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UNIQUENESS OF MEROMORPHIC FUNCTION AND ITS DIFFERENCE POLYNOMIAL OF DIFFERENCE OPERATOR SHARING ONE OR MORE SETS WITH FINITE WEIGHT

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Abstract. In this paper, we mainly investigate the uniqueness property of meromorphic functions together with its difference polynomial of difference operator with finite weight sharing one or two sets. With the help of range set introduced in Banerjee and Chakraborty (Jordan J Math Stat 117-139. 2016), we have improved the result of Goutam Haldar (J Anal 2022. 1-17) and obtain the unique range set corresponding to shift operators. The examples is exhibited to validate certain claims of the main result.

1. Introduction, Definitions and Results

Let ϕ and $\S$ be two non-constant meromorphic functions defined on the set of complex numbers $\mathbb{C}$, and $a \in \mathbb{C} \cup \{\infty\}$. we say that ϕ and $\S$ share the value a CM (counting multiplicities) if $\phi - a$ and $\S - a$ have the same set of zeros with the same multiplicities, and if we do not count the multiplicities, then ϕ and $\S$ are said to share the value a IM (ignoring multiplicities).

Throughout the paper, we have used the standard notations and definitions of value distribution theory of meromorphic functions introduced in [8]. We recall that $T(r, \phi)$ denotes the Nevanlinna characterstic function of the non-constant meromorphic function. Also we denote by $S(r, \phi)$ any quantity satisfying $T(r, \phi) = o(T(r, \phi))$ as $r \to \infty$ possibly outside a finite set of logarithmic measure and $N(r, a; \phi)(\overline{N}(r, a; \phi))$ denotes the counting function (reduced counting function) of $a-$points of meromorphic functions ϕ. A meromorphic function α is said to be a small function of ϕ if $T(r, \alpha) = o(T(r, \phi))$. Let $S(\phi)$ be the set of all small functions of ϕ. For a set $S \subset \mathbb{C}$, we define

$$E_\phi(S) = \bigcup_{a \in S} \{z | \phi(z) - a(z) = 0\},$$

where each zero is counted according to its multiplicity and $\overline{E}_\phi(S) = \bigcup_{a \in S} \{z | \phi(z) - a(z) = 0\}$, where each zero is counted only once.

If $E_\phi(S) = E_\S(S)$, we say that $\phi, \S$ share the set S CM and if $\overline{E}_\phi(S) = \overline{E}_\S(S)$, we say that $\phi, \S$ share the set S IM.

In 2001, Lahiri [10] introduced a remarkable notion called weighted sharing of values and sets which renders a useful tool in the literature. We explain the notion in the following.

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Definition 1.1. *(5, p.196,* [10]*) Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a,\phi)$ the set of all $a-$points of ϕ, where an a point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,\phi) = E_k(a,\S)$, we say that $\phi,\S$ share the value a with weight k.*

We write $\phi,\S$ share (a,k) to mean that $\phi,\S$ share the value a with weight k. Clearly if $\phi,\S$ share (a,k) then $\phi,\S$ share (a,p) for any integer $p, 0 \leq p \leq k$. Also we note that $\phi,\S$ share a value a IM or CM if and only if $\phi,\S$ share $(a,0)$ or (a,∞) respectively.

Definition 1.2. *(6, p.196,* [10]*) Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_\phi(S,k)$ the set $\bigcup_{a\in S} E_k(a,\phi)$.*

Clearly $E_\phi(S) = E_\phi(S,\infty)$ and $\overline{E}_\phi(S) = E_\phi(S,0)$.

In 1977 [6], Gross posed the following question.

Question 1.1. Can one find two finite sets $S_j, j = 1,2$ such that any two non constant entire functions ϕ and $\S$ satisfying $E_\phi(S_j) = E_\S(S_j)$ for $j = 1,2$ must be identical?

In 2003 [19], Yi and Lin asked the following question corresponding to meromorphic functions.

Question 1.2. Can one find two finite sets $S_j, j = 1,2$ such that any two non constant meromorphic functions ϕ and $\S$ satisfying $E_\phi(S_j) = E_\S(S_j)$ for $j = 1,2$ must be identical?

In connection to the Question 1.2, Li and Yang in [26] obtained the following result.

Theorem A Let $m \geq 2$ and $n > 2m+6$ with n and $n-m$ having no common factors. Let a and b be two non-zero constants such that the equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots. Let $S = \{\omega | \omega^n + a\omega^{n-m} + b = 0\}$. Then for any two non constant meromorphic functions ϕ and $\S$, the conditions $E_\phi(S,\infty) = E_\S(S,\infty)$ and $E_\phi(\{\infty\},\infty) = E_\S(\{\infty\},\infty)$ imply $\phi \equiv \S$.

Let us explain some standard definitions and notations of the value distribution theory available in [8] which will be used in the paper.

Definition 1.3. *(2, p.85,* [9]*) For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r,a;\phi|=1)$ the counting function of simple $a-$ point of ϕ. For a positive integer m, we denote by $N(r,a;\phi|\leq m)(N(r,a;\phi|\geq m))$ the counting function of those $a-$point of ϕ whose multiplicities are not greater (less) than m, where each $a-$ point is counted according to its multiplicity.*

$\overline{N}(r,a;\phi|\leq m)(\overline{N}(r,a;\phi|\geq m))$ are defined similarly except that in counting the $a-$points of ϕ we ignore the multiplicity. Also $N(r,a;\phi|< m)(N(r,a;\phi|> m)), N(r,a;\phi|< m)$ and $N(r,a;\phi|> m)$ are defined similarly.

Definition 1.4. *(1, p.142,* [11]*) For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_2(r,a;\phi) = \overline{N}(r,a;\phi) + \overline{N}(r,a;\phi|\geq 2)$.*

Definition 1.5. *(3, p.142,* [11]*) Let ϕ and $\S$ share a value a IM. We denote by $\overline{N}_*(r,a;\phi,\S)$ the counting function of those $a-$points of ϕ whose multiplicities differ from the multiplicities of the corresponding $a-$points of $\S$.*

Suppose p be a non-zero complex constant. We define the shift of $\phi(z)$ by $\phi(z+p)$ and define the difference operators by

$$\triangle_p\phi(z)=\phi(z+p)-\phi(z),$$

$$\triangle_p^n\phi(z)=\triangle_p^{n-1}(\triangle_p\phi(z)),n\in\mathbb{N},n\geq 2.$$

In 2010, Zhang [21] considered a meromorphic function $\phi(z)$ sharing sets with its shift $\phi(z+p)$ and obtained the following result.

Theorem B Let $m\geq 2$ and $n\geq 2m+4$ with n and $n-m$ having no common factors. Let $S=\{\omega|\omega^n+a\omega^{n-m}+b=0\}$, a and b be two non-zero constants such that the equation $\omega^n+a\omega^{n-m}+b=0$ has no multiple roots. Suppose that $\phi(z)$ is a non-constant meromorphic function of finite order. Then $E_{\phi(z)}(S,\infty)=E_{\phi(z+p)}(S,\infty)$ and $E_{\phi(z)}(\{\infty\},\infty)=E_{\phi(z+p)}(\{\infty\},\infty)$ imply $\phi(z)\equiv\phi(z+p)$.

For an analogue result in difference operator, Chen and Chen [5] proved the following result.

Theorem C Let $m\geq 2$ and $n>2m+4$ with n and $n-m$ having no common factors. Let a and b be two non-zero constants such that the equation $\omega^n+a\omega^{n-m}+b=0$ has no multiple roots. Let $S=\{\omega|\omega^n+a\omega^{n-m}+b=0\}$. Suppose that $\phi(z)$ is a non-constant meromorphic function of finite order satisfying $E_{\phi(z)}(S,\infty)=E_{\phi(\triangle_p\phi)}(S,\infty)$ and $E_{\phi(z)}(\{\infty\},\infty)=E_{\phi(\triangle_p\phi)}(\{\infty\},\infty)$. If

$$N(r,0;\triangle_p\phi)=T(r,\phi)+S(r,\phi),$$

then

$$\phi(z)\equiv\phi(\triangle_p\phi).$$

In 2014, Li and Chen [25] considered a linear difference polynomial of ϕ in the following manner

$$L(z,\phi)=b_k\phi(z+c_k)+...+b_0(z)\phi(z+c_0), \tag{1.1}$$

where $b_k(\not\equiv 0),...,b_0(z)$ are small functions of $\phi,c_0,c_1,...,c_k$ are complex constants and k is a non-negative integer satisfying one of the following conditions:

$$b_0(z)+...+b_k(z)\equiv 1, \tag{1.2}$$

$$b_0(z)+...+b_k(z)\equiv 0 \tag{1.3}$$

obtained the following theorem.

Theorem D Let $m\geq 2$ and $n\geq 2m+4$ with n and $n-m$ having no common factors. Let a and b be two non-zero constants such that the equation $\omega^n+a\omega^{n-m}+b=0$ has no multiple roots. Let

$S = \{\omega | \omega^n + a\omega^{n-m} + b = 0\}$. Suppose that $\phi(z)$ is a non-constant meromorphic function of finite order and $L(z,\phi)$ is of the form (1.1) satisfying the conditions (1.2) and (1.3). If $E_{\phi(z)}(S,\infty) = E_{L(z,\phi)}(S,\infty)$, $E_{\phi(z)}(\{\infty\},\infty) = E_{L(z,\phi)}(\{\infty\},\infty)$ and $N(r,0;L(z,\phi)) = T(r,\phi) + S(r,\phi)$, then

$$\phi(z) \equiv L(z,\phi).$$

Remark 1.1. *From the above discussions, it is to be observed that in Theorem B, Theorem C and Theorem D, the minimum cardinality of the main range set S is 9 under the environment of CM sharing hypothesis.*

In 2022, G. Haldar [22] proved the following result.

Theorem E Let $S = \{z | P(z) = 0\}$, where $P(z)$ polynomial and $n(\geq 2m+3), m(\geq 1)$ be two positive integers such that $gcd(n,m) = 1, a, b, c, d$ are non zero complex numbers, $\frac{c^2}{4bd} = \frac{n(n-2m)}{(n-m)^2} \neq 1$ and $a \neq \gamma_j$ for $j = 1, 2, .., m$. Let ϕ be a transcendental meromorphic function of finite order. Suppose $E_{\phi(z)}(S,3) = E_{L(z,\phi)}(S,3)$, $E_{\phi(z)}(\{\infty\},0) = E_{L(z,\phi)}(\{\infty\},0)$, where $L(z,\phi)$ is defined in (1.1). Then

$$\phi(z) \equiv L(z,\phi).$$

Definition 1.6. *(1.3, p.122, [1]) The q-th order difference operator $\triangle_\eta^q \phi(z)$ is defined by $\triangle_\eta^q \phi(z) = \triangle_\eta^{q-1}(\triangle_\eta \phi(z))$, where $q(\geq 2) \in \mathbb{N}$ and $\eta \in \mathbb{C} \setminus \{0\}$, while the difference polynomial of difference operator is given by $\mathcal{L}(\triangle_\eta \phi) = \sum_{i=1}^{q} a_i \triangle_\eta^i \phi$, where $a_i (i = 1, 2, ..., q)$ are nonzero constants.*

We can also deduce that,

$$\triangle_\eta^q \phi = \sum_{i=1}^{q} \binom{q}{i} \phi(z + (q-i)\eta). \tag{1.4}$$

Definition 1.7. *(1.3, p.381, [2]) Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S,k)$ the set $\cup_{a \in S} E_k(a;f)$. Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.*

If $E_f(S,k) = E_g(S,k)$, then we say that f, g share the set S with weight k and write it as f, g share (S,k).

By $N(r,a;f| < m)$ we mean the counting function of those $a-$points of f whose multiplicities are less than m where each $a-$point is counted according to its multiplicity and by $\overline{N}(r,a;f| \geq m)$ we mean the counting function of those $a-$points of f whose multiplicities are not less than m where each $a-$points is counted ignoring multiplicity. We also denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f| \geq 2)$.
Usually, $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside of a possible exceptional set of finite linear measure. Also $S_1(r,f)$ denotes any quantity satisfying $S_1(r,f) = o(T(r,f))$ for all r on a set of logarithmic density 1, where the logarithmic density of a set F is defined by

$$\lim_{r \longrightarrow \infty} \sup \frac{1}{logr} \int_{[1,r] \cap F} \frac{dt}{t}.$$

Let the positive integer l, S, S^* and S_2 represents respectively the sets $\{1, \omega, ..., \omega^{l-1}\}$, $\{\alpha_1, \alpha_2, ..., \alpha_l\}$ and $\{\infty\}$, where $\omega = cos\frac{2\pi}{l} + isin\frac{2\pi}{l}$ and $\alpha_i, i = 1, 2, ..., l$ are non zero constants.
Let $a_{t-1}(\neq 0), a_{t-2}, ..., a_0$ and $C(\neq 0)$ be complex numbers. We define

$$P(z) = CzQ(z) = Cz(a_{t-1}z^{t-1} + a_{t-2}z^{t-2} + ... + a_1 z + a_0). \tag{1.5}$$

For the polynomial $P(z)$ as given in (1.5), let us define two functions:

$$\chi_0^{t-1} = \begin{cases} 1, & if \quad a_0 \neq 0 \\ 0, & if \quad a_0 = 0. \end{cases}$$

and

$$\mu_0^{t-1} = \begin{cases} 1, & if \quad a_0 = 0, a_1 \neq 0 \\ 0, & otherwise. \end{cases}$$

In view of (1.5), corresponding to the set S^*, let us consider the polynomial $P_*(z)$ as follows:

$$P_*(z) = CzQ_*(z). \tag{1.6}$$

where $C = \dfrac{1}{(-1)^{l+1}\alpha_1\alpha_2...\alpha_l}$ and $Q_*(z) = \Sigma_{r=0}^{l-1}(-1)^r \Sigma\alpha_1\alpha_2...\alpha_r z^{l-r-1}$,

$\Sigma\alpha_1\alpha_2...\alpha_r$ = sum of the products of the value $\alpha_1, \alpha_2, ..., \alpha_l$ taking r into account. We also denote by m_1 and m_2 as the number of simple and multiple zeros of $Q_*(z)$ respectively.
Therefore, naturally one may ask the following question.

Question 1.3. Is it possible to get the uniqueness of the meromorphic function ϕ with its difference polynomial of difference operator $\mathcal{L}(\triangle_\eta \phi)$ under sharing of the range sets can further be relaxed?

To seek the possible answer of the above question is the motivation of the paper. We now recall following polynomial introduced by A. Banerjee and G. Haldar [3] renders an useful resource. Let for $d \in \mathbb{C}$,

$$Q(z) = z^n - \frac{2n}{n-m}z^{n-m} + \frac{n}{n-2m}z^{n-2m} - d. \tag{1.7}$$

Then

$$Q'(z) = nz^{n-2m-1}(z^m - 1)^2 = nz^{n-2m-1}\prod_{j=0}^{m-1}(z - \mu_j)^2,$$

where $\mu_j = cos\frac{2j\pi}{m} + isin\frac{2j\pi}{m}, j = 0, 1, ..., m-1$.
Therefore,

$$Q(0) = -d$$

and

$$Q(\mu_j) = \mu_j^n - \frac{2n}{n-m}\mu_j^{n-m} + \frac{n}{n-2m}\mu_j^{n-2m} - d$$

$$= \mu_j^n \left(1 - \frac{2n}{n-m} + \frac{n}{n-2m}\right) - d$$
$$= \frac{2m^2\mu_j^n}{(n-m)(n-2m)} - d$$
$$= \lambda_j - d,$$

where $\lambda_j = \frac{2m^2\mu_j^n}{(n-m)(n-2m)}, j = 0, 1, 2, ..., m-1$. Therefore, if $d \neq 0, \lambda_j, j = 0, 1, ..., m-1$ all the zeros of the polynomial $Q(z)$ given by (1.7) are simple.

Now it is clear that $Q(z) - Q(\mu_j) = (z - \mu_j)^3 \Phi_{n-3}(z)$, where $\Phi_{n-3}(z)$ is poynomial of degree $n-3, j = 0, 1, ..., m-1$. Hence,

$$Q(\phi) - Q(\mu_j) = (\phi - \mu_j)^3 \Phi_{n-3}(\phi).$$

i.e.,

$$dF - d - (\lambda_j - d) = (\phi - \mu_j)^3 \Phi_{n-3}(\phi),$$

where

$$F = \frac{\phi(z)^{n-2m}(\phi(z)^{2m} - \frac{2n}{n-m}\phi(z)^m + \frac{n}{n-2m})}{d}$$

i.e.,

$$F - \frac{\mu_j}{d} = \frac{1}{d}(\phi - \mu_j)^3 \Phi_{n-3}(\phi). \tag{1.8}$$

$$F - \tilde{\psi}_j = \frac{1}{d}(\phi - \mu_j)^3 \Phi_{n-3}(\phi), \tag{1.9}$$

where

$$\tilde{\psi}_j = \frac{\mu_j}{d}, j = 0, 1, ..., m-1. \tag{1.10}$$

Throughout the paper we shall denote by $a = 3m + 2, b = 4 + 2m + \frac{(4m+2)(7n-3)}{(n-1)(3n-1)}, p = 2m + \frac{8m+2}{n-1} + \frac{(n-2m-q+2)(4m+1)}{(n-2m-q-1)(nk+n+q-1)}, r = 2m + \frac{(n-2m-q+2)(4m+1)}{(n-2m-q-1)(nk+n+q-1)} + \frac{4m+1}{n-q-1}$.

Let us define Γ_n as follows:

$$\Gamma_n = \begin{cases} 1, & if \quad n \geq 11 \\ 0, & otherwise. \end{cases}$$

The following theorem is the main result of the paper.

Theorem 1.1. *Let $S = \{z | Q(z) = 0\}$, where $Q(z)$ is a polynomial given by (1.7) and $n(\geq 1), m(\geq 1), q(\geq 1), k, t$ with $gcd(n, m) = 1$ be five positive integers. Let $\phi(z)$ transcendental meromorphic function of finite order and η be a nonzero complex constant. Suppose $\phi(z), \mathcal{L}(\triangle_\eta \phi)$ share $(0, 0), (\infty, k), E_{\phi(z)}(S, t) = E_{\mathcal{L}(\triangle_\eta \phi)}(S, t)$, where $1 \leq k < \infty, t > \frac{3}{2} - \frac{3}{n-2m-q-1} - \frac{2}{n-q-1} - \frac{n-2m-q+2}{(n-2m-q-1)(nk+n+q-1)}$.*

If one of the following conditions hold:

(i) $m = 1, n \geq 5 + q$, where $q = 1$ and $d \neq 0, \frac{2}{(n-1)(n-2)}, \frac{1}{(n-1)(n-2)}$ or

(ii) $m \geq 2, q \geq 1, n > max\{3m, p\}$ *and* $d \in \mathbb{C}\ \{0, \mu_0, \mu_1, ..., \mu_{m-1}\}$ *then*

$$\phi(z) \equiv \mathcal{L}(\triangle_\eta \phi).$$

Putting $m = 1, t = 4, q = 1$ and $k = 5$ in the above theorem we obtain the following corollary.

Corollary 1.1. *Let* $n(\geq 5)$ *be a positive integer and* $d_1 = \frac{(n-1)(n-2)}{2} d$*, where* d *is a nonzero complex constant such that* $d_1 \neq 0, 1, \frac{1}{2}$*. Let* $S = \{z : \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - d_1 = 0\}$*. Let* $\phi(z)$ *be a transcendental meromorphic function of finite order and* η *be a nonzero complex constant. Suppose* $\phi(z), \phi(z+c)$ *share* $(0, 0), (\infty, 5)$ *and* $E_{\phi(z)}(S, 4) = E_{\mathcal{L}(\triangle_\eta \phi)}(S, 4)$*. Then*

$$\phi(z) \equiv \mathcal{L}(\triangle_\eta \phi).$$

Remark 1.2. *The following example shows that in the main result, the polynomial* $Q(z)$ *can not be chosen arbitrarily.*

Example 1.1. Suppose $\phi(z) = \frac{e^z}{1+e^z}, \S(z) = \frac{1}{1+e^z}$. Then ϕ and g are finite order sharing $(0, \infty), (\infty, \infty)$. Also ϕ and $\S$ share the set S as well as S_* CM. But $\phi \not\equiv \S$.

However, we were not able to find the case when $\S(z) = \mathcal{L}(\triangle_\eta \phi), \eta$ is a non-zero complex constant. Next we consider the case when ϕ is of infinite order. It is interesting to investigate whether in Theorem 1.1 and Corollary 1.1, respectively for the case $d = \frac{1}{12}$ and $d_1 = \frac{1}{2}$ such counter example exists at all. The following example shows that such situation is feasible.

Example 1.2. Let $\phi(z) = \frac{1}{1+e^{z}}$. Then $\mathcal{L}(\triangle_\eta \phi) = \frac{e^{e^z}}{e^{e^z}+1}$, where η is chosen such that $e^\eta = -1$. Clearly $\phi(z), \mathcal{L}(\triangle_\eta \phi)$ share $(0, \infty), (\infty, \infty)$ and sets S and S_* CM, but $\phi(z) \not\equiv \mathcal{L}(\triangle_\eta \phi)$.

However, unfortunately when $d \neq \frac{1}{12}$ or $d_1 \neq \frac{1}{2}$, we were again unsuccessful to find out the counter example in this case.

2. **Lemmas**

In this section, we present some lemmas which will be neede in the sequel. let ϕ and $\S$ be two non-constant meromorphic functions defined in $\mathbb{C}$. Let us also define two functions, F and G, in $\mathbb{C}$ by

$$F = \frac{\phi(z)^{n-2m}(\phi(z)^{2m} - \frac{2n}{n-m}\phi(z)^m + \frac{n}{n-2m})}{d} \tag{2.1}$$

$$G = \frac{[\mathcal{L}(\triangle_\eta \phi)]^{n-2m}([\mathcal{L}(\triangle_\eta \phi)]^{2m} - \frac{2n}{n-m}[\mathcal{L}(\triangle_\eta \phi)]^m + \frac{n}{n-2m})}{d} \tag{2.2}$$

We also denote by Ω, W, Ω_1, W_1 and Ψ, the following functions

$$\Omega = \left(\frac{\phi''}{\phi'} - \frac{2\phi'}{\phi - 1}\right) - \left(\frac{\S''}{\S'} - \frac{2\S'}{\S - 1}\right),$$

$$W = \frac{\phi'}{\phi(\phi - 1)} - \frac{\S'}{\S(\S - 1)},$$

$$\Omega_1 = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$
$$W_1 = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}$$

and

$$\Psi = \frac{F'}{(F-1)} - \frac{G'}{(G-1)}$$

Lemma 2.1. *(2.1, p.127,* [4]*) Let* F, G *be two non-constant meromorphic functions such that they share* $(1,1)$ *and* $\Omega \not\equiv 0$*. Then*

$$N(r,1;F|=1) = \overline{N}(r,1;G|=1) \leq N(r,\Omega) + S(r,F) + S(r,G).$$

Lemma 2.2. *(2.5, p.385,* [2]*) Let* F, G *be two non-constant meromorphic functions such that they share* $(1,t)$ *where* $1 \leq t < \infty$*. Then*

$$\overline{N}(r,1;F) + \overline{N}(r,1;G) - \overline{N}(r,1;F|=1) + (t - \frac{1}{2})\overline{N}_*(r,1;F,G) \leq \frac{1}{2}[N(r,1;F) + N(r,1;G)].$$

Lemma 2.3. *Suppose* $\phi, \S$ *share* $(1,0), (\infty,0), (0,0)$ *and* $\tilde{\psi}_j$*, defined as in (1.10), are non-zero complex numbers. If* $\Omega \not\equiv 0$*, then*

$$N(r,\Omega) \leq \overline{N}_*(r,0;\phi,\S) + \sum_{j=0}^{m-1} \overline{N}(r,\tilde{\psi}_j;\phi|\geq 2) + \sum_{j=0}^{m-1} \overline{N}(r,\tilde{\psi}_j;\S|\geq 2)$$
$$+ \overline{N}_*(r,1;\phi,\S) + \overline{N}_*(r,\infty;\phi,\S) + \overline{N}(r,0;\phi') + \overline{N}(r,0;\S') + S(r,\phi) + S(r,\S),$$

where $\overline{N}(r,0;\tilde{\phi}')$ *is reduced counting function of those zeros of* ϕ' *which are not the zeros of* $\phi(\phi - 1)\prod_{j=0}^{m-1}(\phi - \tilde{\psi}_j)$ *and* $\overline{N}(r,0;\S')$ *is similarly defined.*

Proof. By the definition of Ω we verify that the possible poles of Ω occur from the following six cases: (i) The common zeros of ϕ and $\S$ of different multiplicities. (ii) the multiple $\tilde{\psi}_j$-points of ϕ and $\S$ for each $j = 0, 1, 2, ..., m-1$. (iii) Those common poles of ϕ and $\S$, where each such pole of ϕ and $\S$ has different multiplicities related to ϕ and $\S$. (iv) those common 1-points of ϕ and $\S$, where each point has different multiplicities related to ϕ and $\S$. (v) The zeros of ϕ' which are not zeros of $\phi(\phi-1)\prod_{j=0}^{m-1}(\phi - \tilde{\psi}_j)$. (vi) The zeros of $\S'$ which are not zeros of $\S(\S-1)\prod_{j=0}^{m-1}(\S - \tilde{\psi}_j)$. Since all poles of Ω are simple, the lemma follows. □

Lemma 2.4. *(Clunie [2, p.68,* [23]*]) Let* ϕ *be a non-constant meromorphic function and* $Q(\phi) = a_0 + a_1\phi + a_2\phi^2 + ... + a_n\phi^n$*, where* $a_0, a_1, a_2, ..., a_n$ *are constants and* $a_0 \neq 0$*. Then*

$$T(r, Q(\phi) = nT(r,\phi)) + O(1).$$

Lemma 2.5. *(3.2, p.26,* [1]*) Let* ϕ *be a non-constant meromorphic function of finite order and* $c \in \mathbb{C} - \{0\}$ *be fixed. Then*

$$T(r, \mathcal{L}(\triangle_\eta \phi)) = qT(r,\phi(z)) + S(r,\phi(z))$$

Lemma 2.6. *Let ϕ be a non-constant meromorphic function of finite order and $c \in \mathbb{C} - \{0\}$ be fixed. Then*

$$S(r, \mathcal{L}(\triangle_\eta \phi)) = S(r, \phi(z)).$$

Proof. Using Lemma 2.4, it can be easily seen that

$$S(r, \mathcal{L}(\triangle_\eta \phi)) = o(T(r, \mathcal{L}(\triangle_\eta \phi))) = o(T(r, \phi(z))) = S(r, \phi(z)).$$

□

Lemma 2.7. *Let F and G be given by (2.1) and (2.2), $n(\geq 1)$ an integer and $\Psi \not\equiv 0$. If F and G share $(1, m), \phi(z)$ and $\mathcal{L}(\triangle_\eta \phi)$ share $(0, u), (\infty, k)$, where $0 \leq k < \infty$, then*

$$\begin{aligned}
&\{(n - 2m)(u + 1) - 1\}\overline{N}(r, 0; \phi| \geq u + 1)\\
&= \{(n - 2m)(u + 1) - 1\}\overline{N}(r, 0; \mathcal{L}(\triangle_\eta \phi)| \geq u + 1)\\
&\leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + S(r, \phi) + S(r, \mathcal{L}(\triangle_\eta \phi)).
\end{aligned}$$

Proof. Suppose 0 is an exceptional value of Picard(e.v.p) of $\phi(z)$ and $\mathcal{L}(\triangle_\eta \phi)$. Then the lemma follows immediately. Next suppose 0 in not an e.v.p of $\phi(z)$ and $\mathcal{L}(\triangle_\eta \phi)$. Let z_0 be a zero of ϕ with multiplicity p and a zero of $\phi(z)$ and $\mathcal{L}(\triangle_\eta \phi)$ with multiplicity r. Then from (2.1) and (2.2), we know that z_0 is a zero of F and G have no zero multiplicity t, where $(n - 2m)u < t < (n - 2m)(u + 1)$.

So, from definition of Ψ, it is clear that z_0 of Ψ with multiplicity at least $(n - 2m)(u + 1) - 1$. So, we have,

$$\begin{aligned}
&\{(n - 2m)(u + 1) - 1\}\overline{N}(r, 0; \phi(z)| \geq u + 1)\\
&= \{(n - 2m)(u + 1) - 1\}\overline{N}(r, 0; \mathcal{L}(\triangle_\eta \phi)| \geq u + 1)\\
&\leq \overline{N}_*(r, \infty; \phi(z), \mathcal{L}(\triangle_\eta \phi)) + \overline{N}_*(r, 1; F, G) + S(r, \phi) + S(r, \mathcal{L}(\triangle_\eta \phi)).
\end{aligned}$$

□

Lemma 2.8. *Let F and G be given by (2.1) and (2.2), where $n(\geq 8)$ is an integer and $\Omega_1 \not\equiv 0$. Suppose $\phi_1, \phi_2, ..., \phi_{2m}$ are the roots of the equation $z^{2m} - \frac{2n}{n-m}z^m + \frac{n}{n-2m} = 0$. Suppose also that F, G share $(1, t)$ and $\phi(z), \mathcal{L}(\triangle_\eta \phi)$ share $(\infty, k), (0, 0)$, where $2 \leq t < \infty$. Then, for the complex numbers $\tilde{\psi}_j$ given by (1.10), we have*

$$\begin{aligned}
&n\left(m + \frac{q}{2}\right)\{T(r, \phi(z)) + T(r, \mathcal{L}(\triangle_\eta \phi))\}\\
&\leq \overline{N}(r, 0; \phi(z)) + \overline{N}(r, 0; \mathcal{L}(\triangle_\eta \phi)) + \overline{N}_*(r, 0; \phi(z), \mathcal{L}(\triangle_\eta \phi)) + \overline{N}(r, \infty; \phi)\\
&+ \sum_{j=1}^{2m} N_2(r, \tilde{\phi}_j; \phi(z)) + \sum_{j=1}^{2m} N_2(r, \tilde{\phi}_j; \mathcal{L}(\triangle_\eta \phi)) + \overline{N}(r, \infty; \mathcal{L}(\triangle_\eta \phi))\\
&+ \sum_{j=1}^{m-1} N_2(r, \tilde{\psi}_j; \phi(z)) + \sum_{j=1}^{m-1} N_2(r, \tilde{\psi}_j; \mathcal{L}(\triangle_\eta \phi)) + \overline{N}_*(r, \infty; \phi(z), \mathcal{L}(\triangle_\eta \phi))
\end{aligned}$$

$$-\left(t-\frac{3}{2}\right)+\overline{N}_*(r,1;F,G)+S(r,\phi)+S(r,\mathcal{L}(\triangle_\eta\phi)).$$

Proof. By the Second fundamental theorem of Nevanlinna, we have

$$\begin{aligned}
&(m+1)\{T(r,F)+T(r,G)\}\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;F)+\sum_{j=1}^{m-1}\overline{N}(r,\psi_j;F)\\
&+\overline{N}(r,0;G)+\overline{N}(r,1;G)+\overline{N}(r,\infty;G)+\sum_{j=1}^{m-1}\overline{N}(r,\psi_j;G)-\overline{N}(r,0;F')-\overline{N}(r,0;G')\\
&+S(r,F)+S(r,G).
\end{aligned}$$

Now using Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}
&n\left(m+\frac{q}{2}\right)\{T(r,\phi(z))+T(r,\mathcal{L}(\triangle_\eta\phi))\}\\
&\leq \overline{N}(r,0;\phi(z))+\overline{N}(r,0;\mathcal{L}(\triangle_\eta\phi))+\overline{N}_*(r,0;\phi(z),L(\triangle_\eta f))+\overline{N}(r,\infty;\phi)\\
&+\sum_{j=1}^{2m}N_2(r,\tilde{\phi}_j;\phi(z))+\sum_{j=1}^{2m}N_2(r,\tilde{\phi}_j;\mathcal{L}(\triangle_\eta\phi))+\overline{N}(r,\infty;\mathcal{L}(\triangle_\eta\phi))\\
&+\sum_{j=1}^{m-1}N_2(r,\tilde{\psi}_j;\phi(z))+\sum_{j=1}^{m-1}N_2(r,\tilde{\psi}_j;\mathcal{L}(\triangle_\eta\phi))+\overline{N}_*(r,\infty;\phi(z),\mathcal{L}(\triangle_\eta\phi))\\
&-\left(t-\frac{3}{2}\right)+\overline{N}_*(r,1;F,G)+S(r,\phi)+S(r,\mathcal{L}(\triangle_\eta\phi)).
\end{aligned}$$

□

Lemma 2.9. *Let F and G be given by (2.1) and (2.2), where $n(\geq 8)$ is an integer and $W_1\not\equiv 0$. Suppose also F,G share $(1,t)$ and $\phi(z),\mathcal{L}(\triangle_\eta\phi)$ share $(\infty,k),(0,0)$, where t,k and u are non-negative integers. Then the poles of F and G are zeros of W_1 and*

$$\begin{aligned}
&(nk+n+q-1)\overline{N}(r,\infty;\phi(z)|\geq k+1)\\
&=(nk+n+q-1)\overline{N}(r,\infty;\mathcal{L}(\triangle_\eta\phi)|\geq k+1)\\
&\leq \overline{N}_*(r,0;\phi(z),\mathcal{L}(\triangle_\eta\phi))+\sum_{j=1}^{2m}N_2(r,\tilde{\phi}_j;\phi(z))+\sum_{j=1}^{2m}N_2(r,\tilde{\phi}_j;\mathcal{L}(\triangle_\eta\phi))\\
&+\overline{N}_*(r,1;F,G)+S(r,\phi(z))+S(r;\mathcal{L}(\triangle_\eta\phi)),
\end{aligned}$$

where $\phi_j, j=1,2,...,2m$ has the same meaning as in Lemma 2.8.

Proof. Since $\phi(z),\mathcal{L}(\triangle_\eta\phi)$ share (∞,k), it follows that F,G share (∞,nk) so a pole of ϕ with multiplicity $u(\geq nk+q+1)$ is a pole of G with multiplicity $r(\geq nk+q+1)$ and vice versa. We note that F and G have no pole multiplicity u where $nk<u<nk+n$. Now using the Milloux Theorem [[**?**],p. 55], we get from the definition of W_1,

$$m(r,W_1)=S(r,\phi(z))+S(r,\mathcal{L}(\triangle_\eta\phi).$$

Hence

$$\begin{aligned}
&(nk+n+q-1)\overline{N}(r,\infty;\phi(z)|\geq k+1)\\
&=(nk+n+q-1)\overline{N}(r,\infty;\mathcal{L}(\triangle_\eta\phi)|\geq k+1)\\
&\leq N(r,0;W_1)\\
&\leq T(r,W_1)+O(1)\\
&\leq N(r,\infty;W_1)+m(r,W_1)+O(1)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,F)+S(r,G)\\
&\leq \overline{N}_*(r,0;\phi(z),\mathcal{L}(\triangle_\eta\phi))+\sum_{j=1}^{2m}N_2(r,\tilde{\phi}_j;\phi(z))+\sum_{j=1}^{2m}N_2(r,\tilde{\phi}_j;\mathcal{L}(\triangle_\eta\phi))\\
&+\overline{N}_*(r,1;F,G)+S(r,\phi(z))+S(r;\mathcal{L}(\triangle_\eta\phi)),
\end{aligned}$$

where $\phi_j, j=1,2,...,2m$ has the same meaning as in Lemma 2.8. □

Lemma 2.10. *Let F and G be given by (2.1) and (2.2). Then $FG\not\equiv 1$ for $n\geq 5+q$.*

Proof. Suppose on the contrary $FG\equiv 1$. Then by Mokhon'ko's Lemma

$$T(r,\phi(z))=T(r,\mathcal{L}(\triangle_\eta\phi))+O(1).$$

Also

$$(\phi(z))^{n-2m}\prod_{j=1}^{2m}(\phi(z)-\tilde{\phi}_j)(\mathcal{L}(\triangle_\eta\phi))^{n-2m}\prod_{j=1}^{2m}(\mathcal{L}(\triangle_\eta\phi)-\tilde{\phi}_j)\equiv d^2,$$

where $\tilde{\phi}_j, j=1,2,...,2m$ has the same meaning as in Lemma 2.8.
Let z_0 be a $\tilde{\phi}_j$-point of $\phi(z)$ of order u. Then z_0 is a pole of $L(\triangle_\eta\phi$ of order p such that $u=np\geq n$. Therefore,

$$\overline{N}(r,\tilde{\phi}_j;\phi(z))\leq\frac{1}{n}N(r,\tilde{\phi}_j;\phi(z)).$$

Again let z_0 be a zero of $\phi(z)$ of order t. Then z_0 is a pole of $\mathcal{L}(\triangle_\eta\phi)$ of order Θ such that

$$(n-2m)t=n\Theta.$$

This implies $t>\Theta$ and $2m\Theta=(n-2m)(t-\Theta)\geq(n-2m)$. Therefore, $(n-2m)t=n\Theta$ gives $t\geq\frac{n}{2m}$. So

$$\overline{N}(r,0;\phi(z))\leq\frac{2m}{n}N(r,0;\phi(z)).$$

Again

$$\overline{N}(r,\infty;\phi(z))\leq\overline{N}(r,0;\mathcal{L}(\triangle_\eta\phi)+\sum_{j=1}^{2m}\overline{N}(r,\tilde{\phi}_j;\mathcal{L}(\triangle_\eta\phi))$$

$$\leq \frac{2m}{n}N(r,0;\mathcal{L}(\triangle_\eta\phi)) + \frac{1}{n}\sum_{j=1}^{2m} N(r,\tilde{\phi}_j;\mathcal{L}(\triangle_\eta\phi))$$
$$\leq \frac{4m+q}{n}T(r,\mathcal{L}(\triangle_\eta\phi)).$$

Therefore, by the Second Fundamental Theorem of Nevanlinna, we get

$$2mT(r,\phi(z)) \leq \overline{N}(r,\infty;\phi(z)) + \overline{N}(r,0;\phi(z)) + \sum_{j=1}^{2m}\overline{N}(r,\tilde{\phi}_j;\phi(z)) + S(r,\phi(z))$$
$$\leq \frac{8m+q}{n}T(r,\phi) + S(r,\phi),$$

which is a contradiction for $n \geq 5+q$. □

Lemma 2.11. *Let $m(\geq 1)$ and $n(> 2m)$ be two positive integers. then the polynomial*

$$\vartheta(h) = (n-m)^2(h^n-1)(h^{n-2m}-1) - n(n-2m)(h^{n-m}-1)^2$$

of degree $2n-2m$ has m roots of multiplicity 4 and all other zeros are simple.

Proof. Let $F(t) = \frac{1}{2}\vartheta(e^t)e^{-(n-m)t}$ for $t \in \mathbb{C}$.
An elementary calculation gives

$$F(t) = m^2 cosh(n-m)t - (n-m)^2 coshmt + n(n-2m).$$

Assume that $\vartheta(\mu) = \vartheta'(\mu) = 0$ for some $\mu \in \mathbb{C}$.
Then $F(t) = F'(t) = 0$ for every $t \in \mathbb{C}$ satisfying $e^t = \mu$. From $F(t) = 0$, we get

$$m^2 cosh(n-m)t = (n-m)^2 coshmt - n(n-2m) \tag{2.3}$$

From $F'(t) = 0$, we get

$$m^2 sinh(n-m)t = m(n-m)sinhmt. \tag{2.4}$$

Therefore, from (2.3) and (2.4) we have

$$m^4 = \{(n-m)^2 coshmt - n(n-2m)\}^2 - \{m(n-m)sinhmt\}^2$$
$$(n-m)^4 cosh^2 mt + \{n(n-2m)\}^2 - 2n(n-2m)(n-m)^2 coshmt$$
$$-\{m(n-m)\}^2(cosh^2 mt - 1)$$
$$= \{(n-m)^4 - m^2(n-m)^2\}cosh^2 mt + \{n(n-2m)\}^2 + m^2(n-m)^2$$
$$-2n(n-2m)(n-m)^2 coshmt$$

or,

$$(n-m)^2 n(n-2m)cosh^2 mt + \{n(n-2m)\}^2 + m^2(n-m)^2$$
$$-2n(n-2m)(n-m)^2 coshmt - m^4 + \{n(n-2m)\}^2 = 0$$

or,

$$(n-m)^2 n(n-2m)(coshmt-1)^2-(n-m)^2 n(n-2m)+m^2(n-m)^2$$
$$-m^4+\{n(n-2m)\}^2=0$$

or,

$$(n-m)^2 n(n-2m)(coshmt-1)^2-n(n-2m)\{(n-m)^2-n(n-2m)\}$$
$$+m^2 n(n-2m)=0$$

or,

$$(n-m)^2 n(n-2m)(coshmt-1)^2=0$$

or,

$$(coshmt-1)^2=0$$

or,

$$\left(\frac{e^{mt}+e^{-mt}}{2}-1\right)^2=0$$

or,

$$(\mu^m-1)^4=0,$$

which shows that the roots of the equation $\mu^m=1$ are of multiplicity 4.
Therefore, $\vartheta(h)$ has m zeros of multiplicity 4 and all other zeros are simple. □

Lemma 2.12. *(3, p.130,* [20]*) Let $\phi, \S$ share $(\infty,0)$ and $W\equiv 0$. Then $\phi\equiv\S$.*

3. **Main Result**

Proof of Theorem 1.1. Let F and G be two functions defined in (2.1) and (2.2).
Since $E_{\phi(z)}(S,t)=E_{\mathcal{L}(\triangle_\eta\phi)}(S,t)$ and $E_{\phi(z)}(\{\infty\},k)=E_{\mathcal{L}(\triangle_\eta\phi)}(\{\infty\},k)$, it follows that F,G share $(1,t)$ and (∞,nk).
Since

$$F-\tilde{\psi}_j=\frac{1}{d}(\phi-\mu_j)^3\Phi_{n-3}(\phi),$$

where $\Phi_{n-3}(\phi)$ is a polynomial in ϕ of degree $n-3$, for $j=0,1,2,...,m-1$, we have

$$N_2(r,\tilde{\psi}_j;F)\le 2\overline{N}(r,\mu_j;\phi)+N(r,0;\Phi_{n-3}(\phi))\le 2\overline{N}(r,\mu_j;\phi)+(n-3)T(r,\phi)+S(r,\phi). \tag{3.1}$$

Similarly,

$$N_2(r,\psi_j;F)\le 2\overline{N}(r,\mu_j;\mathcal{L}(\triangle_\eta\phi))+(n-3)T(r,\mathcal{L}(\triangle_\eta\phi))+S(r,\mathcal{L}(\triangle_\eta\phi)). \tag{3.2}$$

for $j = 0, 1, 2, ..., m-1$.

Case 1: Suppose $H_1 \not\equiv 0$. Then $F \not\equiv G$. So, it follows from Lemma 2.12 that $W_1 \not\equiv 0$. Hence using (3.1), (3.2) and Lemmas 2.1, 2.2, 2.3, 2.7, 2.8 and Lemma 2.9, we have

$$\begin{aligned}
&n\left(m+\frac{q}{2}\right)\{T(r,\phi(z))+T(r,\mathcal{L}(\triangle_\eta\phi))\}\\
&\le 3\overline{N}(r,0;\phi)+2\overline{N}(r,\infty;\phi)+\{2m+q+m(n-1)\}\{T(r,\phi(z))+T(r,\mathcal{L}(\triangle_\eta\phi))\}\\
&+\overline{N}_*(r,\infty;\phi(z),\mathcal{L}(\triangle_\eta\phi))-\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,\phi)+S(r,\mathcal{L}(\triangle_\eta\phi))
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left(\frac{q(n-1)}{2}-m\right)\{T(r,\phi(z))+T(r,\mathcal{L}(\triangle_\eta\phi))\}\\
&\le \frac{3}{n-2m-q-1}\{\overline{N}_*(r,\infty;\phi(z),\mathcal{L}(\triangle_\eta\phi))+\overline{N}_*(r,1;F,G)\}\\
&+\frac{2}{n-q-1}\{\overline{N}_*(r,\infty;\phi(z),\mathcal{L}(\triangle_\eta\phi))+\overline{N}_*(r,1;F,G)+2mT(r,\phi(z))+2mT(r,\mathcal{L}(\triangle_\eta\phi))\}\\
&+\overline{N}_*(r,\infty;\phi(z),\mathcal{L}(\triangle_\eta\phi))-\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,\mathcal{L}(\triangle_\eta\phi))\\
&\le \left(1+\frac{3}{n-2m-q-1}\right)\{\overline{N}_*(r,\infty;\phi(z),\mathcal{L}(\triangle_\eta\phi))\\
&+\left(2m+\frac{1}{2}\right)\frac{2}{n-q-1}\{T(r,\phi(z))+T(r,\mathcal{L}(\triangle_\eta\phi))\}-\left(t-\frac{3}{2}-\frac{3}{n-2m-q-1}-\frac{2}{n-q-1}\right)\overline{N}_*(r,1;F,G)\\
&+S(r,\phi)+S(r,\mathcal{L}(\triangle_\eta\phi))\\
&\le \frac{n-2m-q+2}{(n-2m-q-1)(nk+n+q-1)}\{\overline{N}_*(r,0;\phi(z),\mathcal{L}(\triangle_\eta\phi))+2m(T(r,\phi(z))\\
&+T(r,\mathcal{L}(\triangle_\eta\phi)))+\overline{N}_*(r,1;F,G)\}+\frac{4m+1}{n-q-1}\{T(r,\phi(z))+T(r,\mathcal{L}(\triangle_\eta\phi))\}\\
&-\left(t-\frac{3}{2}-\frac{3}{n-2m-q-1}-\frac{2}{n-q-1}\right)\overline{N}_*(r,1;F,G)+S(r,\phi)+S(r,\mathcal{L}(\triangle_\eta\phi))\\
&\le \frac{n-2m-q+2}{(n-2m-q-1)(nk+n+q-1)}\{\overline{N}_*(r,0;\phi(z),\mathcal{L}(\triangle_\eta\phi))\\
&+2m(T(r,\phi(z))+T(r,\mathcal{L}(\triangle_\eta\phi)))\}+\frac{4m+1}{n-q-1}\{T(r,\phi(z))+T(r,\mathcal{L}(\triangle_\eta\phi))\}\\
&-\left(t-\frac{3}{2}-\frac{3}{n-2m-q-1}-\frac{2}{n-q-1}-\frac{n-2m-q+2}{(n-2m-q-1)(nk+n+q-1)}\right)\overline{N}_*(r,1;F,G)\\
&+S(r,\phi)+S(r,\mathcal{L}(\triangle_\eta\phi)).
\end{aligned}$$

Therefore, from the condition over t and k in the theorem, we get from above

$$\begin{aligned}
&\left\{\frac{q(n-1)}{2}-m-\frac{4m+1}{n-q-1}-\frac{(n-2m-q+2)(4m+1)}{2(n-2m-q-1)(nk+n+q-1)}\right\}\{T(r,\phi(z))\\
&+T(r,\mathcal{L}(\triangle_\eta\phi))\}\le S(r,\phi)+S(r,\mathcal{L}(\triangle_\eta\phi)),
\end{aligned}$$

which is a contradiction.

Case 2: Suppose $\Omega_1 \equiv 0$. then by integration we have

$$F = \frac{AG+B}{CG+D}, \tag{3.3}$$

where A, B, C, D are complex constants satisfying $AD - BC \neq 0$.

Therefore, from (3.3), F, G share $(1, \infty)$. Since F, G share (∞, nk), it follows that F, G share (∞, ∞). Also from Lemma 2.7, we obtain $\overline{N}(r, 0; \phi(z)) = \overline{N}(r, 0; \mathcal{L}(\triangle_\eta \phi)) = S(r, \phi) + S(r, \mathcal{L}(\triangle_\eta \phi))$.

Subcase 2.1: Suppose $AC \neq 0$. Then $F - \frac{A}{C} = \frac{(AD-BC)}{C(CG+D)} \neq 0$. So F omits the value $\frac{A}{C}$.

Therefore, by the Second Fundamental Theorem, we have

$$\begin{aligned} nT(r, \phi) &\leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, \frac{A}{C}; F) + S(r, F) \\ &\leq (2m+1)T(r, \phi) + S(r, \phi). \end{aligned}$$

i.e.,

$$(n - 2m - 1)T(r, \phi) \leq S(r, \phi),$$

which is a contradiction.

Subcase 2.2: Suppose $AC = 0$. Since $AD - BC \neq 0$, both A and C cannot be simultaneously zero.

Subcase 2.2.1: Suppose $A \neq 0$ and $C = 0$. Then (3.3) becomes

$$F = \tilde{\phi} G + \tilde{\psi}, \tag{3.4}$$

where $\tilde{\phi} = \frac{A}{D}$ and $\tilde{\psi} = \frac{B}{D}$.

If F has no 1-point, then by the Second Fundamental Theorem of Nevanlinna, we have

$$T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, 1; F) + \overline{N}(r, \infty; F) + S(r, F)$$

or,

$$(n - 2m - 1)T(r, \phi) \leq S(r, \phi),$$

which is not possible.

Let F has some 1-points. Then $\phi + \psi = 1$. Therefore from (3.4), we have $F = \tilde{\phi} G + 1 - \tilde{\phi}$.

Subcase 2.2.1.1: Suppose $\tilde{\psi} \neq 1$. We consider the following subcases.

Subcase 2.2.1.1.1: Suppose $m = 1$. So $\mu_0 = 1$. Noting that $n \geq 5 + q$ where $q = 1$, from (1.10), we have $\tilde{\phi}_0 = \frac{\lambda_0}{d} = \frac{2\mu_0^n}{(n-1)(n-2)d} = \frac{2}{(n-1)(n-2)d}$ and therefore, in view of (1.2) we must have

$$F - \tilde{\psi}_0 = \frac{1}{d}(\phi(z) - 1)^3 \Phi_{n-3}(\phi(z)),$$

where $\Phi_{n-3}(\phi(z))$ is a polynomial in $\phi(z)$ of degree $n - 3$. Therefore, we have

$$\overline{N}(r, \tilde{\psi}_0; F) \leq \overline{N}(r, 1; \phi(z)) + (n-3)T(r, \phi(z)) + S(r, \phi(z)).$$

In a similar manner, we write $G - \tilde{\psi}_0 = \frac{1}{d}(\phi(z)-1)^3\Phi^*_{n-3}(\mathcal{L}(\triangle_\eta\phi)), \Phi^*_{n-3}(\mathcal{L}(\triangle_\eta\phi))$ is a polynomial in $\mathcal{L}(\triangle_\eta\phi)$ of degree $n-3$ and

$$\overline{N}(r,\tilde{\psi}_0;F) \le \overline{N}(r,1;\mathcal{L}(\triangle_\eta\phi)) + (n-3)T(r,\mathcal{L}(\triangle_\eta\phi)) + S(r,\mathcal{L}(\triangle_\eta\phi)).$$

If $1-\tilde{\phi} \ne \tilde{\psi}_0$, then by the Second Fundamental Theorem, Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
2T(r,F) &\le \overline{N}(r,0;F) + \overline{N}(r,1-\phi;F) + \overline{N}(r,\psi_0;F) + \overline{N}(r,\infty;F) + S(r,F)\\
&\le \overline{N}(r,0;\phi(z)) + 2T(r,\phi(z)) + \overline{N}(r,0;\mathcal{L}(\triangle_\eta\phi)) + (2+q)T(r,\mathcal{L}(\triangle_\eta\phi))\\
&+ \overline{N}(r,1;\phi(z)) + (n-3)T(r,\phi(z)) + \overline{N}(r,\infty;\phi(z)) + S(r,\phi)\\
&(n+q+3)T(r,\phi) + S(r,\phi).
\end{aligned}$$

i.e.,

$$(n-q-3)T(r,\phi) \le S(r,\phi),$$

which is not possible.

If $1-\tilde{\phi} = \tilde{\psi}_0$, then we have from (3.4) that $F = (1-\tilde{\psi}_0)G + \tilde{\psi}_0$. Since $d \ne \frac{1}{(n-1)(n-2)}$, by the Second Fundamental Theorem, Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
2T(r,G) &\le \overline{N}(r,0;G) + \overline{N}(r,\frac{\tilde{\psi}_0}{\tilde{\psi}_0-1};G) + \overline{N}(r,\tilde{\psi}_0;G) + \overline{N}(r,\infty;G) + S(r,G)\\
&\le \overline{N}(r,0;\mathcal{L}(\triangle_\eta\phi)) + (2+q)T(r,\mathcal{L}(\triangle_\eta\phi)) + \overline{N}(r,0;\phi) + 2T(r,\phi)\\
&+ \overline{N}(r,1;\mathcal{L}(\triangle_\eta\phi)) + (n-3)T(r,\mathcal{L}(\triangle_\eta\phi)) + \overline{N}(r,\infty;\mathcal{L}(\triangle_\eta\phi)) + S(r,\phi) + S(r,\mathcal{L}(\triangle_\eta\phi)).
\end{aligned}$$

i.e.,

$$(n-q-3)T(r,\phi) \le S(r,\phi),$$

which is not possible.

Subcase 2.2.1.1.2: Next suppose $m \ge 2, q \ge 1$. Then by the Second Fundamental Theorem, Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
&(m+1)T(r,F)\\
&\le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1-\tilde{\phi};F) + \sum_{j=0}^{m-1}\overline{N}(r,\tilde{\psi}_j;F) + S(r,F)\\
&\le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \sum_{j=0}^{m-1}\overline{N}(r,\tilde{\psi}_j;F) + S(r,F)\\
&\le \overline{N}(r,0;\phi) + 2mT(r,\phi) + T(r,\phi) + \overline{N}(r,0;\mathcal{L}(\triangle_\eta\phi)) + (2m+q)T(r,\mathcal{L}(\triangle_\eta\phi)) + m(n-2)T(r,\phi) + S(r,\phi).
\end{aligned}$$

i.e.,

$$(n-2m-q-1)T(r,\phi) \le S(r,\phi),$$

which is not possible.

Subcase 2.2.1.2: Suppose $\tilde{\phi} = 1$. Then $F \equiv G$.

i.e.,

$$\phi(z)^{n-2m}\left(\phi(z)^{2m} - \frac{2n}{n-m}\phi(z)^m + \frac{n}{n-2m}\right)$$
$$\equiv [\mathcal{L}(\triangle_\eta \phi)]^{n-2m}\left([\mathcal{L}(\triangle_\eta \phi)]^{2m} - \frac{2n}{n-m}[\mathcal{L}(\triangle_\eta \phi)]^m + \frac{n}{n-2m}\right)$$

i.e.,

$$[\mathcal{L}(\triangle_\eta \phi)]^n - \frac{2n}{n-m}[\mathcal{L}(\triangle_\eta \phi)]^{n-m} + \frac{n}{n-2m}[\mathcal{L}(\triangle_\eta \phi)]^{n-2m}$$
$$\equiv \phi(z)^n - \frac{2n}{n-m}\phi(z)^{n-m} + \frac{n}{n-2m}\phi(z)^{n-2m}.$$

Suppose that $h(z) = \frac{\mathcal{L}(\triangle_\eta \phi)}{\phi(z)}$. Then we have from above,

$$(h^n - 1)f^{2m} - \frac{2n}{n-m}(h^{n-m} - 1)f^m + \frac{n}{n-2m}(h^{n-2m} - 1) = 0.$$

i.e.,

$$\frac{(n-m)(n-2m)}{2}(h^n - 1)g_1^2 - n(n-2m)(h^{n-m} - 1)g_1 + \frac{n(n-m)}{2}(h^{n-2m} - 1) = 0, \quad (3.5)$$

where $g_1 = \phi^m$.

Suppose $h(z)$ is not constant. Then from (3.5) we have,

$$\{(n-m)(n-2m)(h^n - 1)g_1 - n(n-2m)(h^{n-m} - 1)\}^2 = -n(n-2m)\Psi(h), \quad (3.6)$$

where $\Psi(h) = (n-m)^2(h^n - 1)(h^{n-2m} - 1) - n(n-2m)(h^{n-m} - 1)^2$ is a polynomial of degree $2n - 2m$. Therefore, in view of Lemma 2.11, (3.6) can be written as

$$\{(n-m)(n-2m)(h^n - 1)g_1 - n(n-2m)(h^{n-m} - 1)\}^2$$
$$= -(n-2m)\prod_{j=1}^{m}(h - \mu_j)^4 \prod_{i=1}^{2n-6m}(h - \gamma_i),$$

where $\mu_j = cos\frac{2j\pi}{m} + isin\frac{2j\pi}{m}, j = 0, 1, 2, ..., m-1$ and $\gamma_1, \gamma_2, ..., \gamma_{2n-6m}$ are the simple zeros of $\Psi(h)$.

It can easily be seen from the above equation that all the zeros of $h - \gamma_j$ have order at least 2. Since $\phi(z), \mathcal{L}(\triangle_\eta \phi)$ share $(0, \infty)$ and (∞, ∞), it follows that h omits the value 0 and ∞.

Therefore, applying the Second Fundamental Theorem to h, we have

$$(2n - 6m)T(r, h) \leq \sum_{j=1}^{2n-6m} \overline{N}(r, \gamma_j; h) + \overline{N}(r, 0; h) + \overline{N}(r, \infty; h) + S(r, h)$$
$$\leq \frac{1}{2}\sum_{j=1}^{2n-6m} \overline{N}(r, \gamma_j; h) + +S(r, h)$$
$$\leq (n - 3m)T(r, h) + S(r, h).$$

i.e.,

$$(n-3m)T(r,h) \leq S(r,h),$$

which is impossible.

So, h is constant. Hence, from (3.5), we have $h^n - 1 = 0, h^{n-m} - 1 = 0$ and $h^{n-2m} - 1 = 0$. Since $gcd(n,m) = 1$, we must have $h \equiv 1$.

i.e.,

$$\mathcal{L}(\triangle_\eta \phi) = \phi(z).$$

Subcase 2.2.2: Suppose $A = 0$ and $C \neq 0$.

Then (3.1) becomes

$$F \equiv \frac{1}{\sigma G + \tau},$$

where $\sigma = \frac{C}{B}$ and $\tau = \frac{D}{B}$.

If ϕ has no 1-point, the case can be treated in the same way as done in Subcase 2.2.1.

So let ϕ has some 1-point. Then $\sigma + \tau = 1$.

Now, σ cannot be equal to 1. For otherwise $FG \equiv 1$ which is not possible by Lemma 2.10.

Therefore,

$$F \equiv \frac{1}{\sigma G + 1 - \sigma}.$$

Since $C \neq 0, \sigma \neq 0, G$ omits the value $-\frac{1-\sigma}{\sigma}$.

By the Second Fundamental Theorem, we have

$$T(r,G) \leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,-\frac{1-\sigma}{\sigma};G) + S(r,G).$$

i.e.,

$$(n-2m-q-1)T(r,\mathcal{L}(\triangle_\eta \phi)) \leq S(r,\mathcal{L}(\triangle_\eta \phi)),$$

which is a contradiction. This completes the proof of the theorem.

Declarations

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of this paper.

Human/animals participants The authors declare that there is no research involving human participants and/or animals in the contained of this paper.

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