Proceedings of the Jangjeon Mathematical Society
28 (2025), No. 1, pp. 73 - 84
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2025.28.1.73

Note on complete convergence for weighted sums of widely negative dependent random variables under sub-linear expectations

Kyo-Shin Hwang
Graduate School of Education, Yeungnam University, Gyeongsan 38541, Korea
E-mail address: kshwang@yu.ac.kr

Submitted October 25, 2024

Abstract. Suppose that $\{X, X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of row-wise widely negative dependent random variables, There exists a random variable X and a constant C satisfying $\widehat{\mathbb{E}}[h(X_{ni})] \leq C\widehat{\mathbb{E}}[h(X)]$, $\widehat{\mathbb{E}}[|X|] \leq C_V(|X|) < \infty$. If $\{a_{ni}; 1 \leq i \leq n, n \geq 1\}$ is an array of positive real numbers satisfying certain conditions, we obtain the complete convergence of weighted sums for widely negative dependent random variables under the sub-linear expectations. This result generalizes and improves the result of Sung(2012a) and Yi(2021) for widely negative dependent random variables under sub-linear expectations.

Keywords: Complete convergence; widely negative dependent; weighted sum; sub-linear expectation; array of random variables.

Mathematics Subject Classification: 60F15; 60F05.

1 Introduction

Complete convergence is a significant area of research field in probability limit theory. The concept was first introduced by Hsu and Robbins([2]) as follows: a sequence $\{X_n, n \geq 1\}$ of random variables is said to *converge completely* to some constant c if

$$\sum_{n=1}^{\infty} P(|X_n - c| > \epsilon) < \infty$$

for all $\epsilon > 0$. According to the Borel-Cantelli lemma, this definition implies that $X_n \to c$ almost surely (a.s.), and the converge holds if $\{X_n, n \geq 1\}$ are independent random variables. Thus, complete convergence is a stronger concept than almost sure convergence. Many scholars have explored complete convergence in various random variable contexts. For example, Wu([13]) studied the complete convergence of independent and identically distributed random variables, Hu, Rosalsky and Wang([4]) provided complete convergence theorems for extended negatively dependent random variables. We refer the reader to Peligrad and Gut([7]), Sung([10],[12]), Shen, Xue and Wang([9]), and so on.

The classical limit theorems are based on the linearity of expectations and probability measures. In many real-world applications, such as finance, economics and statistics, the assumption of additivity is not feasible, as uncertainties cannot always be modeled using additive probabilities or expectations. Consequently, various sub-linear theories have been developed to describe and measure these risks. Lin and Feng([5]) studied complete convergence and strong law of large numbers for arrays of random variables under sub-linear expectations. Yi([14]), and Feng, Wang and Wu([3]) investigated complete convergence for weighted sums of negatively dependent random variables under the sub-linear expectations, Yu and Wu([15]) studied the Marcinkiewicz type complete convergence for weighted sums under sub-linear expectations.

This paper focuses on the complete convergence of weighted sums for widely negative dependent random variables under the sub-linear expectations. This result extends and enhances the findings of Sung([11]) and Yi([14]) on widely negative dependent random variables under sub-linear expectations.

This paper is organized as follows: in Section 2, we summarized some basic notations and concepts, related properties under the sub-linear expectations and present the preliminary definitions and lemmas that are useful to obtain the main results. In Section 3, we give the main results including the proof. In Section 4, we give the conclusion.

2 Preliminaries

We use the framework and notations of Peng([8]). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \forall \mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$.

Definition 2.1. A *sub-linear expectation* $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\widehat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$ we have

(i) Monotonicity: If $X \geq Y$ then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;

(ii) Constant preserving: $\widehat{\mathbb{E}}[c] = c$;

(iii) Sub-additivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$; whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(iv) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X], \quad \lambda \geq 0$

Here $\bar{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a *sub-linear expectation space.*

Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] = -\widehat{\mathbb{E}}[-X], \quad \forall X \in \mathcal{H}.$$

From Definition 2.1, it is easily shown that

$$\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X], \quad \widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c \ \text{ and } \ \widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$$

for all $X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathbb{E}}[X]$ and $\widehat{\mathcal{E}}[X]$ are both finite, and if $\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X]$, then $\widehat{\mathbb{E}}[X + aY] = \widehat{\mathbb{E}}[X] + a\widehat{\mathbb{E}}[Y]$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations.

Definition 2.2. Let $\mathcal{G} \subset \mathcal{F}$. A function $\mathbb{V} : \mathcal{G} \to [0, 1]$ is called a *capacity* if

$$\mathbb{V}(\emptyset) = 0, \quad \mathbb{V}(\Omega) = 1 \quad \text{and} \quad \mathbb{V}(A) \leq \mathbb{V}(B) \quad \text{whenever } A \subset B \quad \text{and } A, B \in \mathcal{G}.$$

It is called *sub-additive* if $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear space. We denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \mathcal{V}(A) = 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},$$

where A^c is the complement set of A. Then

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g] \quad \text{and} \quad \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g], \tag{2.1}$$

if $f \leq I_A \leq g,\ f, g \in \mathcal{H}$. It is obvious that $\mathbb{V}$ is sub-additive, i.e., $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$.

In this paper we only consider the capacity generated by a sub-linear expectation. Given a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, we define a capacity:

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A], \quad \forall A \in \mathcal{F}$$

and also define the conjugate capacity:

$$\mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}.$$

It is clear that $\mathbb{V}$ is a sub-additive capacity and $\mathcal{V}(A) = \widehat{\mathcal{E}}[I_A]$.

Definition 2.3 ([16]) (1) A sub-linear expectation $\widehat{\mathbb{E}} : \mathcal{H} \to R$ is called to be *countably sub-additive* if it satisfies

$$\widehat{\mathbb{E}}[X] \leq \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[X_n], \text{ whenever } X \leq \sum_{n=1}^{\infty} X_n, X, X_n \in \mathcal{H},$$

where $X \geq 0, X_n \geq 0$ and $n \geq 1$.

(2) A function $\mathbb{V} : \mathcal{F} \to [0,1]$ is called to be *countably sub-additive* if

$$\mathbb{V}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{V}(A_n), \ \forall A_n \in \mathcal{F}.$$

Definition 2.4. Let X be a random variable on $(\Omega, \mathcal{F})$. The *upper Choquet integral/expectation of X induced by* a capacity $\mathbb{V}$ on $\mathcal{F}$ is defined by $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) = \int_{\Omega} X dV(x) = \int_0^{\infty} V(X > x)\, dx + \int_{-\infty}^{0} (V(X > x) - 1)\, dx,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

The *lower Choquet expectation of X induced by* $\mathbb{V}$ is given by $C_{\mathcal{V}}[X] := -C_{\mathbb{V}}[-X]$, which is conjugate to the upper expectation and satisfies $C_{\mathcal{V}}[X] \leq C_{\mathbb{V}}[X]$.

For simplicity, we only consider the upper Choquet expectation in the sequel, since the lower (conjugate) version can be considered similarly.

Lemma 2.5. ([1]) Let X, Y be two random variables on $(\Omega, \mathcal{F})$ and let $C_{\mathbb{V}}$ be the upper Choquet expectation induced by a capacity $\mathbb{V}$, then, we have

(1) Monotonicity: $C_{\mathbb{V}}[X] \leq C_{\mathbb{V}}[Y]$ for $X \leq Y$;

(2) Positive homogeneity: $C_{\mathbb{V}}[\lambda X] \leq \lambda C_{\mathbb{V}}[X]$ for $\lambda \geq 0$;

(3) translation invariance: $C_{\mathbb{V}}[X + a] \leq C_{\mathbb{V}}[X] + a$ for $\forall a \in \mathbb{R}$.

The following lemmas show that some important inequalities in classical probability theory still hold in sub-linear expectation spaces (See [6]).

Lemma 2.6. (Markov's inequality) *For any $X \in \mathcal{H}$, we have*

$$\mathbb{V}(|X| \geq x) \leq \frac{\widehat{\mathbb{E}}[|X|^p]}{x^p}$$

for any $x > 0$ and $p > 0$.

Now we give the definition of widely negative dependence on the sublinear expectation space. The concept of widely negative dependence is introduced by Lin and Feng([5]) as follows.

Definition 2.7. Let $X_1, X_2, \cdots, X_{n+1}$ be real measurable random variables of $(\Omega, \mathcal{F})$.
(1) X_{n+1} is called *widely negative dependence* of $(X_1, \cdots, X_n)$ under $\widehat{\mathbb{E}}$ if for every nonnegative measure function φ_i with the same monotonicity on $\mathbb{R}$ and $\widehat{\mathbb{E}}[\varphi_i(X_i)] < \infty, i = 1, 2, \cdots, n+1$, there exists a positive finite real function $g(n+1)$ such that

$$\widehat{\mathbb{E}}\left[\prod_{i=1}^{n+1} \varphi_i(X_i)\right] \leq g(n+1)\widehat{\mathbb{E}}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right] \widehat{\mathbb{E}}\left[\varphi_{n+1}(X_{n+1})\right].$$

(2) $\{X_i\}_{i=1}^{\infty}$ is said to be *a sequence of widely negative dependent random variables* under $\widehat{\mathbb{E}}$ if for any $n \geq 1$, X_{n+1} is widely negative dependence of $(X_1, X_2, \cdots, X_n)$.
(3) $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is said to be *an array of rowwise widely negative dependent random variables* under $\widehat{\mathbb{E}}$ if for any $n \geq 1$, $\{X_{ni}, 1 \leq i \leq k_n\}$ is a sequence of widely negative dependent random variables

Remark 2.8. For a sequence of widely negative dependent random variables $\{X_i, i \geq 1\}$, we have

$$\widehat{\mathbb{E}}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right] \leq \tilde{g}(n)\prod_{i=1}^{n} \widehat{\mathbb{E}}\left[\varphi_i(X_i)\right], \text{ where } \tilde{g}(n) = \prod_{i=1}^{n} g(i) \tag{2.2}$$

for any $n \geq 1$ and every nonnegative measurable function $\varphi_i(\cdot)$ with the same monotonicity on $\mathbb{R}$ and $\widehat{\mathbb{E}}[\varphi_i(X_i)] < \infty, i = 1, 2, \cdots, n$, where $g(\cdot)$ is in Definition 2.7(1).

Remark 2.9. Without loss of generality, we will assume that $g(n) \geq 1$ for any $n \geq 1$ in the sequal.

The following lemma is introduced by Lin and Feng([5]).

Lemma 2.10. *Suppose that $\{X_i\}_{i=1}^{\infty}$ is a sequence of widely negative dependent random variable under $\widehat{\mathbb{E}}$, and $\{\psi_i(x)\}_{i=1}^{\infty}$ is a sequence of measurable function with the same monotonicity. Then $\{\psi_i(X_i)\}_{i=1}^{\infty}$ is also a sequence of widely negative dependent random variables.*

Throughout this paper, let $\{X_n, n \geq 1\}$ be a sequence of widely negative dependent random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. C will signify a positive constant that may have different values in different places. $a_n = O(b_n)$ and $a_n \ll b_n$ denote that for a sufficiently large n, there exists $C > 0$ such that $a_n \le Cb_n$ and $I(\cdot)$ denotes an indicator function.

3 Main Results and Proofs

Before we introduce the main results, let us first prove the following lemma.

Lemma 3.1. *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise widely negative dependent random variables with $\widehat{\mathbb{E}}[X_{ni}] = 0$. Let $\tilde{g}(x)$ be a nondecreasing positive function on $[0, \infty)$ such that*

$$\tilde{g}(x) = \tilde{g}(n) \text{ when } x = n, \quad \tilde{g}(0) = 1 \quad \text{and} \quad \frac{\tilde{g}(x)}{x^\tau} \downarrow \quad \text{for some } 0 < \tau < 1. \tag{3.1}$$

Assume that

$$\max_{1 \le i \le n} |X_{ni}| = O(\log^{-1} n) \tag{3.2}$$

and

$$\sum_{i=1}^n \widehat{\mathbb{E}}[X_{ni}^2] = o(\log^{-1} n), \tag{3.3}$$

Then, for any $\epsilon > 0$ we have

$$\sum_{n=1}^\infty \mathbb{V}\left(\sum_{i=1}^n X_{ni} > \epsilon\right) < \infty.$$

Proof. Since $1 + x \le e^x \le 1 + x + \frac{1}{2}x^2 e^{|x|}$ for all $x \in R$, then we have from (3.2) and $\widehat{\mathbb{E}}[X_{ni}] = 0$ that $\forall k > 0$,

$$\begin{aligned}
\widehat{\mathbb{E}}\left[\exp(kX_{ni})\right] &\le \widehat{\mathbb{E}}\left\{1 + kX_{ni} + \frac{1}{2}k^2|X_{ni}|^2\exp(k|X_{ni}|)\right\} \\
&\le 1 + \frac{1}{2}k^2\widehat{\mathbb{E}}\left[|X_{ni}|^2\exp(k|X_{ni}|)\right] \\
&\le \exp\left\{\frac{1}{2}k^2\widehat{\mathbb{E}}\left[|X_{ni}|^2\exp(k|X_{ni}|)\right]\right\} \\
&\le \exp\left\{\frac{1}{2}k^2 e^{ck/\log n}\widehat{\mathbb{E}}\left[|X_{ni}|^2\right]\right\}.
\end{aligned}$$

By (2.2), there are $k_i \ge 0\ (\le 0),\ 1 \le i \le n$ such that

$$\widehat{\mathbb{E}}\left[\exp\left(\sum_{i=1}^n k_i X_{ni}\right)\right] \le \tilde{g}(n)\prod_{i=1}^n \widehat{\mathbb{E}}\left[\exp\left(k_i X_{ni}\right)\right]. \tag{3.4}$$

Note that

$$\exp\left\{\widehat{\mathbb{E}}\left[\sum_{i=1}^n X_{ni}\right]\right\} \le \exp\left\{\sum_{i=1}^n \widehat{\mathbb{E}}[X_{ni}]\right\} = \prod_{i=1}^n \exp\left\{\widehat{\mathbb{E}}[X_{ni}]\right\},$$

we have from Markov's inequality, (3.1), (3.3) and (3.4) that $\forall k > 0$,

$$\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n} X_{ni} > \epsilon\right) &\le e^{-k\epsilon}\widehat{\mathbb{E}}\left[\exp\left(k\sum_{i=1}^{n} X_{ni}\right)\right] \\
&\le \tilde{g}(n)e^{-k\epsilon}\prod_{i=1}^{n}\widehat{\mathbb{E}}\left[\exp\left(kX_{ni}\right)\right] \\
&\ll \tilde{g}(n)e^{-k\epsilon}\exp\left\{\frac{1}{2}k^2 e^{ck/\log n}\sum_{i=1}^{n}\widehat{\mathbb{E}}\left[|X_{ni}|^2\right]\right\} \\
&\le e^{\tau\log n - k\epsilon}\exp\left\{\frac{1}{2}k^2 e^{ck/\log n}o(\log^{-1} n)\right\}.
\end{aligned}$$

Let $k = \frac{a+1}{\epsilon}\log n$ for all $a > 2$ and when n is big enough, we obtain that

$$\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n} X_{ni} > \epsilon\right) &\ll e^{\tau\log n-(a+1)\log n}\exp\left\{\frac{1}{2}\left(\frac{a+1}{\epsilon}\log n\right)^2 e^{\frac{c(a+1)}{\epsilon}}o(\log^{-1} n)\right\} \\
&\le e^{\tau\log n-(a+1)\log n}e^{\log n} = e^{-(a-\tau)\log n} \\
&\le n^{-(2-\tau)},
\end{aligned}$$

and we can show that

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n} X_{ni} > \epsilon\right) < \infty.$$

Thus Lemma 3.1 holds. □

We now introduce our main result as follows.

Theorem 3.2. *Suppose that $\{X, X_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of rowwise widely negative dependent random variables, there exist a r.v. X and a constant C satisfying*

$$\widehat{\mathbb{E}}[h(X_{ni})] \le C\widehat{\mathbb{E}}[h(X)] \quad \text{for all } n \ge 1,\ 1 \le i \le n,\ 0 \le h \in C_{l,Lip}(\mathbb{R}). \tag{3.5}$$

Further assume that

$$\widehat{\mathbb{E}}[|X|] \le C_{\mathbb{V}}(|X|) < \infty. \tag{3.6}$$

Let $\tilde{g}(x)$ be a nondecreasing positive function on $[0,\infty)$ such that

$$\tilde{g}(x) = \tilde{g}(n) \text{ when } x = n, \quad \tilde{g}(0) = 1 \quad \text{and} \quad \frac{\tilde{g}(x)}{x^{\tau}} \downarrow \quad \text{for some } 0 < \tau < 1.$$

Assume that $\{a_{ni}; 1 \le i \le n, n \ge 1\}$ is an array of positive real numbers satisfying

$$\sum_{i=1}^{n} a_{ni} = O(n^{-\gamma}) \tag{3.7}$$

for some $\gamma > 0$ *and*

$$\sum_{i=1}^{n} a_{ni}^2 \widehat{\mathbb{E}}\left[X^2 h(a_{ni}X\log n)\right] = o(\log^{-1} n). \tag{3.8}$$

Then for any $\epsilon > 0$*, any positive integer number* N *with* $N\gamma > 2$*, we have*

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(X_{ni} - \widehat{\mathbb{E}}[X_{ni}]\right) I\left(|a_{ni}X_{ni}| \le \frac{\epsilon}{N}\right) > \epsilon\right) < \infty \tag{3.9}$$

and

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(X_{ni} - \widehat{\mathcal{E}}[X_{ni}]\right) I\left(|a_{ni}X_{ni}| \le \frac{\epsilon}{N}\right) < -\epsilon\right) < \infty. \tag{3.10}$$

Proof. When we replace $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ with $\{-X_{ni}, 1 \le i \le n, n \ge 1\}$ in (3.9), we can have (3.10). If $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of rowwise widely negative dependent random variables, then $\{-X_{ni}, 1 \le i \le n, n \ge 1\}$ is also an array of rowwise widely negative dependent random variables. Therefore, we just need to prove (3.9). Without loss of generality, we assume that $\widehat{\mathbb{E}}[X_{ni}] = 0$. For $n \ge 1$ and $1 \le i \le n$, we define

$$\begin{aligned}
X_{ni}^{(1)} &= X_{ni} I\left(|a_{ni}X_{ni}| \le (\log n)^{-1}\right) + a_{ni}^{-1}(\log n)^{-1} I\left(a_{ni}X_{ni} > (\log n)^{-1}\right) \\
&\qquad - a_{ni}^{-1}(\log n)^{-1} I\left(a_{ni}X_{ni} < -(\log n)^{-1}\right), \\
X_{ni}^{(2)} &= \left(X_{ni} - a_{ni}^{-1}(\log n)^{-1}\right) I\left((\log n)^{-1} < a_{ni}X_{ni} \le \frac{\epsilon}{N}\right), \\
X_{ni}^{(3)} &= \left(X_{ni} + a_{ni}^{-1}(\log n)^{-1}\right) I\left(-\frac{\epsilon}{N} \le a_{ni}X_{ni} < -(\log n)^{-1}\right), \\
X_{ni}^{(4)} &= -a_{ni}^{-1}(\log n)^{-1} I\left(a_{ni}X_{ni} > \frac{\epsilon}{N}\right) + a_{ni}^{-1}(\log n)^{-1} I\left(a_{ni}X_{ni} < -\frac{\epsilon}{N}\right).
\end{aligned} \tag{3.11}$$

It is obvious from (3.11) that

$$a_{ni}X_{ni} I\left(|a_{ni}X_{ni}| < \frac{\epsilon}{N}\right) = a_{ni}X_{ni}^{(1)} + a_{ni}X_{ni}^{(2)} + a_{ni}X_{ni}^{(3)} + a_{ni}X_{ni}^{(4)},$$

which yields

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni} I\left(|a_{ni}X_{ni}| < \frac{\epsilon}{N}\right) > 4\epsilon\right) \\
&\quad \le \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni}^{(1)} > \epsilon\right) + \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni}^{(2)} > \epsilon\right) \\
&\quad + \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni}^{(3)} > \epsilon\right) + \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni}^{(4)} > \epsilon\right).
\end{aligned} \tag{3.12}$$

We define the function $h(x) \in C_{l,Lip}(\mathbb{R})$ as follows. For $0 < \mu < 1$, let $h(x) \in C_{l,Lip}(\mathbb{R})$ be a nonincreasing function such that $0 \le h(x) \le 1$ for all x and $h(x) = 1$ if $|x| \le \mu$, $h(x) = 0$ if $|x| > 1$, then

$$I(|x| \le \mu) \le h(x) \le I(|x| \le 1), \qquad I(|x| > 1) \le 1 - h(x) \le I(|x| > \mu). \tag{3.13}$$

According to (3.5), (3.13) and C_r-inequality, we have

$$\begin{aligned}\widehat{\mathbb{E}}\left[|X_{ni}^{(1)}|^r\right] &\ll \widehat{\mathbb{E}}\left[|X|^r h\left(\mu a_{ni} X \log n\right)\right] + a_{ni}^{-r}(\log n)^{-r}\widehat{\mathbb{E}}\left[1 - h\left(a_{ni} X \log n\right)\right] \\ &\leq \widehat{\mathbb{E}}\left[|X|^r h\left(\mu a_{ni} X \log n\right)\right] + a_{ni}^{-r}(\log n)^{-r}\mathbb{V}\left(|a_{ni}X| > \mu(\log n)^{-1}\right).\end{aligned} \tag{3.14}$$

It follows easily from Lemma 2.10 that $\{a_{ni}X_{ni}^{(1)}\}$ is an array of rowwise widely negative dependent random variables. From Markov's inequality, (3.5) – (3.8) and (3.14), we have

$$\begin{aligned}\sum_{i=1}^{n}\widehat{\mathbb{E}}\left[a_{ni}\left(X_{ni}^{(1)} - \widehat{\mathbb{E}}\left[X_{ni}^{(1)}\right]\right)\right]^2 &\leq \sum_{i=1}^{n} a_{ni}^2\widehat{\mathbb{E}}\left[2\left(X_{ni}^{(1)}\right)^2 + 2\left(\widehat{\mathbb{E}}\left[X_{ni}^{(1)}\right]\right)^2\right] \\ &\leq C\sum_{i=1}^{n} a_{ni}^2\widehat{\mathbb{E}}\left[X_{ni}^{(1)}\right]^2 \\ &\leq C\sum_{i=1}^{n} a_{ni}^2\widehat{\mathbb{E}}\left[|X|^2 h\left(\mu a_{ni} X \log n\right)\right] \\ &+ (\log n)^{-2}\sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X| > \mu(\log n)^{-1}\right) \\ &\ll o((\log n)^{-1}) + (\log n)^{-1}\sum_{i=1}^{n} a_{ni}\widehat{\mathbb{E}}\left[|X|\right] \\ &\leq o((\log n)^{-1}) + (\log n)^{-1}O(n^{-\gamma})C_{\mathbb{V}}(|X|) \\ &= o((\log n)^{-1}).\end{aligned}$$

From Lemma 3.1, we have

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(X_{ni}^{(1)} - \widehat{\mathbb{E}}\left[X_{ni}^{(1)}\right]\right) > \frac{\epsilon}{2}\right) < \infty.$$

Note that if $\left|\sum_{i=1}^{n} a_{ni}\widehat{\mathbb{E}}\left[X_{ni}^{(1)}\right]\right| \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni}^{(1)} > \epsilon\right) < \infty$. It follows from (3.5) and (3.7) that

$$\begin{aligned}\left|\sum_{i=1}^{n} a_{ni}\widehat{\mathbb{E}}\left[X_{ni}^{(1)}\right]\right| &= \left|\sum_{i=1}^{n} a_{ni}\left(\widehat{\mathbb{E}}\left[X_{ni}^{(1)}\right] - \widehat{\mathbb{E}}\left[X_{ni}\right]\right)\right| \\ &\leq \sum_{i=1}^{n} a_{ni}\widehat{\mathbb{E}}\left[|X_{ni} - X_{ni}^{(1)}|\right] \\ &\ll \sum_{i=1}^{n} a_{ni}\widehat{\mathbb{E}}\left[|X|\left(1 - h\left(a_{ni} X \log n\right)\right)\right] \\ &\leq \sum_{i=1}^{n} a_{ni}\widehat{\mathbb{E}}\left[|X|\right] \\ &\leq O(n^{-\gamma}) \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which gives

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni}^{(1)} > \epsilon\right) < \infty. \tag{3.15}$$

We next prove $\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}^{(2)}>\epsilon\right)<\infty$. It follows from $X_{ni}^{(2)}$ of (3.11) that

$$0\le a_{ni}X_{ni}^{(2)}\le\frac{\epsilon}{N},\qquad\left|\sum_{i=1}^{n}a_{ni}X_{ni}^{(2)}\right|=\sum_{i=1}^{n}a_{ni}X_{ni}^{(2)}>\epsilon,$$

which implies that there exists at least one positive integer N such that $X_{ni}^{(2)}\neq 0$. By $(2.1),(2.2),(3.5),(3.13)$ and Markov's inequality, we can obtain

$$\begin{aligned}
&\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}^{(2)}>\epsilon\right)\\
&\le\sum_{1\le k_1<\cdots<k_N\le n}\mathbb{V}\left(X_{nk_1}^{(2)}\neq 0,\cdots,X_{nk_N}^{(2)}\neq 0\right)\\
&\le\sum_{1\le k_1<\cdots<k_N\le n}\mathbb{V}\left(|a_{nk_1}X_{nk_1}|\ge(\log n)^{-1},\cdots,|a_{nk_N}X_{nk_N}|\ge(\log n)^{-1}\right)\\
&\le\sum_{1\le k_1<\cdots<k_N\le n}\widehat{\mathbb{E}}\left[(1-h(a_{nk_1}X_{nk_1}\log n))\cdots(1-h(a_{nk_N}X_{nk_N}\log n))\right]\\
&\le\sum_{1\le k_1<\cdots<k_N\le n}\tilde{g}(N)\prod_{i=1}^{N}\widehat{\mathbb{E}}\left[1-h\left(a_{n_{k_i}}X\log n\right)\right]\\
&\le\tilde{g}(N)\sum_{1\le k_1<\cdots<k_N\le n}\prod_{i=1}^{N}\mathbb{V}\left(|a_{nk_i}X|>\mu(\log n)^{-1}\right)\\
&\le\tilde{g}(n)\left[\sum_{k=1}^{n}\mathbb{V}\left(|a_{nk}X|>\mu(\log n)^{-1}\right)\right]^{N}\\
&\le\tilde{g}(n)\left[\sum_{k=1}^{n}a_{nk}\log n\widehat{\mathbb{E}}\left[|X|\right]\right]^{N}\\
&\ll\tilde{g}(n)(\log n)^{N}n^{-\gamma N}\left[\widehat{\mathbb{E}}\left[|X|\right]\right]^{N}\\
&\le Cn^{-(\gamma N-\tau)}(\log n)^{N},
\end{aligned}$$

which gives

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}^{(2)}>\epsilon\right)<\infty.\tag{3.16}$$

By the same methods as (3.16), we can get

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}^{(3)}>\epsilon\right)<\infty.\tag{3.17}$$

We finally will prove that

$$\sum_{n=1}^{\infty}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}^{(4)}>\epsilon\right)<\infty.\tag{3.18}$$

If follows from (3.11) that

$$a_{ni}X_{ni}^{(4)} \le (\log n)^{-1} I\left(|a_{ni}X_{ni}| > \frac{\epsilon}{N}\right) \le \frac{\epsilon}{N},$$

thus there are at least N subscripts k such that $|a_{ni}X_{ni}| > \epsilon/N$. Therefore, by Markov's inequality, (3.5) − (3.7), we have for $\gamma N > 2$,

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}X_{ni}^{(4)} > \epsilon\right) &\ll \sum_{n=1}^{\infty} \tilde{g}(n)\left[\sum_{i=1}^{n} \mathbb{V}\left(|a_{ni}X| > \frac{\mu\epsilon}{N}\right)\right]^N \\ &\le \sum_{n=1}^{\infty} \tilde{g}(n)\left[\sum_{i=1}^{n} a_{ni}\widehat{\mathbb{E}}\left[|X|\,(\mu\epsilon/N)^{-\gamma}\right]\right]^N \\ &\ll \sum_{n=1}^{\infty} n^{\tau}\left[O\left(n^{-\gamma}\right)\widehat{\mathbb{E}}\left[|X|\right]\right]^N \\ &\ll \sum_{n=1}^{\infty} n^{-(\gamma N-\tau)} < \infty.\end{aligned}$$

Combining (3.12) with (3.15) − (3.18) we get (3.9), which completes the proof.

4 Conclusion

In this article, we establish complete convergence for weighted sums of widely negative dependent random variables under sub-linear expectations. This result generalizes and improves upon the work of Sung (2012a) and Yi (2021) on widely negative dependent random variables under sub-linear expectations. The proof is achieved under significantly weaker conditions, thereby extending the strong limit theorems within the framework of sub-linear expectations or Choquet expectations. This generalization enhances the applicability of sub-linear or Choquet expectation in the simulation of financial phenomena.

Acknowledgement. The author would like to thank to the referees and the editors for their valuable comments and some helpful suggestions that improved the clarity and readability of the paper.

References

[1] J. Chen and Z. Chen, A new proof of the generalized law of large numbers under Choquet expectation, *J. Ineq. & Appli.*, (2020), 2020:158.

[2] P.L. Hsu and H. Robbins, Complete convergence and the law of large numbers, *Proc. Natl. Acad. Sci.*, **33** (1947), 25-31.

[3] F. Feng, D. Wang, Q. Wu, Complete convergence for weigted sums of negatively dependent random variables under the sub-linear expectations, *Comm. Statist.-Theory and Methods*, **48** (6) (2019), 1351–1366.

[4] T.C. Hu, A. Rosalsky and K.L. Wang, Complete convergence theorems for extended negatively dependent random variables, *Sankhya A*, **77** (2015), 1–29.

[5] Y. Lin & X. Feng, Complete convergence and strong law of large numbers for arrays of random variables under sublinear expectations, *Commun. Stati.-Th.* & *Methods*, **49**(23), (2020).

[6] D. Lu and Y. Meng, Complete and completd integral convergence for arrays of row wise widely negative dependent random variables under the sub-linear expectations, *Commun. Stati. - Th. & Methods*, (2020).

[7] M. Peligrad and A. Gut, Almost sure results for a class of dependent random variables, *Journal of Theoretical Probability*, **12** (1999), 87–104.

[8] S.G. Peng, A new central limit theorem under sublinear expectations, *J. Math.*, **53**(8) (2008), 1989–1994.

[9] A.T. Shen, M.X. Xue and W.J. Wang, Complete convergence for weighted sums of extended negatively dependent random variables, *Communications in Statistics-Theory and Methods*, **46**(3) (2017), 1433–1444.

[10] S.H. Sung, Complete convergence for weighted sums of $\tilde{\rho}$-maxing random variables, *Discrete Dynamics in Nature and Society*, (2010), Article ID 630608, 13 pages.

[11] S.H. Sung, Complete convergence for weighted sums of negatively dependent random variables, *Stat. Papers*, **53** (2012a), 73–82.

[12] S.H. Sung, A note on the complete convergence for weighted sums of negatively dependent random variables, *Journal of Inequalities and Applications*, **158** (2012b), 10 pages.

[13] Q.Y. Wu, Complete convergence of isodistributed NA sequences, *Basic Science in China*, **1** (1999), 89–98.

[14] Y.C. Yi, Complete convergence for weighted sums of negatively dependent random variables under sub-linear expectations, *Journal of Frontiers of Society, Science and Technology*, **1**(10) (2021), 28–37.

[15] D. Yu and Q. Wu, Marcinkiewicz type complete convergence for weighted sums under sub-linear expectations, *J. Univ. Sci.* & *Tech. China*, **48**(2) (2018), 89-96.

[16] L.X. Zhang and J. Lin, Marcinkiewicz's strong law of large numbers for non-additive expectation, *Stat.* & *Probab. Lett.*, **137** (2018), 267–276.