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# TENSOR PRODUCT OF BAKER'S THEOREM

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ABSTRACT. In this article, we will give the historical background on Baker's theorem both in qualitative and in the quantitative form, in the homogeneous as well as in the non-homogeneous version. The main aim of this article is to describe the Baker's theorem (homogeneous form) equivalent to that as pointed out by J-P.Serre in his Bourbaki lecture on Baker's work, it means that the natural map from the tensor product $(\mathbb{Q}+\mathcal{L})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$ in $\mathbb{C}$, which extends the injection from $\mathbb{Q}+\mathcal{L}$ to $\mathbb{C}$, is still injective.

## 1. INTRODUCTION

In [1], references to the existence of transcendental numbers go back many centuries. The "transcendental" comes from Leibniz in his 1682 paper where he proved $\sin x$ is not an algebraic function of $x$. Certainly Leibniz believed that, besides rational and irrational numbers (by "irrational" he meant algebraic irrational numbers in modern terminology), there also exist transcendental numbers. In [2], Liouville proved a fundamental theorem concerning approximations of algebraic numbers by rational numbers in 1853. This theorem gives first example of transcendental numbers.

**Theorem 1.1** (J. Liouville, 1853)**.** *If $\alpha$ is algebraic of degree $d$, then there is a positive constant $C(\alpha)$, i.e. depending only on $\alpha$, such that for all rationals $\frac{p}{q}$,*

$$\left|\alpha-\frac{p}{q}\right|>\frac{C(\alpha)}{q^d}.$$

From this theorem, we can find explicit examples of transcendental numbers.

**Corollary 1.2.** *The number*

$$\sum_{n=0}^{\infty}\frac{1}{2^{n!}}$$

*is transcendental number.*

In [3], there appeared Hermite's epoch-making memoir entitled *Sur la fonction exponentielle* in which he established the transcendence of $e$, the natural base of logarithms. Liouville had shown in 1840, directly from the defining series, that in fact neither $e$ nor $e^2$ could be rational or quadratic irrational; but Hermite's work began a new era. In particular, within a decade, Lindemann succeeded in generalizing Hermite's method and, in a classical paper, he proved that $\pi$ is transcendental and solved thereby the ancient Greek problem concerning the quadrature of the circle. The work of Hermite and Lindemann was simplified by Weierstrass in 1885,

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and further simplified by Hilbert, Hurwitz and Gordan in 1893. In [4], the transcendence of e was first proved by Hermite in 1873 by using very different ideas and applying the approximation of analytic functions by rational functions.

**Theorem 1.3** (C. Hermite, 1873). *The number e is transcendental number.*

**Theorem 1.4** (F. Lindemann, 1882). *The number $\pi$ is transcendental number.*

In [4], Lindemann stated more general results. One of them is Hermite-Lindemann Theorem:

**Theorem 1.5** (Hermite-Lindemann). *If $\beta$ is a non-zero complex number. Then at least one of the two numbers $\beta$ and $e^{\beta}$ is transcendental.*

Thus, if $\beta$ is algebraic, then $e^{\beta}$ is transcendental number. Let $\alpha$ be non-zero algebraic number, and if $\lambda$ is any non-zero determination of its logarithm, then $\lambda$ is a transcendental number. Now, we define the set $\mathcal{L}$ of logarithm of non-zero algebraic numbers, that is the inverse image of the multiplicative group $\overline{\mathbb{Q}}^{\times}$ by the exponential map :

$$\mathcal{L} = \exp^{-1}(\overline{\mathbb{Q}}^{\times}) = \{\lambda \in \mathbb{C} : e^{\lambda} \in \overline{\mathbb{Q}}^{\times}\}.$$

**Theorem 1.6** (Lindemann-Weierstrass, 1885). *If $\beta_1, \ldots, \beta_n$ are distinct algebraic numbers, then $e^{\beta_1}, \ldots, e^{\beta_n}$ are linearly independent over $\overline{\mathbb{Q}}$.*

The theorem of Hermite-Lindemann can be written $\overline{\mathbb{Q}} \cap \mathcal{L} = \{0\}$, that is, $\lambda\,(\neq 0) \in \mathcal{L}$ is transcendental number.

This is one of the very few results on algebraic independence of numbers connected with the exponential function. After the contributions of J. Liouville, Ch. Hermite, F. Lindemann and K. Weiertraß, the next important step was provided by the work of C. L. Siegel , A. O. Gel'fond and Th. Schneider, which led to the solution of Hilbert's seventh problem.

The story of this problem is as follows. In his "Introductio in analysin infinitorum", L. Euler defined the exponential and logarithm functions, and said:

From what we have seen, it follows that the logarithm of a number will not be a rational number unless the given number is a power of the base *a*. That is, unless the number *b* is a power of the base *a*, the logarithm of *b* cannot be expressed as a rational number. In case *b* is a power of the base *a*, then the logarithm of *b* cannot be an irrational number. If, indeed, $\log b = \sqrt{n}$, then $a^{\sqrt{n}} = b$, but this is impossible if both *a* and *b* are rational. It is especially desirable to know the logarithms of rational numbers, since from these it is possible to find the logarithms of fractions and also surds. Since the logarithms of numbers which are not the powers of the base are neither rational nor irrational, it is with justice that they are called transcendental quantities. For this reason, logarithms are said to be transcendental.

D. Hilbert proposed this question as the seventh of his problem:

**Question.** *The expression $\alpha^{\beta}$ for an algebraic base $\alpha$ and an irrational algebraic exponent $\beta$, e.g. the number $2^{\sqrt{2}}$ or $e^{\pi} = (-1)^{-i}$, always represents a transcendental or at least an irrational number.*

In 1900, at the International Congress of Mathematicians held in Paris, Hilbert raised, as the seventh of his famous list of 23 problems, the question whether an irrational logarithms of an algebraic number to an algebraic base is transcendental. The question is capable of various alternative formulations; thus one can ask whether an irrational quotient of natural logarithms of algebraic number is transcendental, or whether $\alpha^\beta$ is transcendental for any algebraic number $\alpha \neq 0,1$ and any algebraic irrational $\beta$.

**Theorem 1.7** (Gelfond-Schneider, 1934)**.** *Suppose that* $\alpha \neq 0,1$ *and that* $\beta$ *is irrational. Then* $\alpha, \beta$ *and* $\alpha^\beta$ *cannot all be algebraic.*

In particular, $2^{\sqrt{2}}$ and $e^\pi = (-1)^{-i}$ are transcendental numbers. In the same year, Gelfond published extended his results [5] of the Gelfond-Schneider Theorem without proof.

This shows that $\mathcal{L}$, which is a $\mathbb{Q}$-vector space, is not a $\overline{\mathbb{Q}}$-vector space. More precisely, the quotient $\lambda_1/\lambda_2$ of two non-zero elements of $\mathcal{L}$ is either rational or a transcendental number. For instance $\log 2/\log 3$ is transcendental number. Such a quotient cannot be an algebraic irrational number, like $i = \sqrt{-1}$ or like $\sqrt{2}$. The connection with Hilbert's problem is most easily seen by stating the Theorem of Gel'fond-Schneider as follows:

**Theorem 1.8.** *If* $\lambda$ *and* $\beta$ *are two complex numbers with* $\lambda \neq 0$ *and* $\beta \in \mathbb{Q}$*, then one at least of the three numbers* $e^\lambda, \beta$ *and* $e^{\beta\lambda}$ *is transcendental.*

Gelfond was the first to study algebraic independence of the values of the exponential function at points that are not necessarily algebraic. In 1948, he conjectured that *if* $\alpha, \beta \in \overline{\mathbb{Q}}$, $\alpha \neq 0,1$, $\deg\beta = d \geq 2$, *then* $\alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$ *are algebraically independent.* In general this conjecture is still open. We will discuss it later. Gelfond proved the conjecture for $d = 3$ in 1948. The following result is more general than Gelfond's.

**Theorem 1.9** (R. Tijdemann, 1971)**.** *Let* $p, q$ *be positive integers with* $\frac{pq+p}{p+q} \geq 2$. *Let* $\{a_1, \ldots, a_p\}$ *and* $\{b_1, \ldots, b_q\}$ *be two sets of* $\mathbb{Q}$*-linearly independent complex numbers. Then the transcendence degree of*

$$\mathbb{Q}(a_1, \ldots, a_p, e^{a_1 b_1}, \ldots, e^{a_p b_q}) \geq 2.$$

In 1949, Gelfond proved Theorem 5.1 for the case $p = q = 3$ with some conditions on the numbers $a_i, b_j$ for $1 \leq i \leq p$, $1 \leq j \leq q$. Theorem 5.1 in the present general form was proved by Tijdemann in 1971. We derive some of consequences

**Theorem 1.10** (A. Gelfond, 1948)**.** *Let* $\alpha, \beta \in \overline{\mathbb{Q}}$ *with* $\alpha \neq 0,1$ *and* $\deg\beta = 3$. *Then* $\alpha^\beta, \alpha^{\beta^2}$ *are algebraically independent.*

*Proof.* Take $p = q = 3$, $a_j = \beta^{j-1}$, $b_j = \beta^{j-1}\log\alpha$ for $j = 1,2,3$. Since $\deg\beta = 3$, all the numbers $\beta^j, \alpha^{\beta^j}$ for $j \geq 1$ are algebraic over $\mathbb{Q}(\alpha^\beta, \alpha^{\beta^2})$. Hence by Theorem 5.1, $\alpha^\beta$ and $\alpha^{\beta^2}$ are algebraically independent. □

**Theorem 1.11** (Shmelev, 1968)**.** *Let* $\alpha_1, \alpha_2 \in \overline{\mathbb{Q}}$ *such that* $\log\alpha_1$ *and* $\log\alpha_2$ *are linearly independent over* $\mathbb{Q}$. *Suppose* $\beta \in \overline{\mathbb{Q}}$ *with* $\deg\beta = 2$. *Then at least two of the numbers* $\dfrac{\log\alpha_2}{\log\alpha_1}, \alpha_1^\beta, \alpha_2^\beta$ *are algebraically independent.*

*Proof.* We take $p = 4$, $q = 2$, $\gamma = \frac{\log\alpha_2}{\log\alpha_1}$, $a_1 = 1$, $a_2 = \gamma$, $a_3 = \beta$, $a_4 = \beta\gamma$, $b_1 = \log\alpha_1$, $b_2 = \beta\log\alpha_1$. Then we see that $e^{a_i b_j}$ for $1 \leq i \leq 4$, $1 \leq j \leq 2$ are algebraic over $\mathbb{Q}(\gamma, \alpha_1^\beta, \alpha_2^\beta)$. Now the result follows from Theorem 1.8. □

## 2. Alan Baker

In his book [3], A. O. Gel'fond emphasized the importance of getting a generalization of this statement to more than two logarithms. Let $\lambda_1, \cdots, \lambda_n$ be $n$-logarithms of algebraic numbers which are linearly independent over $\mathbb{Q}$. The question is to prove that they are also linearly independent over the field $\mathbb{Q}$ of algebraic numbers. For $n = 2$, this is Theorem 1.8 of Gelfond-Schneider. This problem was solved in 1966 by A. Baker

**Theorem 2.1** (Baker-Homogeneous Case)**.** . *If $\lambda_1, \cdots, \lambda_n$ are $\mathbb{Q}$-linearly independent element of $\mathcal{L}$, then they are linearly independent over $\overline{\mathbb{Q}}$.*

Shortly later, A. Baker extended his result to a non-homogeneous situation as follows:

**Theorem 2.2** (Baker-General Case)**.** *If $\lambda_1, \cdots, \lambda_n$ are $\mathbb{Q}$-linearly independent elements of $\mathcal{L}$, then the $n+1$ numbers $1, \lambda_1, \cdots, \lambda_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

From Baker's Theorem, one easily deduces that if a number of the form

$$e^{\beta_0}\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n} = \exp\{\beta_0 + \beta_1\lambda_1 + \cdots + \beta_n\lambda_n\}$$

(with $\beta_i \in \overline{\mathbb{Q}}, \lambda_i \in \mathcal{L}$ and $\alpha_i = e^{\lambda_i} \in \overline{\mathbb{Q}}^{\times}$) is algebraic, then $\beta_0 = 0$, and moreover, either $\lambda_1, \cdots, \lambda_n$ are all zero, or else the numbers $1, \beta_1, \cdots, \beta_n$ are lineraly independent over $\mathbb{Q}$. Also Theorem shows that any non-zero element in the $\overline{\mathbb{Q}}$-vector space

$$\left\{\beta_1\lambda_1 + \cdots + \beta_n\lambda_n \; ; \; n \geq 0, \; \beta_i \in \overline{\mathbb{Q}}, \; \lambda_i \in \mathcal{L}\right\}$$

spanned by $\mathcal{L}$ is transcendental. It will be convenient to show that several statements are equivalent to Baker's homogeneous Theorem 1.8. As pointed out by J-P. Serre in his Bourbaki lecture on Baker's work, it means that the natural map from the tensor product $(\mathbb{Q}+\mathcal{L}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ in $\mathbb{C}$, which extends the injection from $\mathbb{Q}+\mathcal{L}$ to $\mathbb{C}$, is still injective.

**Lemma 2.3.** *Let $k \subset K$ be two fields, $\varepsilon$ be a $K$-vector space, and $\mathcal{M}$ be a $k$-vector space subspace in $\varepsilon$. The three following statements are equivalent.*

*(1) Let $m$ be a positive integer and let $\lambda_1, \cdots, \lambda_m$ be elements of $\mathcal{M}$ which are linearly independent over $k$. Then these elements are also linearly independent over $K$ in $\varepsilon$.*

*(2) Let $m$ be a positive integer. Let $\lambda_1, \cdots, \lambda_m$ be elements of $\mathcal{M}$, not all vanishing, and let $\beta_1, \cdots, \beta_m$ be $k$-linearly independent elements of $K$. Then*

$$\beta_1\lambda_1 + \cdots + \beta_m\lambda_m \neq 0.$$

*(3) Let $m$ be a positive integer. Let $\lambda_1, \cdots, \lambda_m$ be $k$-linearly independent element of $\mathcal{M}$ and $\beta_1, \cdots, \beta_m$ be $k$-linearly independent element of $K$. Then*

$$\beta_1\lambda_1 + \cdots + \beta_m\lambda_m \neq 0.$$

*Proof.* First, we can easily show that (1) $\Longrightarrow$ (3).

(2) $\Longrightarrow$ (1). Assume that for some $m \geq 1$ we have a relation $\beta_1\lambda_1 + \cdots + \beta_m\lambda_m = 0$ with $\beta_1, \cdots, \beta_m$ not all zero in $K$. Let $\beta_1', \cdots, \beta_s'$ (with $0 < s \leq m$)be a basis of the $k$-vector space they span. We can write

$$\beta_i = \sum_{j=1}^{s} c_{ij}\beta_j' \quad (1 \leq i \leq m),$$

with $c_{ij} \in k$, which not all zero. Then

$$\sum_{j=1}^{s} \beta_j' \left( \sum_{i=1}^{m} c_{ij}\lambda_i \right) = 0.$$

Since $\beta_1', \cdots, \beta_s'$ are $k$-linearly independent, we deduce from (2)

$$\sum_{i=1}^{m} c_{ij}\lambda_i = 0 \quad \text{for } 1 \le j \le s.$$

Therefore, $\lambda_1, \cdots, \lambda_m$ are $K$-linearly independent.

(3) $\Longrightarrow$ (2) Assume $\beta_1\lambda_1 + \cdots + \beta_m\lambda_m = 0$ with $\beta_1, \cdots, \beta_m$ linearly independent over $k$ in $K$ and $\lambda_1, \cdots, \lambda_m$ in $\mathcal{M}$. We shall conclude $\lambda_1 = \cdots = \lambda_m = 0$. Renumbering $\lambda_1, \cdots, \lambda_m$ if necessary, we may assume that $\lambda_1, \cdots, \lambda_r = 0$. (for some $r$ with $0 \le r \le m$) is a basis of the $k$-vector space spanned by $\lambda_1, \cdots, \lambda_m$:

$$\lambda_i = \sum_{j=1}^{r} c_{ij}\lambda_j, \qquad (r+1 \le i \le m),$$

where $c_{ij}$ are in $K$. We deduce

$$\sum_{j=1}^{r} \gamma_j\lambda_j = 0 \text{ with } \gamma_j = \beta_j + \sum_{i=r+1}^{m} c_{ij}\beta_i, \quad (1 \le j \le r).$$

Using (3) (with $m$ replaced by $r$), we deduce from the linear independence of $\lambda_1, \cdots, \lambda_r$ over $k$ that the $r$ elements $\gamma_1, \cdots, \gamma_r$ are $k$-linearly dependent in $K$. However, since $\beta_1, \cdots, \beta_m$ are linearly independent over $k$, the only possibility is $r = 0$, which means $\lambda_1 = \cdots = \lambda_m = 0$. □

*Remark.* When $k = \mathbb{Q}$, $K = \overline{\mathbb{Q}}$, $\mathcal{M} = \mathcal{L}$ and $\varepsilon = \mathbb{C}$, assertion 1.) is nothing but Theorem.

## 3. Baker's Theorem in tensor product version

In this section, we will prove that Baker's theorem equivalent to the following.

**Theorem 3.1.** *Baker's theorem can be represented the following tensor product: The injection of* $\mathbb{Q} + \mathcal{L}$ *into* $\mathbb{C}$ *extends to an injection of* $(\mathbb{Q} + \mathcal{L}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ *into* $\mathbb{C}$. *The image is the* $\overline{\mathbb{Q}}$*-vector space* $\widetilde{\mathcal{L}} \subset \mathbb{C}$ *spanned by 1 and* $\mathcal{L}$*:* $(\mathbb{Q} + \mathcal{L}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \simeq \widetilde{\mathcal{L}}$

Now, we will prove that the theorem 3.1. First we will claim that the Lemma 2.3 is equivalent to the following statements.

(4) Let $n$ be a non-negative integer, $\lambda_1, \cdots, \lambda_{n+1}$ be elements of $\mathcal{M}$, and $\beta_1, \cdots, \beta_n$ elements of $K$. Assume $\lambda_1, \cdots, \lambda_n$ are $K$-linearly independent and

$$\beta_1\lambda_1 + \cdots + \beta_n\lambda_n = \lambda_{n+1}.$$

Then $\beta_1, \cdots, \beta_n$ are all in $k$.

(5) The natural map $\mathcal{M} \otimes_k K \to \varepsilon$, which extends the injection from $\mathcal{M}$ to $\varepsilon$, is still injective.

Let $(\mu_i)_{i\in I}$ be a basis of the $k$-vector space $\mathcal{M}$, and let $(\gamma_j)_{j\in J}$ be a basis of the $k$-vector space in $\mathcal{M}$, and let $(\gamma_j)_{j\in J}$ be a basis of the $k$-vector space $K$. Then $\mu_i\otimes\gamma_j$ $(i\in I, j\in J)$ is a basis of $\mathcal{M}\otimes_k K$ over $k$:

$$\begin{aligned}\mathcal{M}\otimes_k K &= \left\{\sum_{i\in I}\mu_i\otimes\beta_i;\ \beta_i\in K \text{ with } \mathsf{supp}(\beta_i)_{i\in I} \text{ finite}\right\}\\ &= \left\{\sum_{j\in J}\lambda_j\otimes\gamma_j;\ \lambda_j\in\mathcal{M} \text{ with } \mathsf{supp}(\lambda_j)_{j\in J} \text{ finite}\right\}\\ &= \left\{\sum_{i\in I}\sum_{j\in J}c_{ij}\mu_i\otimes\gamma_j;\ c_{ij}\in k \text{ with } \mathsf{supp}(c_{ij})_{i\in I, j\in J} \text{ finite}\right\}\end{aligned}$$

where finite support means that all but finitely many elements vanish.

The map $\mathcal{M}\otimes_k K\to\varepsilon$ is nothing but

$$\sum_{i\in I}\mu_i\otimes\beta_i\longmapsto\sum_{i\in I}\mu_i\beta_i,\qquad \sum_{j\in J}\lambda_j\otimes\gamma_j\longmapsto\sum_{j\in J}\lambda_j\gamma_j$$

as well as

$$\sum_{i\in I}\sum_{j\in J}c_{ij}\mu_i\otimes\gamma_j\longmapsto\sum_{i\in I}\sum_{j\in J}c_{ij}\mu_i\gamma_j.$$

*Proof.* (1), (2), (3) $\Longrightarrow$ (4) Let $\beta_1\lambda_1+\cdots+\beta_n\lambda_n=\lambda_{n+1}\in\mathcal{M}$. Now, $S$ and $T$ are $K$-vector space, $k$-vector space (respectively) spanned by $\lambda_1,\cdots,\lambda_n$. Put $S\cap\mathcal{M}=U$. Since $T\subset U$ and $\lambda_1,\cdots,\lambda_n$ are $k$-linearly independent, $\dim U\geq n$. If $\mu_1,\cdots,\mu_m\in U$ are $k$-linearly independent then by the a) $K$-linearly independent and $\mu_1,\cdots,\mu_m\in S$. So $m\leq n$. Consequently, $\dim U=n$, $U=T$.

Also, $\lambda_{n+1}\in U$ then $\lambda_{n+1}\in T$. So $\exists\ c_1,\cdots,c_n\in k$ such that $c_1\lambda_1+\cdots+c_n\lambda_n=\beta_1\lambda_1+\cdots+\beta_n\lambda_n$. But $\lambda_1,\cdots,\lambda_n$ are $K$-linearly independent then $c_i=\beta_i$ for all $i$. Therefore, $\beta_i\in K$.

(1) $\Longrightarrow$ (5). The map

$$\mathcal{M}\otimes_k K\longrightarrow\varepsilon \text{ defined by } \sum_{i\in I}\mu_i\otimes\beta_i\longmapsto\sum_{i\in I}\mu_i\beta_i$$

If $\sum_{i\in I}\mu_i\beta_i=0$ then $\mu_i$ is $k$-linearly independent, by the (1) $K$-linearly independent. So $\beta_i=0$ for all $i$. Therefore,

$$\sum_{i\in I}\mu_i\otimes\beta_i=0$$

Thus, this function is injective.

(4) $\Longrightarrow$ (1). Suppose that the set $\{\lambda_1,\cdots,\lambda_m\}$ is $k$-linearly independent, but $K$-linearly independent. Let $n$ be a maximum number such that $\{\lambda_1,\cdots,\lambda_n\}$ $K$-linearly independent. Then $n<m$, $\exists\ \beta_1,\cdots,\beta_n\in K$ such that $\beta_1\lambda_1+\cdots+\beta_n\lambda_n=\lambda_{n+1}$. By the (4), since $\beta_1,\cdots,\beta_n\in k$, $\lambda_1,\cdots,\lambda_n,\lambda_{n+1}$ are $k$-linearly independent. Contradiction.

(5) $\Longrightarrow$ (3). Suppose that $\beta_1\lambda_1+\cdots+\beta_m\lambda_m=0$ with $\beta_1,\cdots,\beta_m$ linearly independent over $k$ in $K$, and $\lambda_1,\cdots,\lambda_n$ be $k$-linearly independent in $\mathcal{M}$. Since $(\mu_i)_{i\in I}$ is a

basis of the $k$-vector space $\mathcal{M}$. Also, $(\gamma_j)_{j\in J}$ be a basis of the $k$-vector space over $K$. So,

$$\lambda_j = \sum_{i\in I} c_{ij}\mu_i \quad (1\le j\le m), \qquad \beta_i = \sum_{j\in J} c_{ij}\gamma_j \quad (1\le i\le m)$$

So,

$$\beta_1\lambda_1 + \cdots + \beta_m\lambda_m = \sum_{i\in I}\sum_{j\in J} c_{ij}\mu_i\gamma_j = 0$$

By the (5), the map is still injective,

$$\sum_{i\in I}\sum_{j\in J} c_{ij}\mu_i \otimes \gamma_j = 0.$$

However, since $\mu_i\otimes\gamma_j$ $(i\in I, j\in J)$ is a basis of $\mathcal{M}\otimes_k K$ over $k$. So contradiction. □

Lastly, we put $k := \mathbb{Q}$, $\mathcal{M} := \mathbb{Q}+\mathcal{L}$ and $\varepsilon := \mathbb{C}$. So we can get the desired result.

Baker's theorem means: The injection of $\mathbb{Q}+\mathcal{L}$ into $\mathbb{C}$ extends to an injection of $(\mathbb{Q}+\mathcal{L})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$ into $\mathbb{C}$. The image is the $\overline{\mathbb{Q}}$-vector space $\widetilde{\mathcal{L}}\subset\mathbb{C}$ spanned by 1 and $\mathcal{L}$: $(\mathbb{Q}+\mathcal{L})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}\simeq\widetilde{\mathcal{L}}$.

$$\widetilde{\mathcal{L}} = \left\{\beta_0 + \sum_{h=1}^{l}\beta_h\log\alpha_h \ : \ l\ge 0, \alpha\in\overline{\mathbb{Q}}^{\times}, \beta\in\overline{\mathbb{Q}}\right\}$$

It will be convenient to show that several statements are equivalent to Baker's homogeneous Theorem 1.8. As pointed out by J-P. Serre in his Bourbaki lecture on Baker's work, it means that the natural map from the tensor product $(\mathbb{Q}+\mathcal{L})\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$ in $\mathbb{C}$, which extends the injection from $\mathbb{Q}+\mathcal{L}$ to $\mathbb{C}$, is still injective.

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