Proceedings of the Jangjeon Mathematical Society
28 (2025), No. 1, pp. 85 - 95
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2025.28.1.85

INTEGRAL REPRESENTATIONS AND FORMULAS FOR THE UNIFIED AND MODIFIED PRESENTATION OF FUBINI NUMBERS AND POLYNOMIALS

NESLIHAN KILAR

ABSTRACT. The aim of this paper is to give some different type integral representations for the unified and modified presentation of Fubini numbers and polynomials. By applying both the p-adic integrals including the Volkenborn integral and the fermionic p-adic integral, and the Riemann integral to the generating functions of the unified and modified presentation of Fubini numbers and polynomials, we give some integral representations including these numbers and polynomials, as well as the Daehee numbers, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, and some special numbers. Using the generating function methods, we also give some formulas for these numbers and polynomials.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 05A15, 11B68, 11B73, 11S80, 26C05.

KEYWORDS AND PHRASES. Bernoulli numbers and polynomials, Euler numbers and polynomials, Daehee numbers, Unified and modified presentation of Fubini numbers and polynomials, p-adic integrals, Riemann integral, Generating functions.

1. INTRODUCTION

It is well known that the Riemann integral, the p-adic integrals with their integral equations, and generating function methods have many applications in mathematics, mathematical physics, engineering science with real world problems, and many areas. Moreover, some special numbers and polynomials and their generating functions are used to solve combinatorial problems. One of them, the Fubini numbers, is also known as the ordered Bell numbers. These numbers count the number of weak orderings on a set of n elements and have many applications in number theory and enumerative combinatorics. Further, many researchers have both investigated these types of numbers and given many results (*cf.* [1–14, 21–23]). Therefore, the goal of this paper is to study the unified and modified presentation of Fubini numbers and polynomials with the aid of the p-adic integral and generating function methods. Before the results of this paper, we briefly introduce some special numbers and polynomials and their generating functions with their properties. We also introduce some definitions and properties of p-adic integrals. Afterwards, using generating functions, the p-adic integrals with their integral equations, and the Riemann integral methods, we obtain some formulas and identities for the unified and modified presentation of Fubini numbers and polynomials and some special numbers and polynomials.

This paper is dedicated to Professor Taekyun Kim on the occasion of his 60th anniversary.

1.1. **Generating functions for some special numbers and polynomials.** Here, we introduce some generating functions for the special numbers and polynomials. We also give some properties of these numbers and polynomials.

Firstly, we give some notations that are used during the paper:

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

Let $\mathbb{Z}, \mathbb{R}$, and $\mathbb{Q}$ denote sets of integer numbers, real numbers, and complex numbers, respectively (*cf.* [1–32]).

The Bernoulli numbers and polynomials are defined, respectively, by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{1}$$

and

$$\frac{te^{wt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(w) \frac{t^n}{n!}, \tag{2}$$

where $|t| < 2\pi$ (*cf.* [1, 28, 31, 32]; and references cited therein).

Setting $w = 0$ in (2), one has

$$B_n(0) = B_n.$$

The Euler numbers and polynomials are defined, respectively, by

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \tag{3}$$

and

$$\frac{2e^{wt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(w) \frac{t^n}{n!}, \tag{4}$$

where $|t| < \pi$ (*cf.* [1, 28, 31, 32]; and references cited therein).

For $w = 0$ in (4), we get

$$E_n(0) = E_n.$$

The numbers $y_{9,n}(\lambda; c)$ and the polynomials $y_{9,n}(w, \lambda; c)$ are defined by

$$\frac{2}{c^t + \lambda} = \sum_{n=0}^{\infty} y_{9,n}(\lambda; c) \frac{t^n}{n!} \tag{5}$$

and

$$\frac{2c^{wt}}{c^t + \lambda} = \sum_{n=0}^{\infty} y_{9,n}(w, \lambda; c) \frac{t^n}{n!} \tag{6}$$

(*cf.* [29, p. 53, Eqs. (5.1)-(5.2)]).

When $w = 0$ in (6), we see that

$$y_{9,n}(0, \lambda; c) = y_{9,n}(\lambda; c).$$

The Fubini type numbers and polynomials of order r are defined, respectively, by

$$\frac{2^r}{(2-e^t)^{2r}} = \sum_{n=0}^{\infty} a_n^{(r)} \frac{t^n}{n!} \tag{7}$$

and

$$\frac{2^r e^{wt}}{(2-e^t)^{2r}} = \sum_{n=0}^{\infty} a_n^{(r)}(w) \frac{t^n}{n!} \tag{8}$$

(*cf.* [9, pp. 1611-1612, Eqs. (18)-(19)]; see also [5–14]).

The unified and modified presentation of Fubini numbers and polynomials of order r are defined, respectively, by

$$\begin{aligned} N(t,r;\mu,\vartheta,b) &= \frac{2^r}{(\mu b^t - \vartheta)^{2r}} \\ &= \sum_{n=0}^{\infty} \mathfrak{a}_n^{(r)}(\mu;\vartheta,b) \frac{t^n}{n!} \end{aligned} \tag{9}$$

and

$$\begin{aligned} N(t,r;w;\mu,\vartheta,b,c) &= \frac{2^r c^{tw}}{(\mu b^t - \vartheta)^{2r}} \\ &= \sum_{n=0}^{\infty} \mathfrak{a}_n^{(r)}(w,\mu;\vartheta,b,c) \frac{t^n}{n!}, \end{aligned} \tag{10}$$

where $b,c \in \mathbb{R}^+$ with $b,c \geq 1$, $\vartheta,\mu,t \in \mathbb{C}$ and $\mu \neq \vartheta$, $|t| < \frac{2\pi}{|\ln b|}$ when $\mu = \vartheta$; $\left|t \ln b + \ln\left(\frac{\mu}{\vartheta}\right)\right| < 2\pi$ when $\mu \neq \vartheta$; $1^r := 1$ (*cf.* [14, p. 6, Eqs. (2.1)-(2.2)]).

Setting $w = 0$ in (10), we get

$$\mathfrak{a}_n^{(r)}(0,\mu;\vartheta,b,c) = \mathfrak{a}_n^{(r)}(\mu;\vartheta,b).$$

When $\mu = 1$, $\vartheta = 2$ and $b = c = e$ in (9) and (10), one has

$$\mathfrak{a}_n^{(r)}(w,1;2,e) = a_n^{(r)}$$

and

$$\mathfrak{a}_n^{(r)}(w,1;2,e,e) = a_n^{(r)}(w).$$

1.2. ***p*-adic integrals.** Here we give some definitions and properties of p-adic integrals.

Let p be an odd prime number. Let $\mathbb{Z}_p$ denote set of p-adic integers. Let $\mathbb{Q}_p$ denote set equipped with the norm $|x|_p$ is a topological completion of $\mathbb{Q}$, and $|x|_p$ is defined by

$$|x|_p = \begin{cases} p^{-ord_p(x)}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases},$$

where $ord_p(m)$ denote the greatest integer v ($v \in \mathbb{N}_0$) such that p^v divides m in $\mathbb{Z}$. Let $\mathbb{C}_p$ is the field of p-adic completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{K}$ be field with a complete valuation and $C^1(\mathbb{Z}_p \to \mathbb{K})$ be set of continuous derivative functions:

$$\left\{ f : \mathbb{Z}_p \to \mathbb{K};\ f(y) \text{ is differentiable and } \frac{d}{dy} f(y) \text{ is continuous} \right\}$$

(*cf.* [27]).

The Volkenborn integral (or the bosonic p-adic integral) is defined by

$$\int_{\mathbb{Z}_p} f(y)\, d\mu_1(y) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{y=0}^{p^N-1} f(y),$$

where $\mu_1(y)$ denotes the Haar distribution

$$\mu_1(y) = \mu_1\left(y + p^N \mathbb{Z}_p\right) = \frac{1}{p^N}$$

(*cf.* [15, 20, 24, 27, 28, 30]).

The fermionic p-adic integral is defined by

$$\int_{\mathbb{Z}_p} f(y)\, d\mu_{-1}(y) = \lim_{N\to\infty} \sum_{y=0}^{p^N-1} (-1)^y f(y),$$

where

$$\mu_{-1}(y) = \mu_{-1}\left(y + p^N \mathbb{Z}_p\right) = (-1)^y$$

(*cf.* [16–19, 28, 30]).

The p-adic integrals are related to many special numbers and polynomials and their generating functions. One of these are the Bernoulli numbers and polynomials, and the other are the Euler numbers and polynomials. With the help of the Volkenborn integral, the Bernoulli numbers and polynomials are also given by

$$\int_{\mathbb{Z}_p} y^n d\mu_1(y) = B_n \tag{11}$$

and

$$\int_{\mathbb{Z}_p} (y+w)^n\, d\mu_1(y) = B_n(w), \tag{12}$$

where $n \in \mathbb{N}_0$. We note that the Eq. (11) is also known as the Witt's formula for the Bernoulli numbers (*cf.* [27]; see also [26, 28, 30]).

By using the fermionic p-adic integral, the Euler numbers and polynomials are also given by

$$\int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) = E_n \tag{13}$$

and

$$\int_{\mathbb{Z}_p} (y+w)^n\, d\mu_{-1}(y) = E_n(w), \tag{14}$$

where $n \in \mathbb{N}_0$ (*cf.* [16–19, 26, 28, 30]).

The integral equation for the Volkenborn integral is given by

$$\int_{\mathbb{Z}_p} E^n \left[f\left(y\right) \right] d\mu_1\left(y\right) - \int_{\mathbb{Z}_p} f\left(y\right) d\mu_1\left(y\right) = \sum_{k=0}^{n-1} \frac{d}{dy} \left\{ f\left(y\right) \right\} |_{y=k}, \tag{15}$$

where

$$E^n \left[f\left(y\right) \right] = f\left(y+n\right)$$

and

$$\frac{d}{dy} \left\{ f\left(y\right) \right\} |_{y=k} = f'\left(k\right)$$

(*cf.* [15, 17, 27, 28, 30]).

By using the same method in work of Simsek [30], when $f\left(y\right) = c^{yt}$ and $n=1$ in (15), we have

$$\int_{\mathbb{Z}_p} c^{yt} d\mu_1\left(y\right) = \frac{t\log c}{c^t - 1}, \tag{16}$$

where

$$c \in \mathbb{C}_p^+ = \left\{ y \in \mathbb{C}_p : \left| 1-y\right|_p < 1 \right\}$$

and $c \neq 1$. Here we also note that Simsek gave some results on the above set; see for detail [30].

The integral equation for the fermionic p-adic integral is given by

$$\int_{\mathbb{Z}_p} E^n \left[f\left(y\right) \right] d\mu_{-1}\left(y\right) + \left(-1\right)^{n+1} \int_{\mathbb{Z}_p} f\left(y\right) d\mu_{-1}\left(y\right) = 2\sum_{k=0}^{n-1} \left(-1\right)^{n-1-k} f\left(k\right) \tag{17}$$

(*cf.* [19]; see also [28, 30]).

When $f\left(y\right) = c^{yt}$ and $n=1$ in (17), after some calculations, one has

$$\int_{\mathbb{Z}_p} c^{yt} d\mu_{-1}\left(y\right) = \frac{2}{c^t + 1} \tag{18}$$

(*cf.* [28, 30]).

2. Formulas for the unified and modified presentation of Fubini numbers and polynomials

In this section, by using functional equations of the generating functions for the unified and modified presentation of Fubini numbers and polynomials of order r, we obtain some formulas for these numbers and polynomials. Moreover, with the aid of the p-adic integral and the Riemann integral methods, we give some results including the Daehee numbers, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Fubini type numbers and polynomials, and the numbers $y_{9,n}\left(\lambda ; c\right)$.

Theorem 2.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{a}_n^{(k+l)}\left(x+y, \mu ; \vartheta, b, c\right) = \sum_{j=0}^{n} \binom{n}{j} \mathfrak{a}_j^{(k)}\left(x, \mu ; \vartheta, b, c\right) \mathfrak{a}_{n-j}^{(l)}\left(y, \mu ; \vartheta, b, c\right). \tag{19}$$

Proof. Using (10), we can write

$$N(t,k+l;x+y;\mu,\vartheta,b,c)=N(t,k;x;\mu,\vartheta,b,c)\,N(t,l;y;\mu,\vartheta,b,c).$$

From the above functional equation, we obtain

$$\sum_{n=0}^{\infty}\mathfrak{a}_n^{(k+l)}(x+y,\mu;\vartheta,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathfrak{a}_n^{(k)}(x,\mu;\vartheta,b,c)\frac{t^n}{n!}\times\sum_{n=0}^{\infty}\mathfrak{a}_n^{(l)}(y,\mu;\vartheta,b,c)\frac{t^n}{n!}.$$

Hence

$$\sum_{n=0}^{\infty}\mathfrak{a}_n^{(k+l)}(x+y,\mu;\vartheta,b,c)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathfrak{a}_j^{(k)}(x,\mu;\vartheta,b,c)\,\mathfrak{a}_{n-j}^{(l)}(y,\mu;\vartheta,b,c)\frac{t^n}{n!}.$$

Thus, when the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation are compared, we get the desired result. □

Remark 1. *When* $x=y=0$*, the Eq.* (19) *is reduced to the following known result:*

$$\mathfrak{a}_n^{(k+l)}(\mu;\vartheta,b)=\sum_{j=0}^{n}\binom{n}{j}\mathfrak{a}_j^{(k)}(\mu;\vartheta,b)\,\mathfrak{a}_{n-j}^{(l)}(\mu;\vartheta,b)$$

(cf. [14, p. 11, Eq. (2.11)]*).*

2.1. **The Volkenborn integral representation for the numbers $\mathfrak{a}_n^{(r)}(\mu;\vartheta,b)$ and the polynomials $\mathfrak{a}_n^{(r)}(w,\mu;\vartheta,b,c)$.** Here by applying the Volkenborn integral to the generating function of the unified and modified presentation of Fubini polynomials, we obtain some formulas for these numbers and polynomials, the Bernoulli numbers and polynomials, and some special numbers.

Theorem 2.2. *Let* $n\in\mathbb{N}$*. Then we have*

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_1(y)=\frac{n\log c}{2}\sum_{j=0}^{n-1}\binom{n-1}{j}\mathfrak{a}_j^{(r)}(\mu;\vartheta,b)\,y_{9,n-j-1}(-1;c).$$

Proof. By applying the Volkenborn integral to the both sides of the Eq. (10), we have

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_1(y)\frac{t^n}{n!}=\frac{2^r}{(\mu b^t-\vartheta)^{2r}}\int_{\mathbb{Z}_p}c^{ty}d\mu_1(y).$$

Using (16), we get

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_1(y)\frac{t^n}{n!}=\frac{2^r}{(\mu b^t-\vartheta)^{2r}}\frac{t\log c}{c^t-1}.$$

Combining the above equation with (5) and (9), we obtain

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_1(y)\frac{t^n}{n!}=\frac{t\log c}{2}\sum_{n=0}^{\infty}\mathfrak{a}_n^{(r)}(\mu;\vartheta,b)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_{9,n}(-1;c)\frac{t^n}{n!}.$$

Hence

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_1(y)\,\frac{t^n}{n!}$$
$$=\frac{\log c}{2}\sum_{n=0}^{\infty}n\sum_{j=0}^{n-1}\binom{n-1}{j}\mathfrak{a}_j^{(r)}(\mu;\vartheta,b)\,y_{9,n-j-1}(-1;c)\,\frac{t^n}{n!}.$$

Thus, when the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation are compared, we arrive at the desired result. □

Theorem 2.3. *Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,d\mu_1(y)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,B_j(w). \tag{20}$$

Proof. Replacing x with $y+w$ in Eq. (2.12) in [14, p. 12, Theorem 2.18.], we have

$$\mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,(y+w)^j. \tag{21}$$

By applying the Volkenborn integral to the both sides of the above equation, we obtain

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,d\mu_1(y)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\int_{\mathbb{Z}_p}(y+w)^j\,d\mu_1(y).$$

Using the above equation and (12), we have

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,d\mu_1(y)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,B_j(w).$$

Thus, the proof of the theorem is complete. □

When $w=0$ in (20), we arrive at the following result:

Corollary 2.4. *Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_1(y)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,B_j.$$

2.2. **The fermionic p-adic integral representation for the numbers $\mathfrak{a}_n^{(r)}(\mu;\vartheta,b)$ and the polynomials $\mathfrak{a}_n^{(r)}(w,\mu;\vartheta,b,c)$.** Here by applying the fermionic p-adic integral to the generating function of the unified and modified presentation of Fubini polynomials, we obtain some formulas for these numbers and polynomials, the Euler numbers and polynomials, and some special numbers.

Theorem 2.5. *Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_{-1}(y)=\sum_{j=0}^{n}\binom{n}{j}\mathfrak{a}_j^{(r)}(\mu;\vartheta,b)\,y_{9,n-j}(1;c).$$

Proof. By applying the fermionic p-adic integral to the both sides of the Eq. (10), we have

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_{-1}(y)\frac{t^n}{n!}=\frac{2^r}{(\mu b^t-\vartheta)^{2r}}\int_{\mathbb{Z}_p}c^{ty}d\mu_{-1}(y).$$

From (18), we get

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_{-1}(y)\frac{t^n}{n!}=\frac{2^r}{(\mu b^t-\vartheta)^{2r}}\frac{2}{c^t+1}.$$

Combining the above equation with (5) and (9), we obtain

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_{-1}(y)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathfrak{a}_n^{(r)}(\mu;\vartheta,b)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_{9,n}(1;c)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_{-1}(y)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathfrak{a}_j^{(r)}(\mu;\vartheta,b)\,y_{9,n-j}(1;c)\frac{t^n}{n!}.$$

Thus, when the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation are compared, we arrive at the desired result. □

Theorem 2.6. *Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,d\mu_{-1}(y)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,E_j(w). \tag{22}$$

Proof. By applying the fermionic p-adic integral to the both sides of the Eq. (21), we obtain

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,d\mu_{-1}(y)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\int_{\mathbb{Z}_p}(y+w)^j\,d\mu_{-1}(y).$$

Combining the above equation with (14), we get

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,d\mu_{-1}(y)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,E_j(w).$$

Thus, the proof of the theorem is complete. □

When $w=0$ in (22), we obtain the following corollary:

Corollary 2.7. *Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,d\mu_{-1}(y)=\sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,E_j.$$

2.3. **The Riemann integral representation for the numbers $\mathfrak{a}_n^{(r)}(\mu;\vartheta,b)$ and the polynomials $\mathfrak{a}_n^{(r)}(w,\mu;\vartheta,b,c)$.** Here by applying Riemann integral to the generating function of the unified and modified presentation of Fubini polynomials, we provide some relations for these numbers and polynomials, the Fubini type numbers and polynomials, and the Daehee numbers.

Theorem 2.8. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_0^1 \mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,dy = \sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\left(\frac{(w+1)^{j+1}-w^{j+1}}{j+1}\right). \tag{23}$$

Proof. By applying the Riemann integral to the both sides of the Eq. (21), we can write

$$\int_0^1 \mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,dy = \sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\int_0^1 (y+w)^j\,dy.$$

Hence,

$$\int_0^1 \mathfrak{a}_n^{(r)}(y+w,\mu;\vartheta,b,c)\,dy = \sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\left(\frac{(w+1)^{j+1}-w^{j+1}}{j+1}\right).$$

Thus, the proof of the theorem is complete. □

When $w=0$ in (23), we have the following result:

Corollary 2.9. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_0^1 \mathfrak{a}_n^{(r)}(y,\mu;\vartheta,b,c)\,dy = \sum_{j=0}^{n}\binom{n}{j}\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,\frac{(\ln c)^j}{j+1}. \tag{24}$$

Substituting $w=-1$ in (23), we get

$$\int_0^1 \mathfrak{a}_n^{(r)}(y-1,\mu;\vartheta,b,c)\,dy = \sum_{j=0}^{n}\binom{n}{j}(\ln c)^j\,\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,\frac{(-1)^j}{j+1}.$$

Combining the above equation with the following the Daehee numbers

$$D_j = \frac{(-1)^j\,j!}{j+1}$$

(*cf.* [20]), we obtain the following result:

Corollary 2.10. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_0^1 \mathfrak{a}_n^{(r)}(y-1,\mu;\vartheta,b,c)\,dy = \sum_{j=0}^{n}\binom{n}{j}\mathfrak{a}_{n-j}^{(r)}(\mu;\vartheta,b)\,\frac{(\ln c)^j D_j}{j!}.$$

Remark 2. *When $\mu = 1$, $\vartheta = 2$ and $b = c = e$ in* (24)*, we have the following known result:*

$$\int_0^1 a_n^{(r)}(y)\,dy = \sum_{j=0}^{n} \binom{n}{j} \frac{a_{n-j}^{(r)}}{j+1}$$

(cf. [6, 9]*).*

3. Conclusions

In this paper, by applying the Volkenborn integral, the fermionic p-adic integral, and the Riemann integral to the generating functions of the unified and modified presentation of Fubini numbers and polynomials, many interesting integral representations and formulas were given. These results also included the Daehee numbers, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, and some special numbers. Moreover, using generating function methods, some identities and relations related to these numbers and polynomials were derived. The aforementioned results have the potential to attract the interest of many researchers, leading to future investigations into these special numbers and polynomials. Therefore, the results presented in this paper can be applicable and useful in mathematics, engineering and also applied science.

Acknowledgement. *The author would like to thank to the referees for their valuable comments.*

References

[1] L. Comtet, Advanced Combinatorics: The Art of Finite and Infinite Expansions, *D. Reidel Publishing Company*, Dordrecht-Holland, Boston, 1974 (Translated from the French by J. W. Nienhuys).

[2] T. Diagana and H. Maiga, *Some new identities and congruences for Fubini numbers,* J. Number Theory **173** (2017), 547–569.

[3] J.-M. De Koninck, Those Fascinating Numbers, *American Mathematical Society*, 2009.

[4] I. J. Good, *The number of ordering of n candidates when ties are permitted*, Fibonacci Quart. **13** (1975), 11–18.

[5] N. Kilar, Fubini Type Numbers and Their Generating Functions, *Akdeniz University, MSc Thesis in Mathematics*, Antalya, 2017.

[6] N. Kilar, *Combinatorial sums and identities associated with functional equations of generating functions for Fubini type polynomials*, GUJ Sci **36**(2) (2023), 807–817.

[7] N. Kilar, *Formulas for Fubini type numbers and polynomials of negative higher order*, Montes Taurus J. Pure Appl. Math. **5**(3) (2023), 23–36.

[8] N. Kilar, *A new class of generalized Fubini polynomials and their computational algorithms*, Appl. Anal. Discrete Math. **17**(2) (2023), 496–524.

[9] N. Kilar and Y. Simsek: *A new family of Fubini numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials*, J. Korean Math. Soc. **54**(5) (2017), 1605–1621.

[10] N. Kilar and Y. Simsek, *Identities and relations for Fubini type numbers and polynomials via generating functions and p-adic integral approach*, Publ. Inst. Math., Nouv. Ser. **106**(120) (2019), 113–123.

[11] N. Kilar and Y. Simsek, *Some relationships between Fubini type polynomials and other special numbers and polynomials*, AIP Conf. Proc. **2116**(1) (2019), 100017-1–100017-4.

[12] N. Kilar and Y. Simsek, *Combinatorial sums involving Fubini type numbers and other special numbers and polynomials: Approach trigonometric functions and p-adic integrals*, Adv. Stud. Contemp. Math. **31**(1) (2021), 75–87.

[13] N. Kilar and Y. Simsek, *Formulae to Fubini type numbers emerge from application of p-adic integrals*, GU J Sci, Part A **8**(4) (2021), 402–410.
[14] N. Kilar and Y. Simsek, *Families of unified and modified presentation of Fubini numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **5**(1) (2023), 1–21.
[15] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys. **19** (2002), 288–299.
[16] T. Kim, *A note on q-Volkenborn integration*, Proc. Jangjeon Math. Soc. **8**(1) (2005), 13–17.
[17] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integral and basic q-zeta function*, Trend Math. Inf. Cent. Math. Sci. **9** (2006), 7–12.
[18] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integrals*, J. Nonlinear Math. Phys. **14**(1) (2007), 15–27.
[19] T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$,* J. Math. Anal. Appl. **331**(2) (2007), 779–792.
[20] D. S. Kim and T. Kim, *Daehee numbers and polynomials,* Appl. Math. Sci. (Ruse) **7**(120) (2013), 5969–5976.
[21] D. S. Kim, T. Kim, H.-I. Kwon and J.-W. Park, *Two variable higher-order Fubini polynomials*, J. Korean Math. Soc. **55**(4) (2018), 975–986.
[22] D. S. Kim, G.-W. Jang, H.-I. Kwon and T. Kim, *Two variable higher-order degenerate Fubini polynomials*, Proc. Jangjeon Math. Soc. **21**(1) (2018), 5–22.
[23] T. Kim, D. S. Kim, G.-W. Jang and J. Kwon, *Symmetric identities for Fubini polynomials*, Symmetry **10**(6) (2018), 1–7.
[24] T. Kim and S. H. Rim, *A note on the q-integral and q-series*, Adv. Stud. Contemp. Math. **2**(1) (2000), 37–45.
[25] T. Kim, S. H. Rim, Y. Simsek and D. Kim, *On the analogs of Bernoulli and Euler numbers, related identities and zeta and L-functions*, J. Korean Math. Soc. **45**(2) (2008), 435–453.
[26] I. Kucukoglu and Y. Simsek, *Unified presentations of the generating functions for a comprehensive class of numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **6**(1) (2024), 40–63.
[27] W. H. Schikhof, Ultrametric Calculus: An Introduction to p-adic Analysis, Cambridge Studies in Advanced Mathematics 4, *Cambridge University Press,* Cambridge, 1984.
[28] Y. Simsek, *Explicit formulas for p-adic integrals: Approach to p-adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1**(1) (2019), 1–76.
[29] Y. Simsek, *Interpolation functions for new classes special numbers and polynomials via applications of p-adic integrals and derivative operator*, Montes Taurus J. Pure Appl. Math. **3**(1) (2021), 38–61.
[30] Y. Simsek, *Formulas for p-adic q-integrals including falling-rising factorials, combinatorial sums and special numbers*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **118** (2024); https://doi.org/10.1007/s13398-024-01592-1
[31] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inf. Sci. **5**(3) (2011), 390–444.
[32] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, *Elsevier Science Publishers*, Amsterdam, London and New York, 2012.

Department of Computer Technologies, Bor Vocational School, Niğde Ömer Halisdemir University, TR-51700 Niğde-Turkey
Email address: neslihankilar@ohu.edu.tr; neslihankilar@gmail.com