli, *Nirmala index*, Computation of Inverse Nirmala Indices of Certain Nanostructures, *International Journal of Mathematical combinatorics*, 2, 2021, 33-40.

[7] V. Lokesha, Sushmitha Jain and A.S.Maragadam, *M-polynomials for subdivision graphs of antiviral drugs using in treatment of covid-19*, International Asian congress on Contemporary Sciences-V, Azerbaijan Nakhchivan State University-İksad Publications, 2021, 1005-1018.

[8] V. Lokesha, A.S. Maragadam, Suvarna, Ismail Naci Cangul, *Topological coindices of phytochemicals examined for covid-19 therapy*, Proceedings of the Jangjeon Mathematical Society, 26(4), 2023, 381-405.

[9] V. Lokesha, Suvarna, A.S. Maragadam, *Bounds for the first Zagreb index of a splice and link graphs*, Advances in Mathematical Sciences and Applications, 33(1), 2024, 97–116.

[10] V. Lokesha, B. Shwetha Shetty, P. S. Ranjini, I. N. Cangul and A. S. Cevik, *New bounds for Randic and GA indices*, Journal of inequalities and applications, 1(180), 2013, 01-07.

[11] V. Lokesha, M. Manjunath, B. Chaluvaraju, K. M. Devendraiah, I. N. Cangul and A. S. Cevik, *Computation of Adriatic indices of certain operators of regular and complete bipartite graphs*, Advanced Studies in Contemporary Mathematics, 28(2), 2018, 231-244.

[12] M. Manjunath, V. Lokesha, Suvarna, Sushmitha Jain, *Bounds for the topological indices of $\wp$-graphs*, European Journal of the Pure and Applied Mathematics, 14(2), 2021, 340-350.

[13] M. Manjunath and V. Lokesha, *S-Corona operations of standard graphs in terms of degree sequences*, Proceedings of the Jangjeon Mathematical society, 23(2), 2020, 149-157.

[14] A. S. Maragadam, V. Lokesha, Sushmitha Jain, On ve-degree and ev-degree based molecular descriptors of copper oxide, European Chemical Bulletin, 12(4), 2023, 10704-10712.

[15] A. Milicevic, S. Nikolic and N. Trinajstic, *On reformulated Zagreb indices*, Molecular Divers, 8, 2004, 393-399.

[16] D. Vukicevic, M. Gašperov, *Bond additive modeling 1. Adriatic indices*, Croatica Chemica Acta, 83(3), 2010, 243-260.

[17] D. Vukicevic, *Bond Additive Modeling 2. Mathematical properties of Max-min rodeg index*, Croatica Chemica Acta, 83(3), 2010, 261-273.

[18] H. Wiener, *Structural determination of paraffin boiling points*, Journal of American Chemical Society, 69, 1947, 17-20.