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A PROBABILISTIC PROOF OF A RECURRENCE RELATION FOR SUMS OF VALUES OF DEGENERATE FALLING FACTORIALS

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ABSTRACT. In this paper, we consider sums of values of degenerate falling factorials and give a probabilistic proof of a recurrence relation for them. This may be viewed as a degenerate version of the recent probabilistic proofs on sums of powers of integers.

1. INTRODUCTION

Jacob Bernoulli considered sums of powers of the first n positive integers, $1^k + 2^k + \cdots + n^k$, which have been a topic of research for centuries. We note that

$$\begin{aligned}
&1+2+3+\cdots+n = \binom{n+1}{2},\\
&1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6},\\
&1^3+2^3+3^3+\cdots+n^3 = \big(1+2+3+\cdots+n\big)^2, \quad (\text{see } [6,7,9-11]).
\end{aligned}$$

The Bernoulli polynomials are defined by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad (\text{see } [1-20]). \tag{1}$$

When $x=0,\ B_n = B_n(0)$ are called the Bernoulli numbers. By (1), we get

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}, \quad (n \geq 0), \quad (\text{see } [10,12]). \tag{2}$$

From (1), we note that

$$\sum_{k=0}^{n-1}(k+x)^m = \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}B_l(x)n^{m+1-l}, \tag{3}$$

where $m,n \in \mathbb{N}$.

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Denoting the sum $1^m + 2^m + \cdots + n^m$ by $S_m(n)$, by (3), we get

$$S_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{l}B_l(n+1)^{m+1-l} = \int_0^{n+1} B_m(u)du, \tag{4}$$

$$S_m(n) = \frac{(n+1)^{m+1}-1}{m+1} - \frac{1}{m+1}\sum_{r=0}^{m-1}\binom{m+1}{r}S_r(n), \tag{5}$$

$$= \frac{n^{m+1}}{m+1} + \sum_{r=0}^{m-1}\binom{m}{r}\frac{(-1)^{m-r+1}}{m-r+1}S_r(n), \tag{6}$$

where m is a positive integer (see [6,7,10]).

For any $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad (\text{see } [11-15]), \tag{7}$$

where the degenerate falling factorials are given by

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1).$$

In particular, for $x = 1$, we denote them by $e_\lambda(t) = e_\lambda^1(t)$.

The degenerate Bernoulli polynomials are defined by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see } [4,16,20]). \tag{8}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

The degenerate Stirling numbers of the second kind are defined by Kim-Kim as

$$(x)_{n,\lambda} = \sum_{k=0}^{n}\left\{ {n \atop k} \right\}_\lambda (x)_k, \quad (n \geq 0), \quad (\text{see } [8]), \tag{9}$$

where the falling factorials are given by

$$(x)_0 = 1,\ (x)_n = x(x-1)(x-2)\cdots(x-n+1),\ (n \geq 1).$$

In this paper, we study sums of values of degenerate falling factorials which are given by

$$S_{k,\lambda}(n) = (1)_{k,\lambda} + (2)_{k,\lambda} + \cdots + (n)_{k,\lambda} = \sum_{j=1}^{n}(j)_{k,\lambda}, \quad (k \in \mathbb{N}). \tag{10}$$

In Section 1, we recall the necessary facts that are needed throughout this paper. After recalling two expressions of $S_{k,\lambda}(n)$, we derive two recurrence relations for them in Section 2. Let X be a nonnegative integer-valued random variable, and let k be a positive integer. Then, in Section 3, we show first that the k-th degenerate moment of X is given by $E\left[(X)_{k,\lambda}\right] = \sum_{x=0}^{\infty}\left((x+1)_{k,\lambda} - (x)_{k,\lambda}\right)P\{X > x\}$. Then we apply this to the uniform random variable X supported on $\{0,1,2,\ldots,n\}$ to derive a recurrence relation for $S_{k,\lambda}(n)$.

2. Some formulas for sums of values of degenerate falling factorials

From (8), we note that

$$\sum_{m=0}^{\infty}\sum_{k=0}^{n-1}(k+x)_{m,\lambda}\frac{t^m}{m!} = \sum_{m=0}^{\infty}\frac{1}{m+1}\Big(\beta_{m+1,\lambda}(n+x) - \beta_{m+1,\lambda}(x)\Big)\frac{t^m}{m!}. \tag{11}$$

Thus, by (11), we get

$$\sum_{k=0}^{n-1}(k+x)_{m,\lambda} = \frac{1}{m+1}\Big(\beta_{m+1,\lambda}(n+x)-\beta_{m+1,\lambda}(x)\Big) = \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}\beta_{l,\lambda}(x)(n)_{m+1-l,\lambda}, \tag{12}$$

where m is nonnegative integer.
From (12), for $m,n\in\mathbb{N}$ and $x=0$, we have the next proposition. This is also obtained in [14, Lemma 7].

Proposition 2.1. *For $m,n\in\mathbb{N}$, we have*

$$S_{m,\lambda}(n-1)=\frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}(n)_{m+1-l,\lambda}\beta_{l,\lambda}.$$

As it is done in [14, Theorem 8], by using (9) and (10), we get the following theorem.

Theorem 2.2. *For $k\in\mathbb{N}$, we have*

$$S_{k,\lambda}(n)=\sum_{l=1}^{k}\left\{\begin{matrix}k\\ l\end{matrix}\right\}_{\lambda}\binom{n+1}{l+1}l!.$$

Now, we observe that

$$(x+y)_{n,\lambda}=\sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}(y)_{n-k,\lambda},\quad (n\geq 0). \tag{13}$$

From (10), we note that, for $k\in\mathbb{N}$,

$$\begin{aligned}\sum_{r=0}^{k}\binom{k+1}{r}(1)_{k+1-r,\lambda}S_{r,\lambda}(n) &= \sum_{j=1}^{n}\sum_{r=0}^{k}\binom{k+1}{r}(1)_{k+1-r,\lambda}(j)_{r,\lambda}\\ &= \sum_{j=1}^{n}\Big(\sum_{r=0}^{k+1}\binom{k+1}{r}(1)_{k+1-r,\lambda}(j)_{r,\lambda}-(j)_{k+1,\lambda}\Big)\\ &= \sum_{j=1}^{n}\Big((j+1)_{k+1,\lambda}-(j)_{k+1,\lambda}\Big)\\ &= (n+1)_{k+1,\lambda}-(1)_{k+1,\lambda}.\end{aligned} \tag{14}$$

By (14), we get

$$\begin{aligned}(n+1)_{k+1,\lambda}-(1)_{k+1,\lambda} &= \sum_{r=0}^{k}\binom{k+1}{r}(1)_{k+1-r,\lambda}S_{r,\lambda}(n)\\ &= \sum_{r=0}^{k-1}\binom{k+1}{r}(1)_{k+1-r,\lambda}S_{r,\lambda}(n)+(k+1)S_{k,\lambda}(n).\end{aligned} \tag{15}$$

Thus, by (15), we obtain the following theorem.

Theorem 2.3. *For $k\in\mathbb{N}$, we have the recurrence relation*

$$S_{k,\lambda}(n)=\frac{(n+1)_{k+1,\lambda}-(1)_{k+1,\lambda}}{k+1}-\frac{1}{k+1}\sum_{r=0}^{k-1}\binom{k+1}{r}(1)_{k+1-r,\lambda}S_{r,\lambda}(n).$$

We note that

$$(n)_{k+1,\lambda} = \sum_{j=1}^{n} \Big((j)_{k+1,\lambda} - (j-1)_{k+1,\lambda} \Big), \quad (k \in \mathbb{N}). \tag{16}$$

From (13), we note that

(17)

$$\begin{aligned}
(j)_{k+1,\lambda} - (j-1)_{k+1,\lambda} &= (j)_{k+1,\lambda} - \sum_{r=0}^{k+1} \binom{k+1}{r} (-1)_{k+1-r,\lambda} (j)_{r,\lambda} \\
&= (j)_{k+1,\lambda} - \sum_{r=0}^{k+1} \binom{k+1}{r} (-1)^{k+1-r} \langle 1 \rangle_{k+1-r,\lambda} (j)_{r,\lambda} \\
&= (j)_{k+1,\lambda} - (j)_{k+1,\lambda} + (k+1)(j)_{k,\lambda} - \sum_{r=0}^{k-1} \binom{k+1}{r} (-1)^{k+1-r} \langle 1 \rangle_{k+1-r,\lambda} (j)_{r,\lambda} \\
&= (k+1)(j)_{k,\lambda} - \sum_{r=0}^{k-1} \binom{k+1}{r} (-1)^{k+1-r} \langle 1 \rangle_{k+1-r,\lambda} (j)_{r,\lambda}, \quad (k \in \mathbb{N}),
\end{aligned}$$

where the degenerate rising factorials are given by

$$\langle x \rangle_{0,\lambda} = 1, \ \langle x \rangle_{k,\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(k-1)\lambda), \ (k \geq 1).$$

Thus, by (16) and (17), we get

$$\begin{aligned}
(n)_{k+1,\lambda} &= \sum_{j=1}^{n} \Big((j)_{k+1,\lambda} - (j-1)_{k+1,\lambda} \Big) \\
&= (k+1) \sum_{j=1}^{n} (j)_{k,\lambda} - \sum_{r=0}^{k-1} \binom{k+1}{r} (-1)^{k+1-r} \langle 1 \rangle_{k+1-r,\lambda} \sum_{j=1}^{n} (j)_{r,\lambda} \\
&= (k+1) S_{k,\lambda}(n) - \sum_{r=0}^{k-1} \binom{k+1}{r} (-1)^{k+1-r} \langle 1 \rangle_{k+1-r,\lambda} S_{r,\lambda}(n).
\end{aligned} \tag{18}$$

From (18), we obtain the following theorem.

Theorem 2.4. *For $k \in \mathbb{N}$, we have*

$$S_{k,\lambda}(n) = \frac{(n)_{k+1,\lambda}}{k+1} + \frac{1}{k+1} \sum_{r=0}^{k-1} \binom{k+1}{r} (-1)^{k+1-r} \langle 1 \rangle_{k+1-r,\lambda} S_{r,\lambda}(n).$$

3. A PROBABILISTIC PROOF OF A RECURRENCE RELATION FOR $S_{k,\lambda}(n)$

Recently, probabilistic methods are used in deriving recurrence formulas for sums of powers of integers, (see [6,7]). In this section, we give a probabilistic proof of a recurrence relation for sums of values of degenerate falling factorials.

Let X be a nonnegative integer-valued random variable, and let k be any positive integer. Then we note that

(19)
$$
\begin{aligned}
&\sum_{x=0}^{\infty}\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big)P\{X>x\}\\
&=(1)_{k,\lambda}P\{X>0\}+\Big((2)_{k,\lambda}-(1)_{k,\lambda}\Big)P\{X>1\}+\Big((3)_{k,\lambda}-(2)_{k,\lambda}\Big)P\{X>2\}\\
&\quad+\Big((4)_{k,\lambda}-(3)_{k,\lambda}\Big)P\{X>3\}+\cdots\\
&=(1)_{k,\lambda}P\{X=1\}+(1)_{k,\lambda}P\{X>1\}-(1)_{k,\lambda}P\{X>1\}+(2)_{k,\lambda}P\{X=2\}\\
&\quad+(2)_{k,\lambda}P\{X>2\}-(2)_{k,\lambda}P\{X>2\}+(3)_{k,\lambda}P\{X=3\}-(3)_{k,\lambda}P\{X>3\}\\
&=(1)_{k,\lambda}P\{X=1\}+(2)_{k,\lambda}P\{X=2\}+(3)_{k,\lambda}P\{X=3\}+\cdots\\
&=\sum_{x=0}^{\infty}(x)_{k,\lambda}P\{X=x\}=E\Big[(X)_{k,\lambda}\Big].
\end{aligned}
$$

Therefore, by (19), we obtain the following theorem.

Theorem 3.1. *Let X be a nonnegative integer-valued random variable. For $k\in\mathbb{N}$, the k-th degenerate moment of X is given by*

$$E\Big[(X)_{k,\lambda}\Big]=\sum_{x=0}^{\infty}\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big)P\{X>x\}.$$

Assume that X has support in $\{0,1,2,\dots,n\}$. Then we have

$$
\begin{aligned}
\sum_{x=0}^{n}(x)_{k,\lambda}P\{X=x\}&=\sum_{x=0}^{n}\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big)P\{X>x\}\\
&=\sum_{x=0}^{n}\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big)\sum_{y=x+1}^{n}P\{X=y\},
\end{aligned}
\tag{20}
$$

where k is a positive integer.

Now, let X be the uniform random variable supported on $\{0,1,2,\dots,n\}$, that is, $P\{X=x\}=\frac{1}{n+1}$, for $x\in\{0,1,2,\dots,n\}$. Then we note that

$$\sum_{y=x+1}^{n}P\{X=y\}=\frac{n-x}{n+1}.$$

From (20), we note that

$$
\begin{aligned}
\sum_{x=0}^{n}(x)_{k,\lambda}\frac{1}{n+1}&=\sum_{x=0}^{n}(x)_{k,\lambda}P\{X=x\}\\
&=\sum_{x=0}^{n}\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big)\sum_{y=x+1}^{n}P\{X=y\}\\
&=\sum_{x=0}^{n}\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big)\frac{n-x}{n+1},\quad(k\in\mathbb{N}).
\end{aligned}
\tag{21}
$$

By (21), we get

$$
\begin{aligned}
S_{k,\lambda}(n) &= \sum_{x=0}^{n}(x)_{k,\lambda} = \sum_{x=0}^{n}\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big)(n-x) \\
&= n\sum_{x=0}^{n}\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big) - \sum_{x=0}^{n}x\Big((x+1)_{k,\lambda}-(x)_{k,\lambda}\Big) \\
&= n(n+1)_{k,\lambda} - \sum_{x=0}^{n}x\left(\sum_{r=0}^{k}\binom{k}{r}(x)_{r,\lambda}(1)_{k-r,\lambda}-(x)_{k,\lambda}\right) \\
&= n(n+1)_{k,\lambda} - \sum_{x=0}^{n}\sum_{r=0}^{k-1}\binom{k}{r}(x-r\lambda+r\lambda)(x)_{r,\lambda}(1)_{k-r,\lambda} \\
&= n(n+1)_{k,\lambda} - \sum_{x=0}^{n}\sum_{r=0}^{k-1}\binom{k}{r}(x)_{r+1,\lambda}(1)_{k-r,\lambda} \\
&\qquad - \lambda\sum_{x=0}^{n}\sum_{r=0}^{k-1}\binom{k}{r}r(x)_{r,\lambda}(1)_{k-r,\lambda} \\
&= n(n+1)_{k,\lambda} - \sum_{x=0}^{n}\sum_{r=0}^{k-2}\binom{k}{r}(x)_{r+1,\lambda}(1)_{k-r,\lambda} - k\sum_{x=0}^{n}(x)_{k,\lambda} \\
&\qquad - \lambda\sum_{x=0}^{n}\sum_{r=0}^{k-1}r\binom{k}{r}(x)_{r,\lambda}(1)_{k-r,\lambda} \\
&= n(n+1)_{k,\lambda} - \sum_{r=0}^{k-2}(1)_{k-r,\lambda}\binom{k}{r}S_{r+1,\lambda}(n) - kS_{k,\lambda}(n) \\
&\qquad - \lambda\sum_{r=1}^{k-1}r\binom{k}{r}(1)_{k-r,\lambda}S_{r,\lambda}(n) \\
&= n(n+1)_{k,\lambda} - \sum_{r=1}^{k-1}(1)_{k+1-r,\lambda}\binom{k}{r-1}S_{r,\lambda}(n) - kS_{k,\lambda}(n) \\
&\qquad - \lambda\sum_{r=1}^{k-1}r\binom{k}{r}(1)_{k-r,\lambda}S_{r,\lambda}(n),
\end{aligned}
\tag{22}
$$

where k is a positive integer.
By (22), we obtain the following theorem.

Theorem 3.2. *For $k\in\mathbb{N}$, we have*

$$
\begin{aligned}
S_{k,\lambda}(n) = \frac{n(n+1)_{k,\lambda}}{k+1} &- \frac{1}{k+1}\sum_{r=1}^{k-1}(1)_{k+1-r,\lambda}\binom{k}{r-1}S_{r,\lambda}(n) \\
&- \frac{\lambda}{k+1}\sum_{r=1}^{k-1}r\binom{k}{r}(1)_{k-r,\lambda}S_{r,\lambda}(n).
\end{aligned}
$$

4. Conclusion

In this paper, we derived three recurrence relations for the sums of values of degenerate falling factorials $S_{k,\lambda}(n)=(1)_{k,\lambda}+(2)_{k,\lambda}+\cdots+(n)_{k,\lambda}, \quad (k\in\mathbb{N})$. They are Theorems 2.3, 2.4 and 3.2. If we let $\lambda\to 0$, then Theorem 2.3 and Theorem 2.4

boil down to (5) and (6), respectively. In addition, we obtain another recurrence relation for $S_k(n)$ by letting $\lambda \to 0$. Namely, we get

$$S_k(n) = \frac{n(n+1)^k}{k+1} - \frac{1}{k+1}\sum_{r=1}^{k-1}\binom{k}{r-1}S_r(n).$$

REFERENCES

[1] Aydin, M. S.; Acikgoz, M.; Araci, S. *A new construction on the degenerate Hurwitz-zeta function associated with certain applications,* Proc. Jangjeon Math. Soc. **25** (2022), no. 2, 195-203.

[2] Barman, K.; Chakraborty, B.; Morthini, R. *Two Classical formulas for the sums of powers of consecutive integers via complex analysis,* Complex Anal. Synerg. **2024** (2024), Article no. 10:5.

[3] Beardon, A. F. *Sums of powers of integers,* Amer. Math. Monthly **103** (1996), no. 3, 201-213.

[4] Carlitz, L. *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51-88.

[5] Dolgy, D. V.; Kim, D. S; Kim, H. K.; Kim, T. *Degenerate harmonic and hyperharmonic numbers,* Proc. Jangjeon Math. Soc. **26** (2023), no. 3, 259–268.

[6] Farhadian, R. *A probabilistic proof of a recursion formula for sums of integer powers,* Integers **23** (2023), Paper No. A88.

[7] Hu, X.; Zhong, Y. *A probabilistic proof of a recursion formula for sums of powers,* Amer. Math. Monthly **127** (2020), no. 2, 166-168.

[8] Kim, D. S.; Kim, T. *A note on a new type of degenerate Bernoulli numbers,* Russ. J. Math. Phys. **27** (2020), no. 2, 227-235.

[9] Kim, G.; Kim, B.; Choi, J. *The DC-algorithm for computing sums of powers of consecutive integers and Bernoulli numbers,* Adv. Stud. Contemp. Math. (Kyungshang) **17** (2008), no. 2, 137-145.

[10] Kim, T. *Sums of powers of consecutive q-integers,* Adv. Stud. Contemp. Math. (Kyungshang) **9** (2004), no. 1, 15-18.

[11] Kim, T.; Kim, D. S. *Generalization of Spivey's recurrence relation,* Russ. J. Math. Phys. **31** (2024), no. 2, 218-226.

[12] Kim, T.; Kim, D. S. *Probabilistic degenerate Bell polynomials associated with random variables,* Russ. J. Math. Phys. **30** (2023), no. 4, 528-542.

[13] Kim, T.; Kim, D. S.; Kim, W.; Kwon. J. *A note on modified degenerate gamma random variables,* Proc. Jangjeon Math. Soc. **27** (2024), no. 1, 91–95.

[14] Kim, T.; Kim, D. S.; Jang, L.-C.; Kim, H. Y. *A note on discrete degenerate random variables,* Proc. Jangjeon Math. Soc. **23** (2020), no. 1, 125–135.

[15] Kim, T.; Rim, S.-H.; Simsek, Y. *A note on the alternating sums of powers of consecutive q-integers,* Adv. Stud. Contemp. Math. (Kyungshang) **13** (2006), no. 2, 159-164.

[16] Kim, T.; Kim, D. S.; Kim, H. K. *λ-Whitney numbers and λ-Dowling polynomials,* Proc. Jangjeon Math. Soc. **26** (2023), no. 3, 227-242.

[17] Park, J.-W.; Pyo, S.-S. *A note on degenerate Bernoulli polynomials arising from umbral calculus,* Adv. Stud. Contemp. Math. (Kyungshang) **32** (2022), no. 4, 509-525.

[18] Simsek, Y. *Identities and relations related to combinatorial numbers and polynomials,* Proc. Jangjeon Math. Soc. **20** (2017), no. 1, 127-135.

[19] Spivey, M. Z. *A combinatorial view of sums of powers,* Math. Mag. **94** (2021), no. 2, 125-131.

[20] Xu, R.; Kim, T.; Kim, D. S.; Ma, Y. *Probabilistic degenerate Fubini polynomials associated with random variables,* J. Nonlinear Math. Phys. **31** (2024), no. 1, Paper No. 47.

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