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Gourava Index Adjacency Polynomial of Some Graphs

Pushpalatha Mahalank[a,1], **A. R. Desai**[b], **H. S. Ramane**[c], **Ismail Naci Cangul**[d]

[a.b]Department of Mathematics,
SDM College of Engineering and Technology, Dharwad - 580002, India
Email: mahalankpushpalatha@gmail.com, ardesai69@rediffmail.com

[c]Department of Mathematics, Karnatak University, Dharwad - 580003, India
Email: hsramane@yahoo.com

[d]Department of Mathematics,
Bursa Uludag University, Gorukle, 16059 Bursa, TURKEY
Email: cangul@uludag.edu.tr

Abstract

The Gourava index adjacency polynomial of a graph G is the characteristic polynomial of the Gourava index adjacency matrix $GIA(G)$ whose (i, j)-th entry is $d(v_i) + d(v_j) + d(v_i)d(v_j)$, whenever the vertex v_i is adjacent to v_j, otherwise it is zero, where d_i is the degree of the vertex v_i. In this article, we obtain the Gourava index adjacency polynomial of graphs obtained from regular graphs.

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1 Introduction

The concept of energy in graph theory arose from chemistry. Detailed information on graph energy can be found in [1] [2] and survey [3].

Throughout this article, let $G = (V, E)$ be a simple, finite and connected graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. For undefined terminologies, we refer [7].

The adjacency matrix of a graph G is a square matrix and is defined as $A(G) = [a_{ij}]$ of order n, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

[1]corresponding author

The characteristic polynomial of G is called as the adjacency polynomial and is defined by

$$\phi(G:\lambda) = det(\lambda I - A(G)),$$

where I is the identity matrix.

The incidency matrix of G is an $n \times m$ matrix $B(G) = [b_{ij}]$ where

$$b_{ij} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident to the edge } e_j \\ 0 & \text{otherwise.} \end{cases}$$

If $x_1, x_2, \ldots, x_n$ are the eigenvalues of $A(G)$, then the energy of a graph G is denoted by $E(G)$ and is defined as the sum of the absolute values of the eigenvalues of $A(G)$, [1]. That is

$$E(G) = \sum_{i=1}^{n} |x_i|.$$

A topological graph index is a mathematical formula that can be applied to any graph which models some molecular structure and it is also known as a molecular descriptor. From such an index, it is possible to analyse the obtained mathematical values and further investigate some physicochemical properties of a molecule.

In chemical graph theory, topological indices have several applications in QSAR/QSPR investigations, pharmaceutical drug design, isomer discrimination and many more. There are a few important classes of topological indices that are extensively studied by a number of researchers. Out of these topological indices, Zagreb indices have many applications in QSPR and QSAR studies.

The first and second Zagreb indices of a graph G were introduced by Gutman and Trinajestić in 1972, [4]. These two indices are denoted by $M_1(G)$ and $M_2(G)$, respectively, and defined as

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

After the great success obtained by these two indices having many applications, many mathematicians and chemists have defined and studied several versions of these indices. As an example, motivated by the definitions of Zagreb indices and their applications, in 2017 V. R. Kulli introduced the first and second

Gourava indices of a molecular graph G [5]. These two indices of a graph G are defined by

$$GO_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v) + d_G(u)d_G(v)]$$

and

$$GO_2(G) = \sum_{uv\in E(G)} [(d_G(u) + d_G(v))(d_G(u)d_G(v))].$$

Gourava index adjacency (GIA) matrix of a graph G is $GIA(G) = [g_{ij}]$ where

$$g_{ij} = \begin{cases} d(v_i) + d(v_j) + d(v_i)d(v_j) & \text{if } v_i \text{ is incident to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $GIA(G)$ is called Gourava index polynomial and is defined as

$$\phi(G : \lambda) = det(\lambda I - GIA(G)).$$

Characteristic polynomial of the adjacency matrix is the most studied graph polynomial. For this reason, we obtain the Gourava index adjacency polynomial of graphs obtained from regular graphs in this article. We are going to make use of below lemmas:

Lemma 1.1. *[6] If G is a regular graph of degree r and $L(G)$ is the line graph then*
(i) *$B(G)B(G)^T = A(G) + rI$ and*
(ii) *$B(G)^T B(G) = 2I + A(L(G))$.*

Lemma 1.2. *[6] If M and/or Q is a non-singular matrix, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N| = |Q||M - NQ^{-1}P|.$$

Theorem 1.3. *If G is an r-regular graph, then*

$$\psi(G : \lambda) = (2r + r^2)^n \ \phi\left(G : \frac{\lambda}{2r + r^2}\right).$$

Proof. We have $GIA(G) = (2r + r^2)A(G)$. Hence

$$\begin{aligned} \psi(G : \lambda) &= det(\lambda I - GIA(G)) \\ &= |\ \lambda I - (2r + r^2)A(G)\ | \\ &= (2r + r^2)^n \left| \frac{\lambda}{2r + r^2} I - A(G) \right| \\ &= (2r + r^2)^n \phi\left(G : \frac{\lambda}{2r + r^2}\right). \end{aligned}$$

□

Definition 1. *The subdivision graph of G is obtained by inserting a new vertex on each edge of G [7] and it is denoted by $S(G)$. If $u \in V(G)$ then $deg_{S(G)}(u) = deg_G(u)$ and if v is a vertex added by subdivision, then $d_{S(G)}(v) = 2$.*

Theorem 1.4. *If G is an r-regular graph on n vertices and m edges. Then*

$$\psi(S(G):\lambda) = (2+3r)^{2n}\lambda^{m-n}\phi\left(G : \frac{\lambda^2 \ - r(2+3r)^2}{(2+3r)^2}\right).$$

Proof. We have

$$GIA(S(G)) = \begin{bmatrix} O & (2+3r)B(G)^T \\ (2+3r)B(G) & O \end{bmatrix},$$

where O is the zero matrix. Therefore

$$\psi(S(G):\lambda) = \begin{vmatrix} \lambda I_m & -(2+3r)B(G)^T \\ -(2+3r)B(G) & \lambda I_n \end{vmatrix}.$$

Using Lemma 1.2, we get

$$\begin{aligned} \psi(S(G):\lambda) &= \lambda^m \left|\lambda \ I_n - (2+3r)^2 \ \frac{BI_mB^T}{\lambda}\right| \\ &= \lambda^{m-n} \mid \lambda^2 \ I_n - (2+3r)^2 \ (A(G)+rI) \mid \\ &= \lambda^{m-n} \mid (\lambda^2 - (r(2+3r)^2)I - (2+3r)^2 \ A(G) \mid \\ &= \lambda^{m-n} \ (2+3r)^{2n} \ \left|\frac{\lambda^2 \ - \ r(2+3r)^2}{(2+3r)^2} \ I - A(G)\right| \\ &= (2+3r)^{2n}\lambda^{m-n}\phi\left(G : \frac{\lambda^2 \ - r(2+3r)^2}{(2+3r)^2}\right). \end{aligned}$$

□

Definition 2. *The semitotal point graph of G denoted by $T_1(G)$ is a graph with vertex set $V(G) \cup E(G)$ so that two vertices in $T_1(G)$ are adjacent if they are both adjacent vertices or one is a vertex and other is an incident edge to it. If $u \in V(G)$, then $deg_{T_1(G)}(u) = 2deg_G(u)$ and if $e \in E(G)$, then $deg_{T_1(G)}(e) = 2$.*

Theorem 1.5. *Let G be an r-regular graph. Then*

$$\psi(T_1(G)\colon \lambda) = \lambda^{m-n}[(4r+4r^2)\lambda + (2+6r)^2]^n \ \phi\left(G\colon \frac{\lambda^2 \ - r(2+6r)^2}{(4r+4r^2)\lambda + \ (2+6r)^2}\right).$$

Proof. We have

$$GIA(T_1(G)) = \begin{bmatrix} O & (2+6r)B(G)^T \\ (2+6r)B(G) & (4r+4r^2)A(G) \end{bmatrix}$$

where O is the zero matrix. Therefore

$$\psi(T_1(G)\colon \lambda) = \begin{vmatrix} \lambda I_m & -(2+6r)B^T \\ -(2+6r)B & \lambda I_n - (4r+4r^2)A \end{vmatrix}.$$

Using Lemma 1.2

$$\begin{aligned}
\psi(T_1(G)\colon \lambda) &= \lambda^m \left| \lambda\ I_n - (4r+4r^2)A(G) - (2+6r^2)\frac{B(G)B(G)^T}{\lambda} \right| \\
&= \lambda^{m-n} \mid \lambda^2\ I_n - (4r+4r^2)\lambda\ A(G) - (2+6r^2)\ (A(G)+rI) \mid \\
&= \lambda^{m-n} \mid \lambda^2 - r(2+6r^2)I - ((4r+4r^2)\lambda) + (2+6r^2)A(G) \mid \\
&= \lambda^{m-n}[(4r+4r^2)\lambda + (2+6r)^2]^n \left| \frac{\lambda^2 - r(2+6r)^2}{(4r+4r^2)\lambda + (2+6r)^2} I - A(G) \right| \\
&= \lambda^{m-n}[(4r+4r^2)\lambda + (2+6r)^2]^n\ \phi\left(G\colon \frac{\lambda^2\ - r(2+6r)^2}{(4r+4r^2)\lambda +\ (2+6r)^2} \right).
\end{aligned}$$

□

Definition 3. *The semitotal line graph of G is $T_2(G)$ with vertex set $V(G) \cup E(G)$ and two vertices in $T_2(G)$ are adjacent if they are adjacent edges in G or one is a vertex and other is an edge incident to it in G.*

If $e = uv \in E(G)$, then

$$deg_{T_2(G)}(e) = deg_G(u) + deg_G(v)$$

and if $u \in V(G)$, then

$$deg_{T_2(G)}(u) = deg_G(u).$$

Theorem 1.6. *Let G be an r-regular graph of order n and size m, then*

$$\psi(T_2(G)\colon \lambda) = \lambda^{n-m}\ [(4r+4r^2)\lambda+(3r+2r^2)^2]^m\ \phi\left(L(G)\colon \frac{\lambda^2\ - 2(3r+2r^2)^2}{(4r+4r^2)\lambda +\ (3r+2r^2)^2} \right),$$

where $L(G)$ is the line graph of G.

Proof. We have

$$GIA(T_2(G)) = \begin{bmatrix} (4r+4r^2)A(L(G)) & (3r+2r^2)B(G)^T \\ (3r+2r^2)B(G) & O \end{bmatrix},$$

where $A(L(G))$ is the adjacency matrix of the line graph of G and O is the zero matrix. Therefore

$$\psi(T_2(G)\colon \lambda) = \begin{vmatrix} \lambda I_m - (4r+4r^2)A(L(G)) & -(3r+2r^2)B(G)^T \\ -(3r+2r^2)B(G) & \lambda I_n \end{vmatrix}.$$

Using Lemma 1.2, we have

$$\begin{aligned}
\psi(T_2(G)\colon\lambda) &= \lambda^n \left|\lambda I_m - (4r+4r^2)A(L(G)) - (3r+2r^2)^2\frac{B(G)^T\ B(G)}{\lambda}\right| \\
&= \lambda^{n-m}\left|\lambda^2\ I_m - (4r+4r^2)\lambda\ A(L(G)) - (3r+2r^2)^2\ (2I + A(L(G)))\right| \\
&= \lambda^{n-m}\ |\ [(\lambda^2 - 2(3r+2r^2)^2]\ I_m - ((4r+4r^2)\ \lambda + (3r+2r^2)^2)A(L(G))\ | \\
&= \lambda^{n-m}\ [(4r+4r^2)\ \lambda + (3r+2r^2)^2]^m\ \left|\frac{\lambda^2\ - 2(3r+2r^2)^2}{(4r+4r^2)\ \lambda +\ (3r+2r^2)^2}\ I\ - A(L(G))\right| \\
&= \lambda^{n-m}\ [(4r+4r^2)\lambda + (3r+2r^2)^2]^m\ \phi\left(L(G)\colon \frac{\lambda^2\ - 2(3r+2r^2)^2}{(4r+4r^2)\lambda +\ (3r+2r^2)^2}\right).
\end{aligned}$$

□

Definition 4. *The total graph of* G *is* $T(G)$ *with vertex set* $V(G)\cup E(G)$ *and two vertices in* $T(G)$ *are adjacent if the corresponding elements are adjacent or incident in* G*. Here the vertices and edges of a graph are referred as its elements, [7].*

If $u \in V(G)$, then $deg_{T(G)}(u) = 2\ deg_G(u)$ and if $e = uv \in E(G)$, then $deg_{T(G)}(e) = deg_G(u) + deg_G(v)$.

Theorem 1.7. *If* G *is a regular graph of order* r *on* n *vertices and* m *edges, then*

$$\psi(T(G)\colon\lambda) = (4r{+}4r^2)^{m+n}(x{+}2)^{m-n}\prod_{i=1}^{n}\left[x^2 - (2\lambda_i + r - 2)x + \lambda_i^2 + (r-3)\lambda_i - r\right],$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of* $A(G)$ *and* $x = \dfrac{\lambda}{(4r+4r^2)}$.

Proof. We have

$$GIA(T(G)) = \begin{bmatrix} (4r+4r^2)A(G) & (4r+4r^2)B(G) \\ (4r+4r^2)B(G)^T & (4r+4r^2)A(L(G)) \end{bmatrix}.$$

Therefore

$$\begin{aligned}
\psi(T(G)\colon\lambda) &= \begin{vmatrix} \lambda I_n - (4r+4r^2)A(G) & -(4r+4r^2)B(G) \\ -(4r+4r^2)B(G)^T & \lambda I_m - (4r+4r^2)A(L(G)) \end{vmatrix} \\
&= \left((4r+4r^2\right)^{m+n} \begin{vmatrix} \frac{\lambda}{(4r+4r^2)} I_n - A(G) & -B(G) \\ -B(G)^T & \frac{\lambda}{4r+4r^2} I_m - A(L(G)) \end{vmatrix} \\
&= (4r+4r^2)^{m+n}(x+2)^{m-n}\cdot \\
&\quad \prod_{i=1}^{n}\left[x^2 - (2\lambda_i + r - 2)x + \lambda_i^2 + (r-3)\lambda_i - r\right].
\end{aligned}$$

□

Conclusion

In this article, we obtain the Gourava index adjacency polynomial of some graphs obtained from regular graphs. The similar work can be done for other topological indices.

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