Proceedings of the Jangjeon Mathematical Society
28 (2025), No. 1, pp. 17 - 32

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2025.28.1.17

A STUDY ON VERTEX ADDITION STRATEGIES FOR PRESERVING GRAPH EIGENVALUES

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Abstract. In this paper, we investigate whether adding vertices to a graph can preserve all eigenvalues of the original graph. We demonstrate that it is impossible to add n or fewer vertices to a simple connected graph of order n while maintaining all eigenvalues. We also examine specific cases in which preserving $n-1$ of the original eigenvalues when adding vertices is feasible or not. Additionally, we present a method for adding vertices that preserves each eigenvalue, and we identify a way to add $n+1$ vertices while preserving all n eigenvalues.

1. Introduction

In this paper, given a set X, $M^{m\times n}(X)$ denote an $m\times n$ matrix consisting of elements from X. For simplicity, if $m = n$, we write this as $M^n(X)$. Similarly, X^n represents an n-dimensional vector composed of elements from X. Unless otherwise stated, i, j, m and n are positive integers.

Date: September 27, 2024.

2020 *Mathematics Subject Classification.* 05C50,15A18.

Key words and phrases. Graph, Adjacency matrix, Eigenvalues, Add Vertex.

A graph $G = (V, E)$ consists of a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and a set of edges $E \subseteq \{\{v_i, v_j\} | v_i, v_j \in V\}$. Throughout this paper, we consider only undirected graphs, where an edge $\{v_i, v_j\} = \{v_j, v_i\}$. The number of vertices, denoted by $|V|$, is referred to as *the order of the graph G*. The edge set E defines the connections between pairs of vertices, where the order of the two vertices is not considered. If there exists an edge between two vertices v_i and v_j, we say that v_i and v_j are adjacent. A graph is called a *simple graph* if it contains no loops - edges from a vertex to itself- and no multiple edges between the same pair of vertices. A *connected graph* is a graph where there exists a path between every pair of vertices. If a graph is both simple and connected, it is referred to as a simple connected graph. For a graph $G = (V, E)$, the *degree of a vertex* $v_i \in V$, denoted $\mathsf{deg}(v_i)$, is the number of edges incident to v_i. In a simple graph, this is equivalent to the number of vertices adjacent to v_i. We denote the maximum degree of graph G as $\Delta(G)$ and the minimum degree of graph G as $\delta(G)$. The set of all vertices adjacent to a vertex $v \in V$ is called the *neighborhood* of v, and it is denoted by $N(v)$. The adjacency matrix of a simple graph G with n vertices is an $n \times n$ matrix $A_G = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the adjacency matrix A_G is a symmetric matrix with diagonal entries of 0 and belongs to $M^n(\{0, 1\})$. The row of the adjacency matrix corresponding to vertex v will be referred to as the *v-row*. The eigenvalues of the adjacency matrix A_G are called the eigenvalues of the graph G. These eigenvalues provide critical information about the structural properties of the graph, such as its connectivity, stability, and expansion characteristics([4]). We represent the spectrum of matrix A, which includes all eigenvalues $\lambda_1, \ldots, \lambda_n$ counting multiplicities, as $\mathsf{spec}(A) = (\lambda_1, \ldots, \lambda_n)$. The spectrum of A $\mathsf{spec}(A)$ is a multiset that allows duplicate elements.

A lot of studies delves into the rich interplay between graph theory and linear algebra, with a focus on eigenvalues and the structural properties of graphs. Eigenvalues of the adjacency matrix play a central role in understanding the underlying geometry and symmetry of graphs. Several works explore different aspects of these connections. For instance, Bahmani and Kiani investigate the multiplicity of adjacency eigenvalues in graphs[1], while Bevis et al. examine the rank of a graph when a vertex is added [3]. In [6], Guo et al. look at the impact of edge addition on the eigenvalues of connected graphs. Distance-regular and strongly regular graphs are key structures in graph theory, often explored for their highly symmetric nature. In this regard, Belousov, Makhnev, and Nirova analyze distance-regular extensions of strongly regular graphs with an eigenvalue of 2 [2], a theme echoed by Kabanov et al. [7] and Zyulyarkina and Makhnev [9]. These extensions reveal deeper insights into the structural characteristics of such graphs. Brouwer and Haemers provide a comprehensive account of graph spectra, offering key knowledge that ties these investigations together [4]. Additionally, the effects of vertex deletion on the multiplicities of eigenvalues, explored by Simic et al. present both classic and novel results on this subject[8]. Moreover, in the realm of applications, Pyo [9] discusses how regular graphs can be applied in cryptography, particularly in the design of public key cryptosystems.

To the best of my knowledge, despite numerous studies, no research has yet been found on whether eigenvalues are preserved when a vertex is added. Therefore, this paper investigates whether the eigenvalues of the original graph are preserved or cannot be preserved when a vertex is added.

2. Preliminaries

We will denote by $\mathbf{0}$ a suitably sized vector consisting of all zero elements, and by O a suitably sized matrix consisting of all zero elements. A square matrix A is called *reducible* if there exist a permutation matrix P such that

$$P^T A P = \begin{pmatrix} C & O \\ D & E \end{pmatrix}$$

where C and E are non vacuous square matrices. A square matrix A is called *irreducible* if A is not reducible matrix. In other words, for any index pair i, j, there exists a sequence of indices such that the corresponding matrix entries form a non-zero path from row i to column j.

For a connected graph, the adjacency matrix is always irreducible because there exists a path between any two vertices. This can be formalized in the following proposition.

Proposition 2.1. *The adjacency matrix of graph G is irreducible if and only if G is connected.*

One might naturally consider whether the adjacency matrix of a connected graph is an invertible matrix, however, the following matrix (2.1) demonstrates that this is not always the case.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

The Perron-Frobenius theorem applies to non-negative matrices, such as the adjacency matrix of a graph. It describes the properties of the largest eigenvalue and the corresponding eigenvector. Let A be a square matrix with non-negative entries. The Perron-Frobenius theorem can be summarized as follows:

Theorem 2.2. (Perron-Frobenius [5]) Let A be a non-negative square matrix. Then the following statements hold:

(1) A has a real eigenvalue λ_{max}, known as the Perron eigenvalue, such that λ_{max} is the largest eigenvalue in absolute value.
(2) The Perron eigenvalue λ_{max} is non-negative.
(3) There exists a non-negative eigenvector $\mathbf{v}$ corresponding to λ_{max} , called the Perron eigenvector.
(4) If A is irreducible, λ_{max} is strictly positive, and the Perron eigenvector has only positive entries.

This theorem is particularly useful when studying the spectral properties of the adjacency matrix of a graph, as it guarantees the existence and uniqueness of a largest eigenvalue under the condition of irreducibility, which holds for connected graphs.

A complex entries square matrix H is called Hermitian if it is equal to its conjugate transpose, i.e., $H = H^*$, where H^* denotes the conjugate transpose of H. Hermitian matrices have several important properties, including the fact that all their eigenvalues are real.

An important result concerning Hermitian matrices is that the eigenvalues of a principal submatrix - a matrix obtained by deleting one or more rows and the corresponding columns- are interlaced with the eigenvalues of the original matrix. This is formalized in the following theorem:

Theorem 2.3. (Interlacing Eigenvalue Theorem [5]) Let A be a Hermitian matrix of size $n \times n$, and let B be a principal submatrix of A of size $(n-1)\times(n-1)$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A, and let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of B. Then, the eigenvalues of B interlace with those of A, meaning:

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

This result is useful when analyzing how modifications to a graph, such as adding or removing vertices, affect the eigenvalues of its adjacency matrix.

The following theorem is useful for understanding eigenvalue variations when adding a vertex to a graph or adding a row and column to a Hermitian matrix.

Theorem 2.4. (Cauchy [5]) Let B be an $n \times n$ Hermition, and $\mathbf{y}$ be an n dimensional vector and r be a real number, and let $A = \begin{pmatrix} B & \mathbf{y} \\ \mathbf{y}^* & rn \end{pmatrix}$. Then

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A)$$

in which $\lambda_i(A) = \lambda_i(B)$ if and only if there is an m -dimensional nonzero vector $\mathbf{v}$ such that $B\mathbf{v} = \lambda_i(B)\mathbf{v}$ and $\mathbf{y}^*\mathbf{v} = 0$; equality in the upper bound occurs for some i if and only if there is an m-dimensional nonzero vector $\mathbf{v}$ such that $B\mathbf{v} = \lambda_{i+1}(A)\mathbf{v}$ and $\mathbf{y}^*\mathbf{v} = 0$. If no eigenvectors of B are orthogonal to $\mathbf{y}$, then every inequality is strict inequality.

The following theorem is an extension of the previous one, applied to the case of expanding a submatrix from a vector extension.

Theorem 2.5. ([5])Let A be $n \times n$ Hermition matrix, partitioned as

$$A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}$$

where B and D are $m \times m$ and $(n-m) \times (n-m)$ square matrices respectively. Let eigenvalues of A and B be ordered as $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$ and $\lambda_1(B) \leq \lambda_2(B) \leq \cdots \leq \lambda_m(B)$ respectively. Then

$$\lambda_i(A) \leq \lambda_i(B) \leq \lambda_{i+n-m}(A), \ \ i = 1, \ldots, m$$

with equality in the lower bound for some i if and only if there is an m -dimensional nonzero vector $\mathbf{v}$ such that $B\mathbf{v} = \lambda_i(B)\mathbf{v}$ and $C^*\mathbf{v} = \mathbf{0}$; equality in the upper bound occurs for some i if and only if there is an m- dimensional nonzero vector $\mathbf{v}$ such that $B\mathbf{v} = \lambda_{i+n-m}(A)\mathbf{v}$ and $C^*\mathbf{v} = \mathbf{0}$.

If $i \in \{1, \ldots, m\}, 1 \leq r \leq i$, and

$$\lambda_{i-r+1}(A) = \cdots = \lambda_i(A) = \lambda_i(B)$$

then $\lambda_{i-r+1}(B) = \cdots = \lambda_i(B)$ and there are m-dimensional orthonormal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ such that $B\mathbf{v}_j = \lambda_i(B)\mathbf{v}_j$ and $C^*\mathbf{v}_j = \mathbf{0}$ for each $j = 1, \ldots, r$.

If $i \in \{1, \ldots, m\}, 1 \leq r \leq m - i + 1$, and

$$\lambda_i(B) = \lambda_{i+n-m}(A) = \cdots = \lambda_{i+n-m+r-1}(A)$$

then $\lambda_i(B) = \cdots = \lambda_{i+n-m+r-1}(B)$ and there are m-dimensional orthonormal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ such that $B\mathbf{v}_j = \lambda_i(B)\mathbf{v}_j$ and $C^*\mathbf{v}_j = \mathbf{0}$ for each $j = 1, \ldots, r$.

A simple graph is called a d-regular graph if the degree of every vertex is d. The following is well known for regular graph.

Proposition 2.6. *A simple connected graph G is d-regular graph if and only if the maximal degree $\Delta(G)$ of G is an eigenvalue of G.*

According to the well-known Gershgorin theorem([5]), we can see that $\Delta(G)$ is the largest eigenvalue. From Theorem 2.2, it follows that the eigenvector corresponding to $\Delta(G)$ consists entirely of positive entries. In fact, the eigenvector is the vector $\mathbf{J} = (1, 1, \cdots, 1)^T$ consisting entirely of ones. The above proposition can be replaced by the following proposition.

Proposition 2.7. *A simple connected graph is a d-regular graph if and only if it has* $\mathbf{J}$*, a vector consisting of all 1s, as an eigenvector.*

3. Fundamental Approach to Eigenvalue Variations with Graph Expansion

From now on, in this article, adding a vertex to a graph means that the added vertex has edges connecting it to the vertices of the original graph. Let $A_G = [a_{ij}]$ be the adjacency matrix of a simple connected graph G of order n, and let $\mathsf{spec}(A_G) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. We consider whether adding a vertex to graph G can result in a new spectrum that contains only 0 in addition to the original spectrum.

Proposition 3.1. *It is impossible to add a vertex to a connected simple graph such that only a zero is added to the original spectrum.*

Proof. Since matrix $A_G = [a_{ij}]$ is symmetric, we can see that the following holds,

$$\mathrm{tr}(A_G^2) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2 = \sum_{i=1}^{n} \lambda_i^2 \tag{3.1}$$

Let $B_{G'}$ be the adjacency matrix if the connected graph G' obtained by adding a vertex G. Then $B_{G'}$ can be represented as $B_{G'} = \begin{pmatrix} A_G & \mathbf{v} \\ \mathbf{v}^* & 0 \end{pmatrix}$ for some n-dimensional nonzero vector $\mathbf{v}$. According to Equation 3.1, we can obtain the result. ■

According to the previous Proposition 3.1, we have determined that it is impossible to maintain the original spectrum and add only 0 by adding a vertex. Adding a vertex to the graph to make it have an eigenvalue of 0 can be done easily.

Therefore, if it is acceptable to add any non-zero real number, we can investigate whether it is possible to maintain the original spectrum by adding a vertex. From Theorem 2.1 and Proposition 3.1, the adjacency matrix of connected graph has positive eigenvector $\mathbf{w}$ corresponding to maximal eigenvalue. Therefore, there is no $\mathbf{v} \in \{0, 1\}^n$ such that $\mathbf{v}^*\mathbf{w} = 0$, the following can be obtained.

Proposition 3.2. *It is not possible to add a vertex to a graph in such a way that the resulting graph maintains the original spectrum.*

The eigenvectors of matrix A are vectors in the column space of matrix A. Therefore, we can ask whether a column vector can be an eigenvector. But we have a negative results as follows.

Lemma 3.3. *A simple connected graph G can not have the columns of its adjacency matrix as eigenvector.*

Proof. Let $A_G = [a_{ij}]$ be the adjacency matrix of simple connected graph G. Suppose k-the column $\mathbf{a}_k = (a_{1k}, a_{2k}, \ldots, a_{nk})^T$ of A_G is an eigenvector of A_G. Since $a_{kk} = 0$, it is clear that $\mathbf{a}$ is not $\mathbf{J}$. Suppose the elements $a_{i_1k}, a_{i_2k}, \ldots, a_{i_mk}$ of

a are 1, and all the others are 0. Let $\alpha = \{i_1, i_2, \ldots, i_m\}$. Then $a_{ji_k} = 0$ for $j \notin \alpha, i_k \in \alpha$. By permutation similarity, we place all the rows and columns corresponding to α at the beginning. Then, the elements corresponding to the rows of α and the columns of α^c in A_G are all equal to 0. This means that A_{12} becomes a zero matrix in the following equation.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{J} \\ \mathbf{0} \end{bmatrix} \tag{3.2}$$

This is a contradiction to the irreducibility of A_G. ■

The following theorem is a direct consequence of Lemma 3.3.

Theorem 3.4. *A simple connected and non-regular graph G has no eigenvector in $\{0,1\}^n$.*

Even without using the above Lemma 3.3, it can be known Theorem 3.4. If $\mathbf{v} \in \{0,1\}^n$ is an eigenvector corresponding to the largest eigenvalue, it is a positive vector, and by the Proposition 2.7, the graph must be regular. If it is not an eigenvector corresponding to the largest eigenvalue, since the matrix is symmetric, the vector $\mathbf{v}$ must be orthogonal to the positive vector corresponding to the largest eigenvalue, which leads to a contradiction.

4. Extending Graphs with Spectral Properties Preservation

Let λ be an eigenvalue of a Hermition matrix A and $\mathbf{w} = (w_1, w_2, \cdots, w_n)^T$ be the eigenvector corresponding to λ. By adding the vector $(w_1, w_2, \cdots, w_n, 0)^T$ as the $(n+1)$-th row and column to matrix A, we create an $(n+1)\times(n+1)$

matrix, denoted as $A^{\mathbf{w}}$. Specifically, it is as follows;

$$A^{\mathbf{w}} = \begin{pmatrix} A & \mathbf{w} \\ \mathbf{w}^* & 0 \end{pmatrix}.$$

In this case, if μ, which is different from λ, is an eigenvalue of A, it becomes an eigenvalue of $A^{\mathbf{w}}$. This is because if $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T$ is the eigenvector corresponding to the eigenvalue μ of matrix A, then the vector $\mathbf{v}^{\{0\}} = (v_1, v_2, \cdots, v_n, 0)^T$ becomes the eigenvector corresponding to the eigenvalue μ of matrix $A^{\mathbf{w}}$. This is easily known because $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. This shows that in the case of an $n \times n$ Hermitian matrix, by adding one row, it is possible to preserve $n-1$ eigenvalues.

Let us examine this fact in the case of graphs. It is clear that a complete graph of order n, with all vertices connected, can maintain $n-1$ eigenvalues -1 by adding one vertex to become a complete graph of order $n+1$. This means adding a vertex of highest degree to a complete graph while preserving $n-1$ eigenvalues. Now we can examine whether it is possible to preserve $n-1$ eigenvalues by adding vertices of a minimum degree of 1. In other words, this means adding a vector $\mathbf{v}$ where only one element is 1 and the rest are 0 as the $(n+1)$-th row and column of the adjacency matrix A_G. If the vector $\mathbf{v}$ added to the graph retains $n-1$ eigenvalues, then $\mathbf{v}$ must be orthogonal to the $n-1$ eigenvectors, which implies that $\mathbf{v}$ itself must also be an eigenvector due to the properties of symmetric matrices. This contradicts the previous Theorem 3.4.

In fact, it is generally impossible to preserve $n-1$ eigenvalues, as shown in the following theorem. From the 3.4, we can get the following.

Theorem 4.1. *Let A_G be an adjacency matrix of non-regular graph of order n. Then there is no way to extending a vertex for preserving $n-1$ eigenvalues.*

Proof. Let $A_G^{\mathbf{v}}$ is the extended adjacency matrix such that

$$A_G^{\mathbf{v}} = \begin{bmatrix} A_G & \mathbf{v} \\ \mathbf{v}^T & 0 \end{bmatrix}.$$

Since A_G is an $n \times n$ symmetric matrix, there exist n orthogonal eigenvectors. From 2.2, $\mathbf{v}$ is orthogonal to $n-1$ eigenvectors of A_G. This leads to $\mathbf{v}$ being eigenvector of A_G. From 3.4, $\mathbf{v}$ is not $(0,1)$- vector. ■

The condition for a special eigenvalue to be preserved when adding a vertex to a simply connected graph is as follows.

Theorem 4.2. *Let G be a simple connected non regular graph G of order n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.*

(1) *Eigenvalue λ_1 cannot be preserved by a vertex is added.*
(2) *For $i > 1$, the necessary and sufficient condition for the eigenvalue λ_i to be preserved is the existence of a $\mathbf{v} \in \{0,1\}^n$ vector that is orthogonal to the eigenvector corresponding to the eigenvalue λ_i.*
(3) *If $\mathbf{v}$ is an eigenvector corresponding to μ with i-th element 0, then by adding i-th row of A_G to the $(n+1)$-th row and column, the extended adjacency matrix has an eigenvalue μ.*

Proof. (1) The eigenvector corresponding to λ_1 consists of positive entries, so it cannot be orthogonal to any $(0,1)$ vector. Therefore, by Proposition 3.1, it cannot be preserved.

(2) This naturally follows from Proposition 3.1.

(3) The fact that the i-th entry of the eigenvector $\mathbf{v}$ corresponding to eigenvalue λ is 0 means that the eigenvector $\mathbf{v}$ is orthogonal to the i-th row of the matrix. Therefore, we can obtain the result. ■

Theorem 4.3. *Let G be a simple connected graph of order n and let G' be a simple connected graph obtained by adding m vertices to G, $1 \leq m \leq n$. Then spec(G) can not contained spec(G').*

Proof. Let $n \times n$ matrix A be the adjacency matrix of G and A' be the adjacency matrix of G'. Then

$$A' = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \text{ where} D \in M_m$$

Since G' is connected graph, $B \neq O$. From 2.4, $\lambda(A) \in \text{spec}(A')$ if and only if equality holds. This means that $\lambda(A) \in \text{spec}(A')$ if and only if $B^T\mathbf{v} = \mathbf{o}$ for eigenvector $\mathbf{v}$ corresponding to $\lambda(A)$. From 2.1, we know that there exist positive eigenvector $\mathbf{w}$ corresponding maximum eigenvalue of A. Thus $B^T\mathbf{w} \neq \mathbf{o}$. This lead maximum eigenvalue of A can not contained in the spectrum of spec(A'). ■

Let G be a graph of order n, H a graph of order m, $\mathbf{v}$ an n-dimensional vector, $\mathbf{w}$ an m-dimensional vector, and let v be a vertex not included in either G or H. Let vertex v be connected to graph G via the vector $\mathbf{v}$, and let the vertex v be connected to graph H via the vector $\mathbf{w}$. We define this as the *v-connection of graph G and H*, and we will denote it as $(v, G, H, \mathbf{v}, \mathbf{w})$. For example, let A_G be the adjacency matrix of G and A_H be adjacency matrix of H, the adjacency matrix of G-connection of G and H is the follows

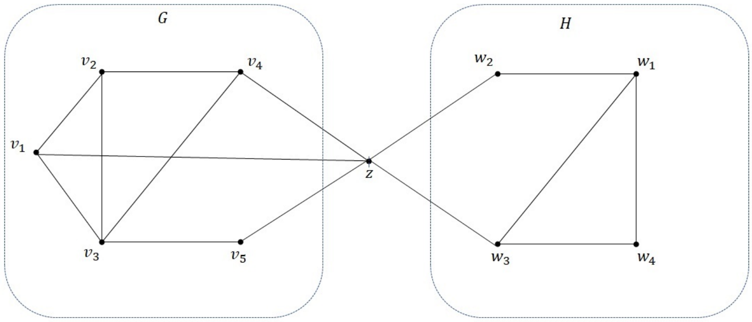

FIGURE 1. The graph of z-connection of G and H

$$A^v_{G,H} = \begin{pmatrix} A_G & \mathbf{v} & O \\ \mathbf{v}^T & 0 & \mathbf{w}^T \\ O^T & \mathbf{w} & A_H \end{pmatrix}.$$

The Figure 1. illustrates an example of such graph formation, and adjacency matrix is as follows;

$$A^z_{G,H} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Theorem 4.4. *The spectrum of the adjacency matrix $A^z_{G,G}$ of a v-connection of G and G, $(v, G, G, \mathbf{v}, \mathbf{w})$ contains all eigenvalues of G. Let $\mathbf{r}$ be a eigenvector of G*

corresponding to the eigenvalue λ*. If* $\mathbf{r}\cdot\mathbf{v}\neq 0$*, then* $n+1$ *dimensional vector* $(-\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{r}}\mathbf{v}, 0, \mathbf{v})$ *is an eigenvector of* $A^z_{G,G}$ *corresponding to the eigenvalue* λ*. If* $\mathbf{r}\cdot\mathbf{v}=0$*, then* $(n+1)$*-dimensional vector* $(\mathbf{v}, 0, \mathbf{o})$ *is an eigenvector of* $A^z_{G,G}$ *corresponding to the eigenvalue* λ*.*

Proof. Let $A^z_{G,G}$ be the adjacency matrix of v-connection $(v, G, G, \mathbf{v}, \mathbf{w})$. According to the matrix determinant expansion along the $(n+1)$-th row, it can be seen that the determinant of this matrix is always a multiple of $det(A^z_{G,G})$. This fact applies similarly to $A^z_{G,G}-\lambda I$, and thus, if λ is an eigenvalue of the graph G, we can see that it becomes an eigenvalue of $(v, G, G, \mathbf{v}, \mathbf{w})$ as well.

Let $\mathbf{r}$ be an eigenvector of the adjacency matrix A_G corresponding to the eigenvalue λ. Then the following equation shows that the eigenvector of $A^z_{G,G}$.

$$\begin{bmatrix} A_G & \mathbf{v} & O \\ \mathbf{v}^T & 0 & \mathbf{v}^T \\ O & \mathbf{w} & A_G \end{bmatrix} \begin{bmatrix} -\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{r}}\mathbf{r} \\ 0 \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} -\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{r}}A_G\mathbf{r} \\ -\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{r}}\mathbf{r}\cdot\mathbf{v}+\mathbf{r}\cdot\mathbf{w} \\ A_G\mathbf{r} \end{bmatrix} = \lambda \begin{bmatrix} -\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{r}}\mathbf{r} \\ 0 \\ \mathbf{r} \end{bmatrix}.$$

■

5. Conclusion

We studied the method of adding vertices to a graph while preserving the eigenvalues of the original graph. As a result, we showed that it is impossible to add up to n vertices to a graph of order n while preserving all n eigenvalues. Furthermore, except for special cases, it is also impossible to preserve $n-1$ eigenvalues. We also found results concerning whether each individual eigenvalue can be preserved. This is clearly related not only to the problem of graph extensions but also to the relationship with the eigenvalues of subgraphs.

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