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AN INTEGRAL PROBLEM SOLUTION VIA FIXED POINTS OF AN EXTENDED TYPE F-KANNAN MAPPING

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Abstract. The primary purpose of this article is to introduce an extended notion of recently introduced $\mathcal{F}$-Kannan mapping. The newly introduced class also subsumes the famous existing class of $\mathcal{F}$-contractions. A new sufficient condition for the existence of a fixed point for such an extended class of mappings is explored. A very interesting example is studied to prove that a mapping can possess a fixed point without demanding any strong conditions as in [9] thereby improving these previous results. Various examples are also included to justify that our extension is non-trivial and that the function $\mathcal{F}$ need not be necessarily continuous. Some very unique examples are developed which assures the closure properties of $\mathcal{F}$ functions under suitable operation thereby generating more such functions in literature. Further, a result is discussed to characterize complete metric spaces on the basis of this new class of mapping. The study is finally concluded with an application to an integral equation problem thereby showing the usability and effectiveness of our results.

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1. Introduction And Background

A point $w \in Z$ is called a fixed point of a self-mapping $\mathcal{T} : Z \to Z$ if $\mathcal{T}w = w$. Banach initiated the fundamental idea of fixed points of contraction mappings in 1922 in his dissertation. He proved the famous contraction mapping theorem in the framework of a complete metric space. Since then many authors have improved this principle with different approaches, some compromised or weakened the contractive condition on the mapping ([2], [11], [12]) while some involved the idea of altering the conditions on the space ([3], [14]). Many independent contractive conditions have also been explored on metric spaces and other general spaces along with fixed point existence results ([1], [7], [8], [10], [16]). Of these, the most famous class is that of the Kannan mappings [17] given by R. Kannan in 1968 with a fixed point result proved for this class in a complete metric space. A very different approach to generalizing a contraction mapping is the development of $\mathcal{F}$-contraction mapping given by Wardowski [13] in 2012 on a metric space. Since then, many researchers have made successful attempts to further improve various contraction mappings using the function $\mathcal{F}$ satisfying certain properties ([9], [5], [6], [4]). This concept of $\mathcal{F}$-contraction introduced by Wardowski is given below along with its fixed point existence result.

Consider a real-valued map $\digamma$ defined for all positive real numbers, $\mathbb{R}^+$ such that:

(Þ1) $\digamma$ is strictly increasing.
(Þ2) For any sequence $(g_k)_{k\in\mathbb{N}}$ of positive real numbers, $(g_k) \to 0 \Leftrightarrow (\digamma g_k)$ diverges to $-\infty$ as $k \to \infty$.
(Þ3) There is some $\lambda \in (0,1)$ such that $u^\lambda \digamma(u) \to 0$ as $u \to 0^+$.

Definition 1. [13] Let $(Z, \tilde{d})$ be a metric space and $\digamma$ be a mapping satisfying (Þ1)-(Þ3). A mapping $\mathcal{T} : Z \to Z$ is said to be a $\digamma$-contraction on $(Z, \tilde{d})$ if there exist $\Upsilon > 0$ such that, for each $\bar{\xi}, \bar{\varrho} \in Z$ with $\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) > 0$, the following contraction condition holds:

$$\Upsilon + \digamma[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] \leq \digamma[\tilde{d}(\bar{\xi}, \bar{\varrho})]$$

For a self-mapping G on a metric space $(Z, \tilde{d})$ and a point $w \in Z$, Rus and Petruşel [15] introduced a very significant iterative sequence called Picard iterative sequence given as $\{G^m w\}_{m\in\mathbb{N}}$ and G is called a Picard Operator if there exists a unique fixed point $\bar{w}$ in Z of G and $\lim_{m\to\infty} G^m w = \bar{w}$ for each w in Z.

Theorem 1. [13] Every $\digamma$-contraction on a complete metric space is a Picard Operator.

Very recently Batra *et al.* [5] introduced the concept of a $\digamma$-Kannan mapping as follows.

Definition 2. [5] Let $\digamma$ be a mapping satisfying (Þ1)-(Þ3) and $(Z, \tilde{d})$ denote a metric space with a metric $\tilde{d}$ on a non empty set Z. A mapping $\mathcal{T} : Z \to Z$ is said to be a $\digamma$-Kannan mapping if the following holds:

(K1) $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho} \Rightarrow \mathcal{T}\bar{\xi} \neq \bar{\xi}$ or $\mathcal{T}\bar{\varrho} \neq \bar{\varrho}$.
(K2) $\exists\ \Upsilon > 0$ such that $\Upsilon + \digamma[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] \leq \digamma\left[\dfrac{\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho}, \mathcal{T}\bar{\varrho})}{2}\right]$ for all $\bar{\xi}, \bar{\varrho} \in Z$ with $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$.

The corresponding fixed point result concerning $\digamma$-Kannan mapping is stated below.

Theorem 2. [5] Every $\digamma$-Kannan mapping on a complete metric space is a Picard Operator.

The following result which characterizes a metric space with the help of a $\digamma$-Kannan mapping was proved and extended by the authors recently. It is further claimed (Corollary 1) that such a characterization remains valid in the case of our newly improved and extended class of mappings.

Theorem 3. [5] A characterization for a metric space $(Z, \tilde{d})$ to be complete is that every $\digamma$-Kannan mapping on Z has a fixed point.

2. Main Results

In this section, we attempt to define a mapping called an extended $\digamma$-Kannan mapping with $\digamma$ satisfying conditions (Þ1)-(Þ3). This new class of mappings is a strict extension of already existing classes of $\digamma$-contractions and F-Kannan mappings and therefore subsumes those classes. Further, the new class of extended F-Kannan mappings strictly contains the united class of F-contractions and F-Kannan mappings since there exists a mapping that is neither a F-contraction nor a F-Kannan mapping as mentioned in Example 8. Further, it is proved that a fixed point of an extended F-Kannan mapping in a complete metric space exists uniquely. It is worth remarking here that our newly defined class of extended F-Kannan mapping is a particular case of F-weak contraction mapping given by Wardowski [9] but there are no corresponding constraints for our mapping to possess a fixed point. This argument is well supported by Example 10.

Throughout this article $(Z, \tilde{d})$ denotes a metric space with a metric $\tilde{d}$ on a non empty set Z, $\mathcal{T} : Z \to Z$ as a self-mapping on Z and for each $\bar{\xi}, \bar{\varrho} \in Z$

$$M_{(\bar{\xi},\bar{\varrho})} = Max\left\{\tilde{d}(\bar{\xi},\bar{\varrho}), \frac{\tilde{d}(\bar{\xi},\mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho},\mathcal{T}\bar{\varrho})}{2}\right\}$$

.

Definition 3. Let F be a mapping satisfying (Þ1)-(Þ3). A mapping $\mathcal{T} : Z \to Z$ is said to be an extended F-Kannan mapping if there exists a $\Upsilon > 0$ such that

$$\Upsilon + F[\tilde{d}(\mathcal{T}\bar{\xi},\mathcal{T}\bar{\varrho})] \leq F\left[M_{(\bar{\xi},\bar{\varrho})}\right] \tag{1}$$

for all $\bar{\xi}, \bar{\varrho} \in Z$ with $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$.

Some crucial remarks are presented below which are often used in our main results.

Remark 1. By (1) and Property (Þ1), the below inequality is true for any extended F-Kannan mapping $\mathcal{T}$:

For each $\bar{\xi}, \bar{\varrho} \in Z$ we have, $\tilde{d}(\mathcal{T}\bar{\xi},\mathcal{T}\bar{\varrho}) \leq M_{(\bar{\xi},\bar{\varrho})}$

Remark 2. If $\mathcal{T}$ is an extended F-Kannan mapping, then it has at most one fixed point. In fact, by Equation (1), the existence of two distinct fixed points results in $\Upsilon \leq 0$.

We now obtain different types of extended F-Kannan mappings corresponding to some known mappings F with properties (Þ1)-(Þ3) in literature.

Example 1. Define F_1 on $\mathbb{R}^+$ as $F_1(t) = \ln t$. Then clearly F_1 satisfies all properties (Þ1), (Þ2) and (Þ3). The corresponding condition given by (1) reduces to:

$$\frac{\tilde{d}(\mathcal{T}\bar{\xi},\mathcal{T}\bar{\varrho})}{M_{(\bar{\xi},\bar{\varrho})}} \leq e^{-\Upsilon} \tag{2}$$

for all $\bar{\xi}, \bar{\varrho} \in Z$ with $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$.

Example 2. Corresponding to $\digamma_2(t) = \ln(t^2+t) = \ln t + \ln(t+1)$, defined for all $t > 0$ and satisfying properties (Þ1), (Þ2) and (Þ3), the condition given by (1) takes the following form:

$$\frac{\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})(\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})+1)}{(M_{(\bar{\xi},\bar{\varrho})})(M_{(\bar{\xi},\bar{\varrho})}+1)} \leq e^{-\Upsilon} \tag{3}$$

for all $\bar{\xi}, \bar{\varrho} \in Z$ with $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$.

Example 3. Define $\digamma_3$ on $\mathbb{R}^+$ as $\digamma_3(t) = \ln t + t$. Then again, $\digamma_3$ satisfies all properties (Þ1), (Þ2) and (Þ3). In this case, the corresponding condition given by (1) becomes the following inequality:

$$\frac{\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})}{M_{(\bar{\xi},\bar{\varrho})}} e^{\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) - M_{(\bar{\xi},\bar{\varrho})}} \leq e^{-\Upsilon} \tag{4}$$

for all $\bar{\xi}, \bar{\varrho} \in Z$ with $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$.

A very crucial example is mentioned below to illustrate that the mapping $\digamma$ is not necessarily continuous.

Example 4. Define $\digamma_4$ on $\mathbb{R}^+$ as

$$\digamma_4(t) = \begin{cases} \ln t & 0 < t < 1 \\ t & t \geq 1 \end{cases}$$

Then clearly, $\digamma_4$ is strictly increasing. Further, it also satisfies property (Þ2). Property (Þ3) also holds for $\lambda = 0.5$.

The following example illuminates ways to generate new mappings $\digamma$ from already known ones.

Example 5. If $\digamma_a$ and $\digamma_b$ are two mappings defined on $\mathbb{R}^+$ satisfying (Þ1)-(Þ3), then:

(1) $\digamma_a + \digamma_b$

(2) $k\digamma_a$ for all $k \in \mathbb{Z}^+$

also satisfy these properties.

The following are some examples of extended $\digamma$-Kannan mappings. The first two examples follow from the corresponding contraction like conditions satisfied by the respective mappings.

Example 6. *Every $\digamma$-contraction mapping is an extended $\digamma$-Kannan mapping.*

Example 7. *Every $\digamma$-Kannan mapping is an extended $\digamma$-Kannan mapping.*

The next example shows that the converse of the statements of Example 6 and Example 7 need not be true. This illuminates the fact that our newly developed class of mappings is a strictly larger class as compared to the classes of $\digamma$-contractions and $\digamma$-Kannan mappings.

Example 8. Let $\mathcal{T} : [0,1] \to [0,1]$ be defined by

$$\mathcal{T}\bar{\xi} = \begin{cases} 0.3\bar{\xi}, & 0 \leq \bar{\xi} < 1 \\ 0.4 & \bar{\xi} = 1 \end{cases}$$

It is proved that $\mathcal{T}$ strictly belongs to the class of extended $\digamma$-Kannan corresponding to $\digamma = \digamma_1$ of Example 1. Observe that $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$ if and only if $\bar{\xi} \neq \bar{\varrho}$. The following case analysis leads to the existence of a positive number Υ satisfying (2).

(i): Exactly one of $\bar{\xi}$ and $\bar{\varrho}$ is 1, say $\bar{\varrho} = 1$ and $0 \leq \bar{\xi} < 1$.

Then, $\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) = 0.4 - 0.3\bar{\xi} \leq 0.4$ and

$$M_{(\bar{\xi},\bar{\varrho})} = \max\left\{1 - \bar{\xi}, \frac{0.7\bar{\xi} + 0.6}{2}\right\} = \max\left\{1 - \bar{\xi}, 0.35\bar{\xi} + 0.3\right\} \geq 0.5.$$

So

$$\frac{\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})}{M_{(\bar{\xi},\bar{\varrho})}} \leq \frac{0.4}{0.5} = 0.8 \tag{5}$$

(ii): $0 \leq \bar{\xi} < 1$ and $0 \leq \bar{\varrho} < 1$ such that $\bar{\xi} \neq \bar{\varrho}$

Then,

$$M_{(\bar{\xi},\bar{\varrho})} = \max\left\{|\bar{\xi} - \bar{\varrho}|, \frac{0.7(\bar{\xi} + \bar{\varrho})}{2}\right\} = \max\left\{|\bar{\xi} - \bar{\varrho}|, 0.35(\bar{\xi} + \bar{\varrho})\right\} \geq |\bar{\xi} - \bar{\varrho}|$$

and $\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) = 0.3|\bar{\xi} - \bar{\varrho}|$. Therefore,

$$\frac{\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})}{M_{(\bar{\xi},\bar{\varrho})}} \leq 0.3 \tag{6}$$

It can be concluded from (5) and (6) that a suitable choice of Υ can be obtained by setting $e^{-\Upsilon} = \max\{0.8, 0.3\} = 0.8$. Being a discontinuous mapping, $\mathcal{T}$ is not a $\digamma$-contraction for any $\digamma$. Further, assume there exists an $\digamma$ satisfying (Þ1)-(Þ3) and $\Upsilon > 0$ such that $\Upsilon + \digamma[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] \leq \digamma\left[\frac{\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho}, \mathcal{T}\bar{\varrho})}{2}\right]$ for every pair $\bar{\xi}, \bar{\varrho} \in Z$ satisfying $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$. Then it follows that $\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) \leq \frac{\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho}, \mathcal{T}\bar{\varrho})}{2}$ $\forall \bar{\xi}, \bar{\varrho} \in Z : \mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$. But this inequality fails for $\bar{\varrho} = 1$ and $\bar{\xi} = 0.1$. Indeed, we have

$$\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) = 0.37 > \frac{\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho}, \mathcal{T}\bar{\varrho})}{2} = 0.335.$$

This justifies that $\mathcal{T}$ is not a $\digamma$-Kannan mapping for any $\digamma$ satisfying (Þ1)-(Þ3) .

The example mentioned below supports the fact that a mapping may be an extended $\digamma$-Kannan mapping with respect to different $\digamma$ maps. In particular, in the below example, it is proved that the above mapping $\mathcal{T}$ is also an extended $\digamma$-Kannan mapping with respect to a discontinuous $\digamma$ map.

Example 9. Define $\mathcal{T}$ on $[0, 1]$ as:

$$\mathcal{T}\bar{\xi} = \begin{cases} 0.3\bar{\xi} & \bar{\xi} \in [0, 1) \\ 0.4 & \bar{\xi} = 1 \end{cases}$$

It is asserted that $\mathcal{T}$ is strictly an extended $\digamma$-Kannan mapping. In fact, since $\mathcal{T}$ is discontinuous it is not $\digamma$-contraction for any F.

Further, assume there exist some $\digamma$ satisfying (Þ1)-(Þ3) and $\Upsilon > 0$ such that inequality (1) holds.
Then it is implied that
$\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) \leq \dfrac{\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho}, \mathcal{T}\bar{\varrho})}{2} \quad \forall \bar{\xi}, \bar{\varrho} \in Z : \mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$
But the above inequality fails for $\bar{\varrho} = 1$ and $\bar{\xi} = 0.1$. Indeed, we have

$$\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) = 0.37 > \frac{\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho}, \mathcal{T}\bar{\varrho})}{2} = 0.335$$

Hence, it follows that $\mathcal{T}$ is also not $\digamma$-Kannan mapping for any $\digamma$.
Next, it is required to find $\Upsilon > 0$ and a mapping $\digamma$ such that

$$\Upsilon \leq \digamma[M_{(\bar{\xi}, \bar{\varrho})}] - \digamma[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] \tag{7}$$

for all $\bar{\xi}, \bar{\varrho} \in [0, 1]$ such that $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$.

Choose $\digamma : \mathbb{R}^+ \to \mathbb{R}$ as in Example 4. The following cases arise for pairs of $(\bar{\xi}, \bar{\varrho})$ such that $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$.

(i) Exactly one of $\bar{\xi}$ and $\bar{\varrho}$ is 1. Let $\bar{\varrho} = 1$ and $\bar{\xi} \in [0, 1)$.

Then $\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) = 0.4 - 0.3\bar{\xi}$ and $\digamma[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] = \ln[0.4 - 0.3\bar{\xi}]$
Further,

$$\begin{aligned} M_{(\bar{\xi}, \bar{\varrho})} &= \max\left\{1 - \bar{\xi}, \frac{0.7\bar{\xi} + 0.6}{2}\right\} \\ &= \max\left\{1 - \bar{\xi}, 0.35\bar{\xi} + 0.3\right\} \\ &= \begin{cases} 1 & \texttt{if} \quad \bar{\xi} = 0 \\ \max\left\{1 - \bar{\xi}, 0.35\bar{\xi} + 0.3\right\} & \texttt{if} \quad \bar{\xi} \neq 0 \end{cases} \end{aligned}$$

Thus,

$$\begin{aligned} \digamma[M_{(\bar{\xi}, \bar{\varrho})}] - \digamma[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] &= \begin{cases} 1 - \ln 0.4 & \texttt{if} \quad \bar{\xi} = 0 \\ \ln[\frac{\max\{1 - \bar{\xi}, 0.35\bar{\xi} + 0.3\}}{\{0.4 - 0.3\bar{\xi}\}}] & \texttt{if} \quad \bar{\xi} \neq 0 \end{cases} \\ &\geq 0.2 \end{aligned} \tag{8}$$

(ii) Both $\bar{\xi}$ and $\bar{\varrho} \in [0, 1)$ and $\bar{\xi} \neq \bar{\varrho}$. Then,

$$\begin{aligned} \digamma[M_{(\bar{\xi}, \bar{\varrho})}] - \digamma[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] &= \ln[\max\left\{|\bar{\xi} - \bar{\varrho}|, 0.35(\bar{\xi} + \bar{\varrho})\right\}] - \ln[0.3|\bar{\xi} - \bar{\varrho}|] \\ &\geq \ln[|\bar{\xi} - \bar{\varrho}|] - \ln[0.3|\bar{\xi} - \bar{\varrho}|] \\ &= \ln(10/3) > 0.2 \end{aligned} \tag{9}$$

Equations (8) and (9) lead to a choice of $\Upsilon = 0.2$ thereby satisfying (7). Hence $\mathcal{T}$ is an Extended $\digamma$-Kannan Mapping with $\digamma$ being a discontinuous map.

We now propose a result that may be used to conclude that the class of extended $ln(\beta)$-Kannan mappings is a subclass of extended $(ln(\beta) + \beta)$-Kannan mappings as well as of the extended $ln(\beta^2 + \beta)$-Kannan mappings.

Proposition 1. *Consider $\digamma, H$ be mappings satisfying* (Þ1)-(Þ3) *such that $\digamma \leq H$ and also $H - \digamma$ is non-decreasing. Then, if $\mathcal{T}$ is an extended $\digamma$-Kannan mapping it is also an extended H-Kannan contraction.*

Proof. Using Remark 1, we have $\tilde{d}(\mathcal{T}\bar{\xi},\mathcal{T}\bar{\varrho}) \le M_{(\bar{\xi},\bar{\varrho})}$ for all $\bar{\xi},\bar{\varrho}\in Z$.
Since $H-F$ is assumed to be non-decreasing, so we obtain:

$$H[\tilde{d}(\mathcal{T}\bar{\xi},\mathcal{T}\bar{\varrho})] \le F[\tilde{d}(\mathcal{T}\bar{\xi},\mathcal{T}\bar{\varrho})] + H[M_{(\bar{\xi},\bar{\varrho})}] - F[M_{(\bar{\xi},\bar{\varrho})}]$$

provided $\mathcal{T}\bar{\xi}\neq\mathcal{T}\bar{\varrho}$.
Further, using the fact that $\mathcal{T}$ is an extended F-Kannan map, we have some positive number Υ satisfying:

$$H[\tilde{d}(\mathcal{T}\bar{\xi},\mathcal{T}\bar{\varrho})] \le H[M_{(\bar{\xi},\bar{\varrho})}] - \Upsilon$$

for every $\bar{\xi},\bar{\varrho}\in Z$, $\mathcal{T}\bar{\xi}\neq\mathcal{T}\bar{\varrho}$. This proves the above proposition. □

Lemma 1. *Let $\mathcal{T}: Z\to Z$ be an extended F-Kannan mapping for some F satisfying* (Þ1)-(Þ3)*. Then for all $\bar{\xi}\in Z$,*

$$\lim_{n\to\infty}\tilde{d}(\mathcal{T}^n\bar{\xi},\mathcal{T}^{n+1}\bar{\xi}) = 0.$$

Proof. Let $\bar{\xi}\in Z$ be fixed. If $\mathcal{T}^n\bar{\xi}=\mathcal{T}^{n+1}\bar{\xi}$ for some $n\in\mathbb{N}$, then, the sequence $\{\mathcal{T}^n\bar{\xi}\}_{n\in\mathbb{N}}$ becomes eventually constant and hence the sequence $\{\tilde{d}(\mathcal{T}^n\bar{\xi},\mathcal{T}^{n+1}\bar{\xi})\}$ converges to zero. Consider the case when we have

$$\mathcal{T}^n\bar{\xi}\neq\mathcal{T}^{n+1}\bar{\xi} \text{ for all } n\in\mathbb{N}.$$

Then by Equation (1), we obtain:
$\Upsilon + F[\tilde{d}(\mathcal{T}^n\bar{\xi},\mathcal{T}^{n+1}\bar{\xi})] \le F[M_{(\mathcal{T}^{n-1}\bar{\xi},\mathcal{T}^n\bar{\xi})}]$ for every $n\in\mathbb{N}$.
Letting $A_n=\tilde{d}(\mathcal{T}^{n-1}\bar{\xi},\mathcal{T}^n\bar{\xi})$, we have for every $n\in\mathbb{N}$,

$$\Upsilon + F[A_{n+1}] \le F\left[\max\{A_n, \frac{A_n+A_{n+1}}{2}\}\right] \tag{10}$$

Further, using property (Þ1) of F, it can be asserted that

$$\max\{A_n, \frac{A_n+A_{n+1}}{2}\} = A_n$$

for each $n\in\mathbb{N}$. In fact if for some $n\in\mathbb{N}$, we have:

$$A_n \le \frac{A_n+A_{n+1}}{2} \le A_{n+1}$$

Then (10) reduces to $\Upsilon\le 0$.
Therefore, we must have for each $n\in\mathbb{N}$,

$$\Upsilon + F[A_{n+1}] \le F[A_n]. \tag{11}$$

A repeated application of (11) results in the following relation

$$F[A_{n+1}] \le F[A_1] - n\Upsilon \quad \text{for all } n. \tag{12}$$

Letting n tending to infinity and using property (Þ2), we get the desired conclusion.

□

Theorem 4. *On a metric space $(Z,\tilde{d})$ that is complete, any extended F-Kannan operator $\mathcal{T}$, satisfies the following:*

(1) *$\mathcal{T}$ has a unique fixed point ξ^* in Z.*
(2) *For any $\bar{\xi}\in Z$, the sequence $\{\mathcal{T}^n\bar{\xi}\}$ converges to ξ^*.*

Proof. Let $\bar{\xi} \in Z$. We claim that the sequence $\{\mathcal{T}^n\bar{\xi}\}_{n\in\mathbb{N}}$ converges in Z. By Lemma 1, $\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi}) \to 0$ as $n \to \infty$. If $\mathcal{T}^n\bar{\xi} = \mathcal{T}^{n+1}\bar{\xi}$ for some $n \in \mathbb{N}$ then the sequence $\{\mathcal{T}^n\bar{\xi}\}_{n\in\mathbb{N}}$ being an eventually constant sequence is convergent, otherwise $\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi}) > 0$ for all $n \in \mathbb{N}$ and then, by using property (Þ3) of F, there exists $\beta \in (0, 1)$ such that

$$\lim_{n\to\infty}(\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi}))^\beta F[\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi})] = 0. \tag{13}$$

From Equation (12), we have for every $n \in \mathbb{N}$,

$$[\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi})]^\beta F[\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi})] - [\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi})]^\beta F[\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi})] \leq -n\Upsilon[\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi})]^\beta$$

Letting $n \to \infty$ in the above inequality, and using Lemma 1 and Equation (13), we obtain

$$\lim_{n\to\infty} n[\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi})]^\beta = 0.$$

Thus, there is at least one number $m_1 \in \mathbb{N}$ such that $n[\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi})]^\beta \leq 1$ for all $n \geq m_1$. Consequently, we have

$$\tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi}) \leq \frac{1}{n^{1/\beta}}$$

for all $n \geq m_1$.

$$\text{Therefore, } \sum_{n=m_1}^{\infty} \tilde{d}(\mathcal{T}^n\bar{\xi}, \mathcal{T}^{n+1}\bar{\xi}) \quad \text{converges.} \tag{14}$$

Consider any $n, m \in \mathbb{N}, n > m \geq m_1$. Then,

$$\begin{aligned}\tilde{d}(\mathcal{T}^m\bar{\xi}, \mathcal{T}^n\bar{\xi}) &\leq \tilde{d}(\mathcal{T}^m\bar{\xi}, \mathcal{T}^{m+1}\bar{\xi}) \\ &\quad + \tilde{d}(\mathcal{T}^{m+1}\bar{\xi}, \mathcal{T}^{m+2}\bar{\xi}) + \cdots + \\ &\quad + \cdots + \tilde{d}(\mathcal{T}^{n-1}\bar{\xi}, \mathcal{T}^n\bar{\xi}) \\ &\quad \to 0 \text{ as } m \to \infty\end{aligned}$$

Thus the sequence $\{\mathcal{T}^n\bar{\xi}\}$ is a Cauchy sequence in Z and hence converges in Z by the completeness property of Z. So, there exists a point $\bar{\xi}*$ in Z to which the sequence $\{\mathcal{T}^n\bar{\xi}\}$ converges. It remains to be proved that $\bar{\xi}*$ is a fixed point of $\mathcal{T}$ and then the uniqueness of the fixed point of $\mathcal{T}$ follows from Remark 2. Consider

$$\tilde{d}(\bar{\xi}*, \mathcal{T}\bar{\xi}*) \leq \tilde{d}(\bar{\xi}*, \mathcal{T}^{n+1}\bar{\xi}) + \tilde{d}(\mathcal{T}^{n+1}\bar{\xi}, \mathcal{T}\bar{\xi}*) \text{ for all } n \in \mathbb{N}$$

Further, by Remark 1,

$$\tilde{d}(\bar{\xi}*, \mathcal{T}\bar{\xi}*) \leq \tilde{d}(\bar{\xi}*, \mathcal{T}^{n+1}\bar{\xi}) + M_{(\mathcal{T}^n\bar{\xi}, \bar{\xi}*)} \text{ for all } n \in \mathbb{N}$$

Letting $n \to \infty$ in the above inequality and by using Lemma 1, we have

$$\tilde{d}(\bar{\xi}*, \mathcal{T}\bar{\xi}*) = 0.$$

□

The example considered below illustrates that Theorem 4 generalizes previously established results in literature, particularly, Theorem 1 and Theorem 2. Also, it is noted that the mapping $\mathcal{T}$ is an extended F-Kannan (and hence F-weak contraction of [9]) which possesses a fixed point with no condition imposed on $\mathcal{T}$ and F. Whereas Wardowski in [9] demands the continuity of at least one of these maps.

Example 10. *Let $\mathcal{T}$ be as in Example 9. Then clearly this mapping possesses a unique fixed point $x = 0$ but it is worth noting that the existence of this unique fixed point is guaranteed by Theorem 4 since $\mathcal{T}$ is an extended F-Kannan mapping whereas Theorem 1 and Theorem 2 are not applicable, indeed $\mathcal{T}$ is neither a F-Kannan mapping nor a F-contraction for any F. Further, this unique fixed point is not guaranteed by Wardowski* [9] *as neither $\mathcal{T}$ nor F is continuous.*

Corollary 1. *A necessary and sufficient condition for a metric space $(Z, \tilde{d})$ to be complete is that every extended F-Kannan mapping on Z has a fixed point where F is a mapping that satisfies conditions* (Þ1)-(Þ3).

Proof. The proof follows immediately by combining Theorem 3 and Theorem 4. □

Corollary 2. *Let $(Z, \tilde{d})$ be a complete metric space and $\mathcal{T}$, a self-mapping on Z satisfying*

$$\Upsilon + F[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] \leq F\left[a\tilde{d}(\bar{\xi}, \bar{\varrho}) + b\frac{\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho}, \mathcal{T}\bar{\varrho})}{2}\right] \tag{15}$$

for all $\bar{\xi}, \bar{\varrho} \in Z$ with $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$, for some $\Upsilon > 0$, F satisfying (Þ1)-(Þ3) *and $a, b \geq 0$ such that $a + b \leq 1$. Then $\mathcal{T}$ has a unique fixed point.*

Proof. Using property (Þ1), we have for each $\bar{\xi}, \bar{\varrho} \in Z$ with $\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) > 0$

$$\begin{aligned}\Upsilon + F[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] &\leq F\left[a\tilde{d}(\bar{\xi}, \bar{\varrho}) + b\frac{\tilde{d}(\bar{\xi}, \mathcal{T}\bar{\xi}) + \tilde{d}(\bar{\varrho}, \mathcal{T}\bar{\varrho})}{2}\right] \\ &\leq F\left[(a+b)M_{(\bar{\xi}, \bar{\varrho})}\right] \\ &\leq F[M_{(\bar{\xi}, \bar{\varrho})}] \text{ since } a + b \leq 1.\end{aligned}$$

Therefore, condition (1) is obtained proving that such a mapping is an extended F-Kannan mapping. Hence the proof follows from Theorem 4. □

3. Application to Integral Equations

Let $I = [0, 1]$ and $Z = C(I, \mathbb{R})$ be the set of all continuous real-valued functions on I. Consider the metric $\tilde{d}$ on Z as

$$\tilde{d}(\bar{\xi}, \bar{\varrho}) = \sup_{\theta \in I} |\bar{\xi}(\theta) - \bar{\varrho}(\theta)|.$$

Then clearly, $(Z, \tilde{d})$ is a complete metric space. We seek a unique solution in Z of the integral equation given below, as an application of our work.

$$z(\theta) = h(\theta) + \int_0^1 f(\theta, t, z(t))dt \tag{16}$$

where $\theta \in I$ and $h : I \to \mathbb{R}$ is continuous. Also $f : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Define a self-mapping $\mathcal{T}$ on Z as:

$$(\mathcal{T}z)(\theta) = h(\theta) + \int_0^1 f(\theta, t, z(t))dt \text{ for all } \theta \in I.$$

It is observed that $z \in Z$ is the unique solution of the integral equation (16) if and only if z is the unique fixed point of $\mathcal{T}$.

Define a function $L_{(\bar{\xi},\bar{\varrho})} : I \to \mathbb{R}$ by

$$L_{(\bar{\xi},\bar{\varrho})}(t) = \max\left\{\bar{\xi}(t) - \bar{\varrho}(t), \frac{\left(\bar{\xi}(t) - (\mathcal{T}\bar{\xi})(t)\right) + (\bar{\varrho}(t) - (\mathcal{T}\bar{\varrho})(t))}{2}\right\}$$

It can be easily observed that $L_{(\bar{\xi},\bar{\varrho})}$ is continuous on I and $L_{(\bar{\xi},\bar{\varrho})}(t) \leq M_{(\bar{\xi},\bar{\varrho})}$ for all $t \in I$.

Theorem 5. *Let $I = [0,1]$ and $\beta : I \times I \to \mathbb{R}$, $f : I \times I \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that*

(i) $|f(\theta, t, \bar{\xi}(t)) - f(\theta, t, \bar{\varrho}(t))| \leq \beta(\theta, t)L_{(\bar{\xi},\bar{\varrho})}(t)$ *for all* $t, \theta \in I$ *and* $\bar{\xi}, \bar{\varrho} \in Z = C(I, \mathbb{R})$,

(ii) *There exists a number* $0 < k < 1$ *satisfying* $\int_0^1 \beta(\theta, t)dt \leq k$ *for all* $\theta \in I$.

Then there exists a unique $z \in Z$ which is a solution to the integral equation(16).

Proof. We show that $\mathcal{T}$ is an extended F-Kannan mapping under the above conditions (i) and (ii).
Consider for a fixed $\theta \in I$ and $\bar{\xi}, \bar{\varrho} \in Z$,

$$\begin{aligned}
|(\mathcal{T}\bar{\xi})(\theta) - (\mathcal{T}\bar{\varrho})(\theta)| &= \left|h(\theta) + \int_0^1 f(\theta, t, \bar{\xi}(t)dt - h(\theta) - \int_0^1 f(\theta, t, \bar{\varrho}(t))dt\right| \\
&= \left|\int_0^1 f(\theta, t, \bar{\xi}(t)) - \int_0^1 f(\theta, t, \bar{\varrho}(t))dt\right| \\
&\leq \int_0^1 \left|f(\theta, t, \bar{\xi}(t) - f(\theta, t, \bar{\varrho}(t))\right| dt \\
&\leq \int_0^1 \beta(\theta, t)L_{(\bar{\xi},\bar{\varrho})}(t)dt \\
&\leq \int_0^1 \beta(\theta, t)M_{(\bar{\xi},\bar{\varrho})}dt \\
&= M_{(\bar{\xi},\bar{\varrho})}\int_0^1 \beta(\theta, t)dt \\
&\leq kM_{(\bar{\xi},\bar{\varrho})}
\end{aligned}$$

So $\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho}) = \sup_{\theta \in I} |(\mathcal{T}\bar{\xi})(\theta) - (\mathcal{T}\bar{\varrho})(\theta)| \leq kM_{(\bar{\xi},\bar{\varrho})}$. This gives $\Upsilon = \ln(1/k) > 0$ such that for $\bar{\xi}, \bar{\varrho} \in Z$ with $\mathcal{T}\bar{\xi} \neq \mathcal{T}\bar{\varrho}$, we have

$$\Upsilon + \ln[\tilde{d}(\mathcal{T}\bar{\xi}, \mathcal{T}\bar{\varrho})] \leq \ln[M_{(\bar{\xi},\bar{\varrho})}]$$

This proves that $\mathcal{T}$ is an extended ln(.)-Kannan mapping. So, by Theorem 4, $\mathcal{T}$ possesses a unique fixed point $z \in Z$ which is the desired unique solution of the integral equation (16). □

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